International Journal of Mathematics And its Applications
Volume 5, Issue 4–C (2017), 277–282.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

On Covering Radius of Codes Over $R = \mathbb{Z}_2 + u\mathbb{Z}_2$, where $u^2 = 0$ Using Bachoc Distance

Research Article

P. Chella Pandian[1*]

1 Department of Mathematics, Srimad Andavan Arts and Science College (Autonomous), Tiruchirappalli, Tamilnadu, India.

Abstract: In this paper, we give lower and upper bounds on the covering radius of codes over the ring $R = \mathbb{Z}_2 + u\mathbb{Z}_2$, where $u^2 = 0$ with bachoc distance and also obtain the covering radius of various Repetition codes, Simplex codes of α-Type code and β-Type code. We give bounds on the covering radius for MacDonald codes of both types over $R = \mathbb{Z}_2 + u\mathbb{Z}_2$.

MSC: 20C05, 20C07, 94A05, 94A24.

Keywords: Covering radius, Codes over finite rings, Simplex code, MacDonald code.

1. Introduction

In recent years, several papers have mentioned, codes over $\mathbb{Z}_4$ received much attention [1–3, 10, 12, 14, 15]. The covering radius of binary linear codes were studied [7, 8]. Recently the covering radius of codes over $\mathbb{Z}_4$ has been investigated with respect to chinese euclidean distance [5]. In 1999, Sole et al gave many upper and lower bounds on the covering radius of a code over $\mathbb{Z}_4$ with chinese euclidean distance. In [4], the covering radius of some particular codes over $\mathbb{Z}_2 + u\mathbb{Z}_2$ have been investigated. In this correspondence, we consider the ring $R = \mathbb{Z}_2 + u\mathbb{Z}_2$, where $u^2 = 0$. In this paper, we investigate the covering radius of the Simplex codes of both types and MacDonald codes and repetition codes over R. We also generalized some of the known bounds in [1]. A *linear code* C of length n over R is an additive subgroup of R^n. An element of C is called a *codeword* of C and a *generator matrix* of C is a matrix whose rows generate C. The Bachoc weight is defined in [6] and the weight of the elements 0, 1, u and 1+u are 0, 1, 2 and 2 respectively.

Definition 1.1. *The Bachoc weight is given by the relation* $wt_B = \sum_{i=1}^{n} wt_B(x_i)$, *where*

$$wt_B(x_i) = \begin{cases} 0 & if \;\; x_i = 0 \\ 1 & if \;\; x_i = 1 \\ 2 & if \;\; x_i = u \; or \; 1+u \end{cases}$$

The Bachoc distance between x and y in R^n is $d_B(x, y) = wt_B(x - y) = \sum_{i=1}^{n} wt_B(x_i - y_i)$. The minimum Bachoc weight d_B of C is the smallest Bachoc weights among all non-zero codewords of C. A linear *Gray map* ϕ from $R \to \mathbb{Z}_2^2$ is defined by

---

[*] *E-mail: chellapandianpc@gmail.com*

$\phi(x + uy) = (y, x + y)$, for all $x + uy \in R$. The image $\phi(C)$, of a linear code C over R of length n by the Gray map, is a binary code of length $2n$ with same cardinality [14]. Any linear code C over R is equivalent to a code with generator matrix G of the form

$$G = \begin{bmatrix} I_{k_0} & A & B \\ \mathbf{0} & uI_{k_1} & 2D \end{bmatrix}, \qquad (1)$$

where A, B and D are matrices over R. Then the code C contain all codewords $[v_0, v_1]G$, where v_0 is a vector of length k_1 over R and v_1 is a vector of length k_2 over $\mathbb{Z}_2$. Thus C contains a total of $4^{k_1}2^{k_2}$ codewords. The parameters of C are given $[n, 4^{k_1}2^{k_2}, d]$ where d repereents the minimum bachoc distance of C. A linear code C over R of length n, 2-dimension k, minimum Bachoc distance d_B is called an $[n, k, d_B]$ or simply an $[n, k]$ code.

2. Covering Radius of Codes

In this section, we introduce the basic notions of the covering radius of codes over R. The covering radius of a code C, denoted $r(C)$, is the smallest number r such that the spheres covering radius of radius r around the codewords of C cover the sets R^n. The covering radius of a code C over R with respect to the Bachoc distance is given by $r_B(C) = \max_{x \in R^n} \left\{ \min_{c \in C} \{d(x, c)\} \right\}$. The following result of Mattson [7] is useful for computing covering radius of codes over rings generalized easily from codes over finite fields.

Proposition 2.1. *If C_0 and C_1 are codes over R generated by matrices G_0 and G_1 respectively and if C is the code generated by*

$$G = \left(\begin{array}{c|c} 0 & G_1 \\ \hline G_0 & A \end{array} \right),$$

then $r_d(C) \leq r_d(C_0) + r_d(C_1)$ and the covering radius of D (concatenation of C_0 and C_1) satisfy the following $r_d(D) \geq r_d(C_0) + r_d(C_1)$, for all distances d over R.

3. Covering Radius of Repetition Codes

A q-ary repetition code C over a finite field $\mathbb{F}_q = \{\alpha_0 = 0, \alpha_1 = 1, \alpha_2, \alpha_3, \cdots, \alpha_{q-1}\}$ is an $[n, 1, n]$ code $C = \{\bar{\alpha} | \alpha \in \mathbb{F}_q\}$, where $\bar{\alpha} = \alpha\alpha\cdots\alpha$. The covering radius of C is $\lfloor \frac{n(q-1)}{q} \rfloor$ [13]. Using this, it can be seen easily that the covering radius of block of size n repetition code $[n(q-1), 1, n(q-1)]$ generated by $G = [\overbrace{11\cdots1}^{n}\overbrace{\alpha_2\alpha_2\cdots\alpha_2}^{n}\cdots\overbrace{\alpha_{q-1}\alpha_{q-1}\cdots\alpha_{q-1}}^{n}]$ is $\lfloor \frac{n(q-1)^2}{q} \rfloor$ since it will be equivalent to a repetition code of length $(q-1)n$. Consider the repetition code over R. There are two types of them of length n viz. unit repetition code $C_\beta : [n, 1, 2n]$ generated by $G_\beta = [\overbrace{11\cdots1}^{n}]$ and zero divisor repetition code $C_\alpha : (n, 2, 4n)$ generated by $G_\alpha = [\overbrace{uu\cdots u}^{n}]$. The following result determines the covering radius with respect to chinese euclidean distance over R.

Theorem 3.1. $2\lfloor \frac{n}{2} \rfloor \leq r_B(C_\alpha) \leq 2n$ *and* $n \leq r_B(C_\beta) \leq 2n$.

Proof. We know that $r_B(C_\alpha) = \max_{x \in R^n} \{d(x, C_\alpha)\}$. Let $x = \overbrace{uu\cdots u}^{\lfloor \frac{n}{2} \rfloor}\overbrace{000\cdots0}^{\lceil \frac{n}{2} \rceil} \in R^n$ and the generator matrix of α-type code is $[uu\cdots u]$ is an $[n, 1, 2n]$ code. We have, $d_B(x, 00\cdots0) = wt_B(\overbrace{uu\cdots u}^{\lfloor \frac{n}{2} \rfloor}\overbrace{00\cdots0}^{\lceil \frac{n}{2} \rceil} - 00\cdots0) = \lfloor \frac{n}{2} \rfloor u = \lfloor \frac{n}{2} \rfloor 2$, since the bachoc weight of u is 2, and $d_B(x, uu\cdots u) = wt_B(\overbrace{uu\cdots u}^{\lfloor \frac{n}{2} \rfloor}\overbrace{000\cdots0}^{\lceil \frac{n}{2} \rceil} - uu\cdots u) = u\lceil \frac{n}{2} \rceil = 2\lceil \frac{n}{2} \rceil$. Therefore, $d_B(x, C_\alpha) =$

$\min\{2\left\lfloor\frac{n}{2}\right\rfloor, 2\left\lceil\frac{n}{2}\right\rceil\} = 2\left\lfloor\frac{n}{2}\right\rfloor$. Thus,

$$r_B(C_\alpha) \geq 2\left\lfloor\frac{n}{2}\right\rfloor. \tag{2}$$

If x be any word in R. Let us take x has ω_0 coordinates as $0'$s, ω_1 coordinates as $1'$s, ω_2 coordinates as u's and ω_3 coordinates as $(1+u)'$s, then $\omega_0+\omega_1+\omega_2+\omega_3 = n$. Since $C_\alpha = \{00\cdots0, uu\cdots u\}$ and the bachoc weight of R : 0 is 0, 1 is 1 and $(1+u)$, u is 2, we have $d_B(x, 00\cdots0) = n-\omega_0+\omega_2+\omega_3$ and $d_B(x, uu\cdots u) = n-\omega_2+\omega_0+\omega_3$. Thus

$$d_B(x, C_\alpha) = \min\{n-\omega_0+\omega_2+\omega_3, n-\omega_2+\omega_0+\omega_3\}.$$

$$d_B(x, C_\alpha) \leq n+n = 2n, \text{since } w_3 \leq n. \tag{3}$$

From the Equations (2) and (3), we get $2\left\lfloor\frac{n}{2}\right\rfloor \leq r_B(C_\alpha) \leq 2n$. Obtain the covering radius of C_β with respect to the bachoc weight. We have $d_B(x, 00\cdots0) = n-\omega_0+\omega_2+\omega_3$, $d_B(x, 11\cdots1) = n-\omega_1+\omega_2+\omega_3, d_B(x, uu\cdots u) = n-\omega_2+\omega_0+\omega_3$ and $d_B(x, 1+u1+u\cdots1+u) = n-\omega_3+\omega_0+\omega_1$ for any $x \in R$. This implies $d_B(x, C_\beta) = \min\{n-\omega_0+\omega_2+\omega_3, n-\omega_1+\omega_2+\omega_3, n-\omega_2+\omega_0+\omega_3, n-\omega_3+\omega_0+\omega_1\} \leq 2n$ and hence $r_B(C_\beta) \leq 2n$. Let $x = \overbrace{00\cdots0}^{t}\overbrace{11\cdots1}^{t}\overbrace{uu\cdots u}^{t}\overbrace{1+u1+u\cdots1+u}^{n-3t} \in R^n$, where $t = \lfloor\frac{n}{4}\rfloor$, then $d_B(x, 00\cdots0) = 2n-3t, d_B(x, 11\cdots1) = 2n-4t, d_B(x, uu\cdots u) = n$ and $d_B(x, 1+u1+u\cdots1+u) = 5t$. Therefore, $r_B(C_\beta) \geq \min\{2n-3t, 2n-4t, n\} \geq n$. □

To determines the covering radius of R three blocks each of size n repetition code $BRep^{3n} : [3n, 1, 4n]$ generated by $G = [\overbrace{11\cdots1}^{n}\overbrace{uu\cdots u}^{n}\overbrace{1+u1+u\cdots1+u}^{n}]$ the block repetition code $BRep^{3n}$: $\{c_0 = (0\cdots0\,0\cdots0\,0\cdots0),\ c_1 = (11\cdots1\ uu\cdots u\ 1+u1+u\cdots1+u), c_2 = (uu\cdots u\ 0\cdots0\ uu\cdots u), c_3 = (1+u1+u\cdots1+u\ uu\cdots u\ 1\cdots1)\}$. Thus $d_B(x, BRep^{3n}) = 2\lfloor\frac{n}{2}\rfloor+2n$ and $r_B(BRep^{3n}) \geq 2\lfloor\frac{n}{2}\rfloor+2n$. Let $x = (u|v|w) \in R^{3n}$, with u, v and w have compositions (r_0, r_1, r_2, r_3), (s_0, s_1, s_2, s_3) and (t_0, t_1, t_2, t_3) respectively such that $\sum_{i=0}^{3} r_i = n$, $\sum_{i=0}^{3} s_i = n$ and $\sum_{i=0}^{3} t_i = n$, then $d_B(x, c_0) = 3n-r_0+r_2+r_3-s_0+s_2+s_3-t_0+t_2+t_3$, $d_B(x, c_1) = 3n-r_1+r_2+r_3-s_2+s_0+s_1+s_3-t_3+t_0+t_1$, $d_B(x, c_2) = 3n-r_2+r_0+r_1-s_0+s_2+s_3-t_2+t_0+t_1$ and $d_B(x, c_3) = 3n-r_3+r_0+r_1-s_2+s_0+s_1-t_1+t_2+t_2$. Thus, $d_B(x, BRep^{3n}) = \min\{3n-r_0+r_2+r_3-s_0+s_2+s_3-t_0+t_2+t_3, 3n-r_1+r_2+r_3-s_2+s_0+s_1+s_3-t_3+t_0+t_1, 3n-r_2+r_0+r_1-s_0+s_2+s_3-t_2+t_0+t_1, 3n-r_3+r_0+r_1-s_2+s_0+s_1-t_1+t_2+t_3\}$. Thus, we have the following theorem

Theorem 3.2. $2\lfloor\frac{n}{2}\rfloor+2n \leq r_B(BRep^{3n}) \leq 4n$.

One can also define a R two blocks each of size n repetition code $BRep^{2n} : [2n, 1, 2n]$ generated by $G = [\overbrace{11\cdots1}^{n}\overbrace{uu\cdots u}^{n}]$. We have following theorem.

Theorem 3.3. $2\lfloor\frac{n}{2}\rfloor+n \leq r_B(BRep^{2n}) \leq \frac{11n}{4}$.

Block code $BRep^{m+n}$ can be generalized to a block repetition code (two blocks of size m and n respectively) $BRep^{m+n}$: $[m+n, 1, \min\{m, 2m+n\}]$ generated by $G = [\overbrace{11\cdots1}^{m}\overbrace{uu\cdots u}^{n}]$. Theorem 3.3 can be easily generalized for two different length using similar arguments to the following

Theorem 3.4. $2\lfloor\frac{n}{2}\rfloor+m \leq r_B(BRep^{2n}) \leq 2m+\frac{3n}{2}$.

4. Simplex Codes of α-type Code and β-type Code Over R

Quaternary Simplex codes of α-type and β-type have been recently studied in [2]. The α-type Simplex code S_k^α is a linear code over R with parameters $[4^k, k]$ and an inductive generator matrix given by

$$G_k^\alpha = \left[\begin{array}{c|c|c|c} 0\,0\cdots0 & 1\,1\cdots1 & u\,u\cdots u & 1+u\,1+u\cdots1+u \\ \hline G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha & G_{k-1}^\alpha \end{array}\right] \tag{4}$$

with $G_1^\alpha = [0\ 1\ u\ 1+u]$. The β-type simplex code S_k^β is a punctured version of S_k^α with parameters $\left[2^{k-1}(2^k-1), k\right]$ and an inductive generator matrix given by

$$G_2^\beta = \left[\begin{array}{cccc|c|c} 1 & 1 & 1 & 1 & 0 & u \\ \hline 0 & 1 & u & 1+u & 1 & 1 \end{array}\right], \tag{5}$$

and for $k > 2$

$$G_k^\beta = \left[\begin{array}{c|c|c} 1\,1\cdots 1 & 0\,0\cdots 0 & u\,u\cdots u \\ \hline G_{k-1}^\alpha & G_{k-1}^\beta & G_{k-1}^\beta \end{array}\right], \tag{6}$$

where G_{k-1}^α is the generator matrix of S_{k-1}^α. For details the reader is refered to [2].

Theorem 4.1. $r_B(S_k^\alpha) \le \frac{2^{2k+2}-1}{3}$.

Proof. From equation 4, the result of Mattson for finite rings and using Theorem 3.2, we get

$$\begin{aligned} r_B(S_k^\alpha) &\le r_B(S_{k-1}^\alpha) + r_B(<\overbrace{11\cdots 1}^{2^{2(k-1)}}\overbrace{uu\cdots u}^{2^{2(k-1)}}\overbrace{1+u1+u\cdots 1+u}^{2^{2(k-1)}}>) \\ &= r_B(S_{k-1}^\alpha) + 4.2^{2(k-1)} \\ &\le 4.2^{2(k-1)} + 4.2^{2(k-2)} + 4.2^{2(k-3)} + \cdots + 4.2^{2.1} + r_B(S_1^\alpha) \\ r_B(S_k^\alpha) &\le \frac{2^{2k+2}-1}{3} \text{ (since } r_B(S_1^\alpha) = 5) \end{aligned}$$

□

Theorem 4.2. $r_B(S_k^\beta) \le \frac{2^{2k+1}+3.4^{k-1}-9.2^{k-2}-20}{3}$.

Proof. By equation 6, Proposition 2.1 and Theorem 3.4, we get

$$\begin{aligned} r_B(S_k^\beta) &\le r_B(S_{k-1}^\beta) + r_B(<\overbrace{11\cdots 1}^{4^{(k-1)}}\ \overbrace{uu\cdots u}^{2^{(2k-3)}-2^{(k-2)}}\ >) \\ &= r_B(S_{k-1}^\beta) + 2^{(2k-2)} + 2^{(2k-3)} - 2^{(k-2)} \\ &\le 2(2^{(2k-2)} + 2^{(2k-4)} + \cdots + 2^4) + \tfrac{3}{2}(2^{(2k-3)} + 2^{(2k-5)} + \cdots + 2^3) - \\ &\quad \tfrac{3}{2}(2^{(k-2)} + 2^{(k-3)} + \cdots + 2) + r_B(S_2^\beta) \\ r_B(S_k^\beta) &\le \frac{2^{2k+1}+3.4^{k-1}-9.2^{k-2}-20}{3} \text{ (since } r_B(S_2^\beta) = 5). \end{aligned}$$

□

5. MacDonald Codes of α-type Code and β-type Code Over R

The q-ary MacDonald code $M_{k,t}(q)$ over the finite field $\mathbb{F}_q$ is a unique $[\frac{q^k-q^t}{q-1}, k, q^{k-1} - q^{t-1}]$ linear code in which every non-zero codeword has weight either q^{k-1} or $q^{k-1} - q^{t-1}$ [11]. In [13], he studied the covering radius of MacDonald codes over a finite field. In fact, he has given many exact values for smaller dimension. In [9], authors have defined the MacDonald codes over a ring using the generator matrices of the Simplex codes. For $2 \le t \le k-1$, let $G_{k,t}^\alpha$ be the matrix obtained from G_k^α by deleting columns corresponding to the columns of G_t^α. That is,

$$G_{k,t}^\alpha = \left[\ G_k^\alpha \ \setminus \frac{\mathbf{0}}{G_t^\alpha}\ \right] \tag{7}$$

and let $G_{k,t}^\beta$ be the matrix obtained from G_k^β by deleting columns corresponding to the columns of G_t^β. That is,

$$G_{k,t}^\beta = \left[\ G_k^\beta \ \setminus \frac{\mathbf{0}}{G_t^\beta}\ \right] \tag{8}$$

where $[A\backslash B]$ denotes the matrix obtained from the matrix A by deleting the columns of the matrix B and $\mathbf{0}$ is a $(k-t)\times 2^{2t}\left((k-t)\times 2^{t-1}(2^t-1)\right)$. The parameters in MacDonald codes of α-type and β-type is $[4^k-4^t,k]$ and $[(2^{k-1}-2^{t-1})(2^k+2^t-1),k]$ code over R. The following Theorem gives a basic bound on the covering radius of above MacDonald codes.

Theorem 5.1. $r_B(G^{\alpha}_{k,t}) \leq \frac{2^{2k+2}-2^{2r+2}}{3} + r_B(G^{\alpha}_{r,t})$ *for* $k \geq r > t$.

Proof. By Proposition 2.1 and Theorem 3.2,

$$\begin{aligned} r_B(G^{\alpha}_{k,t}) &\leq r_B(<\overbrace{11\cdots 1}^{2^{2(k-1)}}\overbrace{uu\cdots u}^{2^{2(k-1)}}\overbrace{1+u1+u\cdots 1+u}^{2^{2(k-1)}}>) + r_B(G^{\alpha}_{r,t}) \\ &= 4\cdot 4^{k-1} + r_B(\ G^{\alpha}_{k-1,t}) \\ &\leq 4\cdot 4^{k-1} + 4\cdot 4^{k-2} + \cdots + 4\cdot 4^{r} + r_B(G^{\alpha}_{r,t}) \text{ for } k\geq r>t \\ r_B(G^{\alpha}_{k,t}) &\leq \frac{2^{2k+2}-2^{2r+2}}{3} + r_B(G^{\alpha}_{r,t}) \text{ for } k\geq r>t. \end{aligned}$$

□

Theorem 5.2. $r_B(G^{\beta}_{k,t}) \leq \frac{2^{2k+2}-2^{2r+2}+3\cdot 2^{2k-2}-3\cdot 2^{2r-2}-9\cdot 2^{k-1}+9\cdot 2^{k-1}}{6} + r_B(G^{\beta}_{r,t})$ *for* $t<r\leq k$

Proof. Using Proposition 2.1 and Theorem 3.4, we have

$$\begin{aligned} r_B(G^{\alpha}_{k,t}) &\leq r_B(<\overbrace{11\cdots 1}^{2^{2(k-1)}}\quad \overbrace{uu\cdots u}^{2^{2(k-1)-1}-2^{(k-1)-1}}\quad >) + r_B(G^{\beta}_{k-1,t}) \\ &\leq 2\cdot 2^{2(k-1)} + \tfrac{3}{2}\cdot 2^{2(k-1)-1} - \tfrac{3}{2}\cdot 2^{(k-1)-1} + r_B(G^{\beta}_{k-1,t}) \\ &= 2\cdot 2^{2(k-1)} + \tfrac{3}{2}\cdot 2^{2(k-1)-1} - \tfrac{3}{2}\cdot 2^{(k-1)-1} + 2\cdot 2^{2(k-2)} \\ &\quad +\tfrac{3}{2}\cdot 2^{2(k-2)-1} - \tfrac{3}{2}\cdot 2^{(k-2)-1} + r_B(G^{\beta}_{k-2,t}) \\ &\leq 2\cdot 2^{2(k-1)} + \tfrac{3}{2}\cdot 2^{2(k-1)-1} - \tfrac{3}{2}\cdot 2^{(k-1)-1} + 2\cdot 2^{2(k-2)} + \tfrac{3}{2}\cdot 2^{2(k-2)-1} \\ &\quad -\tfrac{3}{2}\cdot 2^{(k-2)-1} + \cdots + 2\cdot 2^{2\cdot r} + \tfrac{3}{2}\cdot 2^{2\cdot r-1} + \tfrac{3}{2}\cdot 2^{r-1} + r_B(\ G^{\beta}_{r,t}) \\ &= 2^{2k} - 2^{2r} - 2^{k} + 2^{r} + r_{CE}(G^{\beta}_{r,t}), k\geq r>t \\ r_B(G^{\beta}_{k,t}) &\leq \frac{2^{2k+2}-2^{2r+2}+3\cdot 2^{2k-2}-3\cdot 2^{2r-2}-9\cdot 2^{k-1}+9\cdot 2^{k-1}}{6} + r_B(G^{\beta}_{r,t})\ t<r\leq k. \end{aligned}$$

□

Acknowledgement

This work was done while the author was supported by a grant(F. No: 4-4/2014-15(MRP-SEM/UGC-SERO, Nov.2014)) for the University Grants Commission, South Eastern Regional office, Hyderabad - 500 001.

References

[1] T.Aoki, P.Gaborit, M.Harada, M.Ozeki and P.Solé, *On the covering radius of $\mathbb{Z}_4$ codes and their lattices*, IEEE Trans. Inform. Theory, 45(6)(1999), 2162-2168.

[2] M.C.Bhandari, M.K.Gupta and A.K.Lal, *On $\mathbb{Z}_4$ Simplex codes and their gray images*, Applied Algebra, Algebraic Algorithms and Error-Correcting Codes, AAECC-13, Lecture Notes in Computer Science, (1999), 170-180.

[3] A.Bonnecaze, P.Solé, C.Bachoc and B.Mourrain, *Type II codes over $\mathbb{Z}_4$*, IEEE Trans. Inform. Theory, 43(1997), 969-976.

[4] P.Chella Pandian and C.Durairajan, *On the covering radius of some code over $R=\mathbb{Z}_2+u\mathbb{Z}_2$, where $u^2=0$*, International Journal of Research in Applied, 2(1)(2014), 61-70.

[5] P.Chella Pandian and C.Durairajan, *On the Covering Radius of Codes Over $\mathbb{Z}_4$ with Chinese Euclidean Weight*, International Journal on Information Theory, 4(4)(2015), 1-8.

[6] Christine Bachoc, *Application of coding theory to the construction of modular lattices*, Journal of Combin. Theory Ser., A78(1997), 92-119.

[7] G.D.Cohen, M.G.Karpovsky, H.F.Mattson and J.R.Schatz, *Covering radius-Survey and recent results*, IEEE Trans. Inform. Theory, 31(3)(1985), 328-343.

[8] C.Cohen, A.Lobstein and N.J.A.Sloane, *Further Results on the Covering Radius of Codes*, IEEE Trans. Inform. Theory, 32(5)(1986), 680-694.

[9] C.J.Colbourn and M.K.Gupta, *On quaternary MacDonald codes*, Proc. Information Technology:Coding and Computing (ITCC), (2003), 212-215.

[10] J.H.Conway and N.J.A.Sloane, *Self-dual codes over the integers modulo* 4, Journal of Combin. Theory Ser. A, 62(1993), 30-45.

[11] S.Dodunekov and J.Simonis, *Codes and projective multisets*, The Electronic Journal of Communications, R37(5)(1998).

[12] S.T.Dougherty, M.Harada and P.Solé, *Shadow codes over $\mathbb{Z}_4$*, Journal of Finite Fields and Their Appl., (to appear).

[13] C.Durairajan, *On Covering Codes and Covering Radius of Some Optimal Codes*, Ph.D. Thesis, Department of Mathematics, IIT Kanpur, (1996).

[14] A.R.Hammons, P.V.Kumar, A.R.Calderbank, N.J.A.Sloane and P.Solé, *The $\mathbb{Z}_4$-linearity of kerdock, preparata, goethals, and related codes*, IEEE Trans. Inform. Theory, 40(1994), 301-319.

[15] M.Harada, New extremal Type II codes over $\mathbb{Z}_4$, Journal of Des. Codes and Cryptogr., 13(1998), 271-284.