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Meromorphic Functions and its Sharing Properties

Research Article

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Abstract: In this paper, we study partial sharing of meromorphic functions and its derivatives.Our results improve or generalize the results of K.S. Karak and B.Lal [7] and Yang and Yi [1].

MSC: 30D35.

Keywords: Meromorphic function, shared values, differential polynomials, uniqueness.

1. Introduction

In this paper a meromorphic function will always mean meromorphic for whole complex plane. We shall use the standard notations of value distribution theory such as $T(r,f), m(r,f), N(r,f)$,... (see [10]). We denote by $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ possible outside of a set with finite measure. Let f and g be two non constant meromorphic functions and a be a finite complex number. We denote by $E(a,f)$,the set of zeros of $f-a$, counting multiplicities and $\overline{E}(a,f)$ while ignoring multiplicities. We also say that the functions f and g are said to share the value a CM if $E(a,f) = E(a,g)$ and to share the value a IM if $\overline{E}(a,f) = \overline{E}(a,g)$. We denote $E_{k)}(a,f)$ the set of those zeros of $f-a$ for which multiplicities are not greater than k, counting multiplicities and $\overline{E}_{k)}(a,f)$ is the corresponding one for which multiplicities are not counted. We also denote $N_{k)}(r, \frac{1}{f-a})$ the counting function of those a points of f whose multiplicities are not greater than k counting according to multiplicities and $\overline{N}_{k)}(r, \frac{1}{f-a})$ is similar one when multiplicities are counted only once. Similarly we defined $N_{(k}(r, \frac{1}{f-a})$ when the multiplicities are atleast k and $\overline{N}_{(k}(r, \frac{1}{f-a})$ is the reduced one. We denote class A to those meromorphic functions which satisfies $\overline{N}(r,f) + \overline{N}(r, \frac{1}{f}) = S(r,f)$. Then clearly each member of class A is transcendental meromorphic functions. A function f is said to share the value a partially with a function g CM(IM) if $E(a,f) \subseteq E(a,g)[\overline{E}(a,f) \subseteq \overline{E}(a,g)]$. We also use $N_{1)}(r, \frac{1}{g-a}|f \neq a)$ to denote the simple zeros of $g-a$ that are not the zeros of $f-a$.

2. Main Results

In his book Yang and Yi [1] proved the following theorem:

Theorem 2.1. *Let $f, g \in A$ and a be a non zero complex number.Furthermore, let k be a positive integer.*

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(1). If $\overline{E}_{1)}(a,f)=\overline{E}_{1)}(a,g)$*, then* $f=g$ *or* $fg=a^2$.

(2). If $\overline{E}_{1)}(a,f^{(k)})=\overline{E}_{1)}(a,g^{(k)})$ *then* $f=g$ *or* $f^{(k)}g^{(k)}=a^2$.

In 2014, K.S.Charak and B.Lal [7] proved the following theorems which improved the Theorem 2.1.

Theorem 2.2. *Let* $f,g\in A$*,* a *be a complex number and* k *be a positive integer.*

(1). If $\overline{E}_{1)}(a,f)\subseteq\overline{E}_{1)}(a,g)$ *and* $N_{1)}\left(r,\frac{1}{g-a}|f\neq a\right)=S(r,g)$ *then* $f=g$.

(2). If $\overline{E}_{1)}(a,f^{(k)})\subseteq\overline{E}_{1)}(a,g^{(k)})$ *and* $N_{1)}\left(r,\frac{1}{g^{(k)}-a}|f^{(k)}\neq a\right)=S(r,g)$ *then* $f=g$ *or* $f^{(k)}g^{(k)}=a^2$.

In 2011, Huang and Huang [5] improved the following result of Yang and Hua [2] as

Theorem 2.3. *Let* f *and* g *be two meromorphic functions and* $n\geq 19$ *be an integer. If* $E_1(1,f^nf^{(1)})=E_1(1,g^ng^{(1)})$*, then either* $f=dg$ *for some* $(n+1)^{th}$ *root of unity* d *or* $f(z)=c_1e^{cz}$ *and* $g(z)=c_2e^{-cz}$*, where* c,c_1,c_2 *are constant satisfying* $(c_1c_2)^{n+1}c^2=-1$.

In 2014, K.S.Charak and B.Lal [7] improved Theorem 2.3 for functions of class A as

Theorem 2.4. *Let* $f,g\in A$*, and* $n\geq 2$ *be an integer and* $a(\neq 0)\in C$*. If* $\overline{E}_{1)}(a,f^nf(1))=\overline{E}_{1)}(a,g^ng(1))$*, then either* $f=dg$ *for some* $(n+1)th$ *root of unity* d *or* $f(z)=c_1e^{cz}$ *and* $g(z)=c_2e^{-cz}$*, where* c,c_1,c_2 *are constant satisfying* $(c_1c_2)^{(n+1)}c^2=-a^2$.

In this paper we prove the following theorems which improve and generalise the above mentioned theorems. We extend Theorem 2.3 by incorporating partial sharing-which is our first theorem. The next theorem generalises the Theorem 2.1 for the function of class A.

Theorem 2.5. *Let* $f,g\in A$*,* $n\geq 2$ *be an integer and* $a(\neq 0)\in C$*. If* $\overline{E}_{1)}(a,f^nf^{(1)})\subseteq\overline{E}_{1)}(a,g^ng^{(1)})$ *and* $N_{1)}(r,\frac{1}{g^ng^1-a}|f^nf^{(1)}\neq a)=S(r,g)$ *some* $(n+1)th$ *root of unity* d *or* $f(z)=c_1e^{cz}$ *and* $g(z)=c_2e^{-cz}$ *where* c,c_1,c_2 *are constant satisfying* $(c_1c_2)^{n+1}c^2=-a^2$.

Theorem 2.6. *Let* $f,g\in A$ *and* a *be a non zero complex number. Also let* n *and* k *be two positive integers such that* $n>2k$*. If* $\overline{E}_{1)}(a,(f^n)^{(k)})=\overline{E}_{1)}(a,(g^n)^{(k)})$*, then either* $f=dg$ *for some* n^{th} *root of unity* d *or* $f(z)=c_1e^{cz}$ *and* $g(z)=c_2e^{-cz}$ *where* c,c_1,c_2 *are constant satisfying* $(-1)^k(c_1c_2)^n(2c)^k=a^2$.

Theorem 2.7. *Let* $f,g\in A$ *and* a *be a non zero complex number. Also let* n *and* k *be two positive integers such that* $n>2k$*. If* $\overline{E}_{1)}(a,(f^n)^{(k)})\subseteq\overline{E}_{1)}(a,(g^n)^{(k)})$ *and* $N_{1)}\left(r,\frac{1}{(g^n)^{(k)}-a}|(f^n)^{(k)}\neq a\right)=S(r,g)$*, then either* $f=dg$ *for some* nth *root of unity* d *or* $f(z)=c_1e^{cz}$ *and* $g(z)=c_2e^{-cz}$ *where* c,c_1,c_2 *are constant satisfying* $(-1)^k(c_1c_2)^n(2c)^k=a^2$.

Before going to the proof of the theorems, we need to mention some results in the form of lemmas.

Lemma 2.8 ([2])**.** *Let* f *and* g *be two non constant entire functions,* $n\geq 1$ *and* $a(\neq 0)\in C$*. If* $f^nf^{(1)}g^ng^{(1)}=a^2$ *then* $f(z)=c_1e^{cz}$ *and* $g(z)=c_2e^{-cz}$ *where* c,c_1,c_2 *are constant satisfying* $(c_1c_2)^{(n+1)}c^2=-a^2$.

Lemma 2.9 ([1])**.** *If* $f\in A$ *and* k *is a positive integer then* $f^{(k)}\in A$.

Lemma 2.10 ([1])**.** *If* $f,g\in A$ *and* $f^{(k)}=g^{(k)}$ *where* k *is a positive integer, then* $f=g$.

Lemma 2.11 ([4])**.** *Let* $f(z)$ *be a non constant entire function and let* $k\geq 2$ *be a positive integer. If* $f(z)f^{(k)}(z)\neq 0$ *then* $f(z)=e^{az+b}$ *where* $a\neq 0$*,*b *are constant.*

Lemma 2.12 ([10]). *Suppose that $f_1(z), f_2(z), \ldots, f_n(z)(n \geq 2)$ are meromorphic functions and $g_1(z), g_2(z), \ldots, g_n(z)$ are entire functions satisfying the following conditions*

(1). $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} = 0.$

(2). $g_j(z) - g_k(z)$ *are not constant for* $1 \leq j < k \leq n$,

(3). For $1 \leq j \leq n$, $1 \leq h < k \leq n, T(r, f_j) = o(T(r, e^{g_h - g_k}))(r \to \infty, r \notin E)$.

Then $f_j(z) = 0 \ \ (j = 1, 2, \ldots, n)$.

Proof of Theorem 2.5. Let $F = \frac{f^{n+1}}{n+1}$ and $G = \frac{g^{n+1}}{n+1}$. So, $F^{(1)} = f^n f^{(1)}$ and $G^{(1)} = g^n g^{(1)}$.

$$\begin{aligned}\overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) &= \overline{N}\left(r, \frac{f^{n+1}}{n+1}\right) + \overline{N}\left(r, \frac{n+1}{f^{n+1}}\right)\\ &= \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right)\\ &= S(r, g)\end{aligned}$$

Similarly, $\overline{N}(r, G) + \overline{N}\left(r, \frac{1}{G}\right) = S(r, g)$. Therefore, $F, G \in A$. By Lemma 2.9, we can get $F^{(1)}, G^{(1)} \in A$. By hypothesis,we have, $\overline{E}_{1)}(a, F^{(1)}) \subseteq \overline{E}_{1)}(a, G^{(1)})$ and $N_{1)}\left(r, \frac{1}{G^{(1)}-a}|F^{(1)} \neq a\right) = S(r, g)$. So by Theorem 2.2 we have, $F = G$ or $F^{(1)}G^{(1)} = a^2$. The remains of the proof are in same line as the proof of Theorem 2.4. □

Proof of Theorem 2.6. Let $F = f^n$ and $G = g^n$. So, $\overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) = S(r, f)$ and $\overline{N}(r, G) + \overline{N}\left(r, \frac{1}{G}\right) = S(r, g)$. Therefore, $F, G \in A$. So, $\overline{E}_{1)}(a, F^{(k)}) \subseteq \overline{E}_{1)}(a, G^{(k)})$. By Theorem (2) of 2.2, we have, $F = G$ or $F^{(k)}G^{(k)} = a^2$. If $F = G$ i.e., $f^n = g^n$ then $f = dg$ for some n^{th} root of unity d. If $F^{(k)}G^{(k)} = a^2$ i.e., $[f^n(z)]^{(k)}[g^n(z)]^{(k)} = a^2$. Let, $f(z)$ has a zero of multiplicity p at z_0, then z_0 must be a pole of $g(z)$ of multiplicity q(say). So, $np - k = nq + k$ i.e., $n(p - q) = 2k$. This relation does not hold since $n > 2k$. Therefore, $f(z) \neq 0$ for any z and also $g(z) \neq \infty$ for any z. i.e., Similarly we can say $g(z) \neq 0$ and $f(z) \neq \infty$ for any z. Therefore, $[f^n]^{(k)} \neq 0$ and $[g^n]^{(k)} \neq 0$. By the Lemma 2.11, for $k \geq 2$, we have $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$ where c_1, c_2 and c are constants satisfying $c_1^n e^{ncz}(nc)^k c_2^n e^{-ncz}(-1)^k(nc)^k = a^2$ i.e., $(-1)^k(c_1c_2)^n(nc)^{2k} = a^2$. When $k = 1$, we have $[f^n]^{(1)}[g^n]^{(1)} = a^2$ i.e.,

$$n^2 f^{n-1} g^{n-1} f^{(1)} g^{(1)} = a^2 \tag{1}$$

Suppose that f has a zero of multiplicity p_1 at z_1. Then z_1 is a pole of multiplicity q_1 of g. Therefore, $(n-1)p_1 + p_1 - 1 = (n-1)q_1 + q_1 + 1$ i.e.,$n(p_1 - q_1) = 2$. Since $n > 2k$ i.e.,$n > 2$. So the relation does not hold. Therefore

$$f(z) \neq 0 \ \text{ and } g(z) \neq 0 \text{ for any z and} \tag{2}$$

$$f(z) \neq \infty \ \text{ and } g(z) \neq \infty \text{ for any z} \tag{3}$$

Therefore $f(z)$ and $g(z)$ can be expressed as

$$f(z) = e^{\alpha(z)} \ \text{ and } \ g(z) = e^{\beta(z)} \tag{4}$$

where $\alpha(z)$ and $\beta(z)$ are non constant functions. Putting these value in equation (1) we get,

$$n^2 \alpha^{(1)} \beta^{(1)} e^{n(\alpha+\beta)} = 1 \tag{5}$$

Thus $\alpha^{(1)}$ and $\beta^{(1)}$ have no zeros and we can set $\alpha^{(1)} = e^{\delta(z)}$ and $\beta^{(1)} = e^{\gamma(z)}$ where δ and γ are entire functions. Equation (5) reduces to, $n^2 e^{n(\alpha+\beta)+\delta+\gamma} = 1$. Differentiating we have, $n(\alpha^{(1)} + \beta^{(1)} + \delta^{(1)} + \gamma^{(1)} = 0$ i.e.,

$$n(e^{\delta} + e^{\gamma}) + \delta^{(1)} + \gamma^{(1)} = 0 \tag{6}$$

i.e., $n(e^{\delta-\gamma} + 1)e^{\gamma} + \alpha^{(2)}e^{-\delta} + \beta^{(2)}e^{-\gamma} = 0$. By Lemma 2.12, we get, $e^{\delta-\gamma} + 1 = 0$ i.e., $e^{\delta-\gamma} = -1$ i.e.,$\delta - \gamma = (2m+1)\pi i$. So from the above equalities, we get $\delta^{(1)} = \gamma^{(1)} = 0$. So, δ and γ are constant. Therefore,

$$\alpha^{(1)} \text{ and } \beta^{(1)} \text{ are constant.} \tag{7}$$

From (1), (2), (3), (4) and (7) we obtain, $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$ where c_1,c_2 and c are three constants satisfying $(c_1c_2)^n(2c)^{2k} = -a^2$. □

Proof of Theorem 2.7. Let $F = f^n$ and $G = g^n$. So as in previous theorem we have $F, G \in A$. So we have, $\overline{E}_{1)}(a, F^{(k)}) \subset \overline{E}_{1)}(a, G^{(k)})$. So by Theorem 2.2, we have, $F = G$ or $F^{(k)}G^{(k)} = a^2$. The remains of the proof are in the same line as the proof of the Theorem 2.6. □

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