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Intersection Operator Graph of a Group

Research Article

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Abstract: Let $(G, *)$ be a group with binary operation '$*$'. The Intersection Operator graph $\Gamma_{IO}(G)$ of G is a graph with $V(\Gamma_{IO}(G)) = G$ and two distinct vertices x and y are adjacent in $\Gamma_{IO}(G)$ if and only if $\langle x\rangle \cap \langle y\rangle \subseteq \langle x * y\rangle$. In this paper, we want to explore how the group theoretical properties of G can effect on the graph theoretical properties of $\Gamma_{IO}(G)$. Some characterizations for fundamental properties of $\Gamma_{IO}(G)$ have also been obtained. Finally, we characterize certain classes of Intersection Operator Graph corresponding to finite abelian groups.

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1. Introduction

The study of algebraic structures, using the properties of graphs, becomes an exciting research topic in the last twenty years, leading to many fascinating results and questions. There are many papers on assigning a graph to a ring or group and thereby investigating algebraic properties of the ring or group using the associated graph, for instance, see [1, 2]. In the present article, to any group G, we assign a graph and investigate algebraic properties of the group using the graph theoretical concepts. Before starting, let us introduce some necessary notation and definitions.

We consider simple graphs which are undirected, with no loops or multiple edges. For any graph $\Gamma = (V, E)$, V denote the set of all vertices and E denote the set of all edges in Γ. The degree $deg_\Gamma(v)$ of a vertex v in Γ is the number of edges incident to v and if the graph is understood, then we denote $deg_\Gamma(v)$ simply by $deg(v)$. The order of Γ is defined $|V(\Gamma)|$ and its maximum and its minimum degrees will be denoted, respectively, by $\Delta(\Gamma)$ and $\delta(\Gamma)$. A graph Γ is regular if the degrees of all vertices of Γ are the same. A vertex of degree 0 is known as an *isolated vertex* of Γ. A graph Ω is called a subgraph of Γ if $V(\Omega) \subseteq V(\Gamma)$, $E(\Omega) \subseteq E(\Gamma)$. Let $\Gamma = (V, E)$ be a graph and let $S \subseteq V$. A *subgraph* Ω of Γ is said to be an*induced subgraph* of Γ induced by S, if $V(\Omega) = S$ and each edge of Γ having its ends in S is also an edge of Ω. A simple graph Γ is said to be *complete* if every pair of distinct vertices of Γ are adjacent in Γ. A graph Γ is said to be *connected* if every pair of distinct vertices of Γ are connected by a path in Γ. An *Eulerian* graph has an *Eulerian trail*, a closed trail containing all vertices and edges. The *Union* of two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ is a graph $\Gamma = (V, E)$ with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. The *join* of two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ is a graph denoted by $\Gamma_1 + \Gamma_2 = (V, E)$ with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{$Edges joining every vertex of V_1 with every vertex of $V_2\}$.

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Let G be a group with identity e. The order of the group G is the number of elements in G and is denoted by $O(G)$. The order of an element a in a group G is the smallest positive integer k such that $a^k = e$. If no such integer exists, we say a has infinite order. The order of an element a is denoted $O(a)$. Let p be a prime number. A group G with $O(G) = p^k$ for some $k \in \mathbb{Z}^+$, is called a *p-group.*

2. Preparation of Manuscript

In this section, we observe certain basic properties of Intersection operator graphs.

Definition 2.1. *Let $(G, *)$ be a group with binary operation '$*$'. The Intersection Operator graph $\Gamma_{IO}(G)$ of G is a graph with $V(\Gamma_{IO}(G)) = G$ and two distinct vertices x and y are adjacent in $\Gamma_{IO}(G)$ if and only if $\langle x\rangle \cap \langle y\rangle \subseteq \langle x * y\rangle$.*

Proposition 2.2. *Let $(G, *)$ be a group with n elements. In $\Gamma_{IO}(G)$, the identity element e of G has degree $n - 1$.*

Proof. Let $(G, *)$ be a group with n elements. Let $x \in G$ be any element. Clearly $\langle x\rangle \cap \langle e\rangle = \{e\} \subseteq \langle x * e\rangle = \langle x\rangle$. Hence the result follows. □

Proposition 2.3. *Let $(G, *)$ be a group. For any non self inverse element $x \in G$, x and x^{-1} are non adjacent in $\Gamma_{IO}(G)$.*

Proof. Let $(G, *)$ be a group with identity element e. Let $x \in G$ be non self inverse element. Since $x * x^{-1} = e$ and $\langle x\rangle = \langle x^{-1}\rangle$, $\langle x\rangle \cap \langle x^{-1}\rangle \nsubseteq \langle e\rangle$. Hence the result follows. □

Proposition 2.4. *Let $(G, *)$ be a group. Any two distinct prime order elements are adjacent in $\Gamma_{IO}(G)$.*

Proof. Let $(G, *)$ be a group with identity element e. Let $x, y \in G$ be any two elements such that $O(x) = p$ and $O(y) = q$, where p, q are distinct prime. Clearly $\langle x\rangle \cap \langle y\rangle = \{e\}$. Therefore $\langle x\rangle \cap \langle y\rangle \subseteq \langle x * y\rangle$. Hence the result follows. □

Theorem 2.5. *Let G be any group. $\Gamma_{IO}(G)$ is complete if and only if every element of G is self inverse element.*

Proof. Assume every element of G is self inverse element. Let $x, y \in G$. Clearly $\langle x\rangle \cap \langle y\rangle = \{e\}$. Therefore $\langle x\rangle \cap \langle y\rangle \subseteq \langle x*y\rangle$. Hence x and y are adjacent in $\Gamma_{IO}(G)$. Since x and y are arbitrary , any two elements in G are adjacent in $\Gamma_{IO}(G)$. Hence $\Gamma_{IO}(G)$ is complete. Conversely assume that $\Gamma_{IO}(G)$ is complete. Suppose G has a non self inverse element x, by Proposition 2.3, x and x^{-1} are non adjacent in $\Gamma_{IO}(G)$, which is a contradiction. Hence every element of G is self inverse element. □

Theorem 2.6. *Let G a group. $\Gamma_{IO}(G)$ is a star graph if and only if $G \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$.*

Proof. Clearly if $G \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$, then $\Gamma_{IO}(G)$ is star graph.
Conversely assume that $\Gamma_{IO}(G) = K_{1,n}$. Since the identity element $'e'$ has a full degree, any two non identity elements are non adjacent in $\Gamma_{IO}(G)$. It is enough to prove that $O(G) = 2$ or 3. Suppose G has an element x of order k such that $k\rangle 3$. $\langle x\rangle = \{e, x, x^2, x^3, \dots, x^{k-2}, x^{k-1}\}$. Since $x * x^{k-2} = x^{k-1} = x^{-1}$, $\langle x\rangle = \langle x * x^{k-2}\rangle$ and $\langle x\rangle \cap \langle x^{k-2}\rangle = \langle x^{k-2}\rangle \subseteq \langle x\rangle$. Therefore x and x^{k-2} are adjacent, which is a contradiction. There every non identity element of G has an order either 2 or 3. Suppose G has atleast two distinct subgroup of order either 2 or 3. Let $x, y \in G$ be two element of order 2 such that $\langle x\rangle \neq \langle y\rangle$. Clearly $\langle x\rangle \cap \langle y\rangle = \{e\}$. Therefore $\langle x\rangle \cap \langle y\rangle \subseteq \langle x * y\rangle$. Hence x and y are adjacent in $\Gamma_{IO}(G)$, which is a contradiction. Therefore G has a unique subgroup of order either 2 or 3. Hence $G \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$ □

Proposition 2.7. *Let G be a finite group of order n with no self inverse element and q be number of edges in $\Gamma_{IO}(G)$. Then $q \leq \frac{(n-1)^2}{2}$. Moreover, this bound is sharp.*

Proof. By Proposition 2.2, $deg_{\Gamma_{IO}(G)}(e) = n-1$, where e is the identity element of G. Since G has no self inverse element, by Proposition 2.3, for all $x \in G-e$, $deg_{\Gamma_{IO}(G)}(x) \leq n-2$. From this we get the degree sum $\leq (n-1)+(n-1)(n-2) = (n-1)^2$. Hence $q \leq \frac{(n-1)^2}{2}$. Moreover, for the group $\mathbb{Z}_3$, $\Gamma_{IO}(\mathbb{Z}_3) \cong K_{1,2}$ and for this graph the bound is sharp. □

We now characterize the groups G for which the associated graph $\Gamma_{IO}(G)$ attains this bound.

Theorem 2.8. *Let G be a finite abelian group of order n and q be number of edges in $\Gamma_{IO}(G)$. Then $q = \frac{(n-1)^2}{2}$ if and only if every element of G has an order p, where p is an odd prime.*

Proof. Assume that $\Gamma_{IO}(G)$ is a graph with $\frac{(n-1)^2}{2}$ edges. In view of Proposition 2.4, we get that $deg_{\Gamma_{IO}}(a) = n-2$ for all vertices $a \in G-e$ and $deg_{\Gamma_{IO}}(e) = n-1$. Let $a \in G-e$ be any element of order k.

Claim: k be a prime number: By the assumption, a is adjacent to $a^2, a^3, \ldots, a^{k-2}$. Then by definition $\langle a^2 \rangle \subseteq \langle a^3 \rangle \subseteq \ldots \subseteq \langle a^{k-2} \rangle \subseteq \langle a \rangle$. Also a^{-1} is adjacent to $a^{k-2}, a^{k-3}, \ldots, a^2$. Then by definition $\langle a^{k-2} \subseteq \langle a^{k-3} \rangle \subseteq \ldots \langle a^2 \rangle \subseteq \langle a \rangle$. Hence we have

$$\langle a \rangle = \langle a^2 \rangle = \langle a^3 \rangle = \cdots = \langle a^{k-1} \rangle \tag{1}$$

Now we prove that $O(a) = k$ is a prime number. Suppose not, k is not a prime. Without loss of generality assume that $k = pq$, for some prime p and q. Since $p, q|k$ and $\langle a \rangle$ is a cyclic group, there exists two element $a^l, a^m \in \langle a \rangle$ such that $\langle a^l \rangle = p$ and $\langle a^m \rangle = q$. Which is a contradiction to (1). Hence k is a prime number. Since a is arbitrary, every element of G is of prime order. Now we have to prove this prime order is unique. Suppose that let $a, b \in G$ such that $O(a) = p$,$O(b) = q$, where p and q are distinct prime. Since G is abelian, $O(a * b) = pq$, which is a contradiction to the fact every element of G is of prime order. Therefore every element of G has a unique prime order.

Conversely, assume that every element other than identity has an order p. Since G is abelian, $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \ldots \times \mathbb{Z}_p$. By Proposition 2.2, $deg_{\Gamma_{IO}(G)}(e) = n-1$. Let a and b two element such that $b \neq a^{-1}$. Clearly $\langle a \rangle \cap \langle b \rangle = \langle a \rangle or \{e\}$. Therefore $\langle a \rangle \cap \langle b \rangle \subseteq \langle a * b <$. Therefore a and b are adjacent. Hence a is adjacent to all other elements in G other than its inverse. Therefore $deg_{\Gamma_{IO}(G)}(a^i) = n-2$ for $i = 1, 2, \ldots, n-1$. From this we get the degree sum $= (n-1)+(n-1)(n-2) = (n-1)^2$. Hence $q = \frac{(n-1)^2}{2}$. □

Theorem 2.9. *Let G be a abelian group of order p^n. $\Gamma_{IO}(G) \cong K_{1,2,2,\ldots,ktimes}$, where $k = \frac{p^n-1}{2}$ if and only if $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \ldots \times \mathbb{Z}_p$.*

Proof. Let G be a abelian group of order p^n. Assume that $\Gamma_{IO}(G) \cong K_{1,2,2,\ldots,ktimes}$, where $k = \frac{p^n-1}{2}$. Clearly the number of edges of the graph $\Gamma_{IO}(G)$ is $\frac{(p^n-1)^2}{2}$. Therefore by Theorem 2.8, every element of G has an order p and hence $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \ldots \times \mathbb{Z}_p$.

Conversely, assume that $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \ldots \times \mathbb{Z}_p$. Therefore for every $a \in G-e$ is not adjacent to a^{-1} only. Therefore we can be partition the vertex set of $\Gamma_{IO}(G)$ into $k+1$ set, where $k = \frac{p^n-1}{2}$ such that the identity element e belongs into single partition and the remaining k sets, each set contains the pair elements a and a^{-1}. Clearly each partition is an independent set and every element of one partition is adjacent to every element of other partition. Hence $\Gamma_{IO}(G) \cong K_{1,2,2,\ldots,ktimes}$, where $k = \frac{p^n-1}{2}$ □

Theorem 2.10. *Let G be a cyclic group of order $2p$, where p is a prime and $p \geq 3$. Then the number of edges of $\Gamma_{IO}(G)$ is equal to $\frac{3p^2-4p+3}{2}$.*

Proof. Let G be a cyclic group of order $2p$, where p is a prime and $p \geq 3$. Therefore $G \cong \mathbb{Z}_{2p}$. The vertex set of G can be partition into four sets namely A,B,C and D such that $A = \{0\}$, $B = \{1, 3, 5, \ldots, p-2, p-3, \ldots, 2p-1\}$,

$C = \{2, 4, 6, \ldots, 2p-2\}$ and $D = \{p\}$. Clearly $|A| = 1$, $|B| = p-1$, $|C| = p-1$ and $|D| = 1$. By Proposition 2.2, $deg_{\Gamma_{IO}}(0) = 2p-1$. The elements of B are generator of G, the elements of C have an order p and the element in D has an order 2. Since the sum of two odd integer is even, no two elements of B are adjacent in $\Gamma_{IO}(G)$. Let $x \in B$. Therefore $p - x \in C$ and $x + p - x = p$. So x and $p - x$ are non adjacent in $\Gamma_{IO}(G)$ also the sum of odd integer and even integer is again odd integer. Let $x \in B$, x is adjacent to all element in C other than $p - x$ and the element of A. Therefore $\forall x \in B$, $deg_{\Gamma_{IO}}(x) = 1 + p - 2$. Since G has unique subgroup of order p and sum of two even integer is even, Every element in C is adjacent to all element in C other than its inverse. Since p is odd, every element in C is adjacent to the element in D. Therefore $\forall x \in C$, $deg_{\Gamma_{IO}}(x) = 1 + (p-3) + (p-2) + 1 = 2p-3$.Clearly $deg_{\Gamma_{IO}}(p) = 1 + p - 1 = p$. Therefore

$$\begin{aligned}
\text{The sum of the degrees of all vertices} &= (2p-1) + [(p-1)(p-1)] + [(p-1)(2p-3)] + p \\
&= 2p - 1 + p^2 - 2p + 1 + 2p^2 - 3p - 2p + 3 + p \\
&= 3p^2 - 4p + 3
\end{aligned}$$

Hence the number of edges of $\Gamma_{IO}(G) = \frac{3p^2-4p+3}{2}$. □

Theorem 2.11. *Let G be a cyclic group of order p^2, where p is an odd prime . Then the number of edges of $\Gamma_{IO}(G)$ is equal to $\frac{(p-1)(p^3-1)}{2}$.*

Proof. Let G be a cyclic group of order p^2, where p is an odd prime. Therefore $G \cong \mathbb{Z}_{p^2}$. The vertex set of G can be partition into three sets namely A,B and C such that $A = \{0\}$, $B = \{p, 2p, 3p, \ldots, (p-1)p\}$ and $C = G - A - B$. Clearly $|A| = 1$, $|B| = p-1$ and $|C| = p(p-1)$. By Proposition 2.2, $deg_{\Gamma_{IO}}(0) = p^2 - 1$. The elements of C are generator of G and the elements of B have an order p. Let $x, y \in B$. Since $A \cup B$ is a subgroup of G, either $x + y \in B$ or $x + y = 0$. Hence for all x in B, x is adjacent to all elements other than its inverse in B. Let $x \in B$ and $y \in C$, then $x + y \in C$. Therefore for all x in B, x is adjacent to all elements in C. Therefore $\forall x \in B$, $deg_{\Gamma_{IO}}(x) = 1 + p - 3 + p(p-1) = p^2 - 2$. Clearly the element in C is adjacent to all element in A and B. For each x in C, x is not adjacent to $p - x$, $2p - x$, …, $p^2 - x$, which are in C. Therefore $\forall x \in C$, $deg_{\Gamma_{IO}}(x) = 1 + (p-1) + p^2 - 2p - 1 = p^2 - p - 1$. Therefore

$$\begin{aligned}
\text{The sum of the degrees of all vertices} &= (p^2-1) + [(p-1)(p^2-2)] + [p(p-1)(p^2-p-1)] \\
&= (p-1)[p + 1 + p^2 - 2 + p^3 - p^2 - p] \\
&= (p-1)(p^3-1)
\end{aligned}$$

Hence the number of edges of $\Gamma_{IO}(G) = \frac{(p-1)(p^3-1)}{2}$. □

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