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T-Hardy Rogers Contraction and Fixed Point Results in Cone Metric Spaces with c-Distance

Research Article

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Abstract: In this paper, we introduce the T-Hardy-Rogers's contraction under the concept of c-distance in cone metric spaces and prove the existence and uniqueness fixed point results. The presented theorem extends and generalizes several well known comparable results in literature.

MSC: 54H25, 17H10.

Keywords: Cone metric space, complete cone metric space, c-distance, fixed point, T-HardyRogers contraction.

1. Introduction

In 1922, S. Banach [1], proved fixed point theorem for contraction mappings in complete metric space. It is first important fundamental results in fixed point theory, which is also known as Banach contraction principle or Banach fixed point theorem. After this provital result, many authors have studied various extensions and generalizations of Banach's theorems by considering contractive mappings on several directions in the literature (see [3–11]). In 2007, Huang and Zhang [2] generalized concept of metric space, replacing the set of real numbers by an order Banach space, and showed some fixed point theorems of different type of contractive mappings on cone metric spaces. Later, many authors generalized and studied fixed and common fixed point results in cone metric spaces for normal and non normal cone. The Hardy-Roger's contraction was introduced in the work of Hardy-Rogers [15] which is generalization of Reich contraction. Recently, Cho et al. [12] Wang and Guo [14] defined a concept of the c- distance in a cone metric space, which is a cone version of the w-distance of Kada et al. [11] and proved some fixed point theorems in ordered cone metric spaces. Then Sintunavarat et al. [13] generalized the Banach contraction theorem on c-distance of Cho et al. [12]. After that, several authors studied the existence and uniqueness of the fixed point, common fixed point, coupled fixed point and common coupled fixed point problems using this distance in cone metric spaces and ordered cone metric spaces see for examples [16–27].

In this paper, we studied some fixed point theorems of T-Hardy-Rogers contraction type mappings under the concept of c-distance in complete cone metric spaces depended on another function. Throughout this paper, we do not impose the normality condition for the cones, but the only assumption is that the cone P is solid, that is int $P \neq \emptyset$.

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2. Preliminary Notes

First, we recall some standard notations and definitions in cone metric spaces with some of their properties [2].

Definition 2.1. *Let E be a real Banach space and P be a subset of $E$and θ denote to the zero element in E, then P is called a cone if and only if :*

(1). P is a non-empty set closed and $P \neq \{\theta\}$,

(2). If a, b are non-negative real numbers and $x, y \in P$,then $ax + by \in P$,

(3). $x \in P$ and $-x \in P \Rightarrow x = \theta \Leftrightarrow P \bigcap (-P) = \{\theta\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ on E with respect to P by $x = y$ if and only if $y - x \in P$. We shall write $x \ll y$ if $y - x \in intP$ (where int P denotes the interior of P). If $intP \neq \emptyset$, then cone P is solid. The cone P called normal if there is a number $K > 0$ such that for all $x, y \in E$, $\theta \leq x \leq y \Rightarrow \|x\| = k\|y\|$. The least positive number k satisfying the above is called the normal constant of P.

Definition 2.2. *Let x be a non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies*

(1). $\theta < d(x, y)$ for all $x, y \in X$ and $(x, y) = \theta$ if and only if $x = y$,

(2). $d(x, y) = d(y, x)$ for all $x, y \in X$,

(3). $d(x, y) = d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space. The concept of cone metric space is more general than that of a metric space.

Example 2.3. *Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = R$ and $d : X \times X \to E$ defined by $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.*

Definition 2.4. *Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}_{n\geq1}$ be a sequence in X. then,*

(1). $\{x_n\}_{n\geq1}$ Converges to x whenever for every $c \in E$ with $\theta \ll c$, if there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$, $(n \to \infty)$.

(2). $\{x_n\}_{n\geq1}$ is said to be a Cauchy sequence if for every $c \in E$ with $\theta \ll c$, if there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n.m \geq N$

(3). (X, d) is called a complete cone metric space if every Cauchy sequence in X is Convergent.

Definition 2.5 ([28])**.** *Let (X, d) be a cone metric space, P be a solid cone and $T : X \to X$ then*

(1). T is said to be continuous if $\lim\limits_{n\to\infty} x_n = x$ implies that $\lim\limits_{n\to\infty} Tx_n = Tx$ for all $\{x_n\}$ in X;

(2). T is said to be subsequently convergent, if for every sequence $\{x_n\}$ that $\{Tx_n\}$ is convergent, implies $\{x_n\}$ has a convergent subsequence,

(3). T is said to be sequentially convergent if for every sequence $\{x_n\}$, if $\{Tx_n\}$ is convergent, then $\{x_n\}$ is also convergent.

Lemma 2.6 ([29])**.**

(1). If E is a real Banach space with cone P and $a \leq \lambda a$ where $a \in P$ and $\theta \leq \lambda < 1$, then $a = \theta$.

(2). If $c \in intP$, $\theta \leq a_n$ and $a_n \to \theta$ then there a positive integer N such that $a_n \ll c$ for all $n \geq N$.

Next, we give the definition of c-distance on a cone metric space (X, d) which is generalization of w-distance of Kada et al. [11] with some properties.

Definition 2.7 ([12]). *Let (X, d) be a cone metric space. A function $q : X \times X \to E$ is called a c-distance on X if the following conditions hold:*

(Q_1). $\theta \leq q(x, y)$ for all $x, y \in X$,

(Q_2). $q(x, y) \leq q(x, y) + q(y, z)$ for all $x, y, z \in X$,

(Q_3). for each $x \in X$ and $n \geq 1$, if $q(x, y_n) \leq u$ for some $u = u_x \in P$, then $q(x, y) \leq u$. Whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$,

(Q_4). for all $c \in E$ with $\theta \ll c$, there exist $e \in E$ with $\theta \in e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Example 2.8 ([12]). *Let $E = R$ and $P = \{x \in E : x \geq 0\}$. Let $X = [0, \infty)$ and define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define by $q : X \times X \to E$ by $q(x, y) = y$ for all $x, y \in X$. Then qis a c-distance on$X$.*

Example 2.9 ([17, 18]). *Let $E = R^2$ and $P = \{(x, y) \in E : x, y \geq 0\}$. Let $X = [0, 1]$ and define a mapping $d : X \times X \to E$ by $d(x, y) = (|x - y|, |x - y|)$ for all $x, y \in X$. Then (X, d) is a complete cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = (y, y)$ for all $x, y \in X$. Then q is a c-distance.*

Example 2.10 ([26]). *Let $X = C_{\frac{1}{R}}[0, 1]$ (the set of real valued functions on X which also have continuous derivatives on X), $P = \{\varphi \in E : \varphi(t) \geq 0\}$. A cone metric d on X is defined by $d(x, y)(t) := |x - y| . \varphi(t)$ where $\varphi \in P$ is an arbitrary function. This cone is non normal. Then (X, d) is a complete cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y)(t) = y.e^t$ for all $x, y \in X$. It is easy to see that q is a c-distance.*

Lemma 2.11 ([12]). *Let (X, d) be a cone metric space and q is c-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be a sequences in X and $x, y, z \in X$. Suppose that u_n is sequence in P converging to 0. Then the following conditions hold:*

(1). If $q(x_n, y) \leq u_n$ and $q(x_n, z) \leq u_n$, then $y = z$.

(2). If $q(x_n, y_n) \leq u_n$ and $q(x_n, z) \leq u_n$, then $\{y_n\}$ converges to z.

(3). If $q(x_n, x_m) \leq u_n$ for $m > n$ and $\{x_n\}$ is a Cauchy sequence in X.

(4). If $q(y, x_n) \leq u_n$ then $\{x_n\}$ is a Cauchy sequence in X.

Remark 2.12 ([12]).

(1). $q(x, y) = q(y, x)$ does not necessarily for all $x, y \in X$.

(2). $q(x, y) = \theta$ is not necessarily equivalent to $x - y$ for all $x, y \in X$.

Now, we introduce the T-Hardy-Rogers's contraction under the concept of c-distance in cone metric spaces.

Definition 2.13. *Let (X, d) be a cone metric spaces and $f, T : X \to X$ be any two mappings. A mapping f is said to be T-Hardy-Rogers contraction, if there exists a constant $k, l, m, n, r \in [0, 1)$ with $k + l + m + n + r < 1$ such that*

$$q(Tfx, Tfy) \leq kq(Tx, Ty) + lq(Tx, Tfx) + mq(Ty, Tfy) + nq(Tx, Tfy) + rq(Ty, Tfx) \text{ for all } x, y \in X.$$

3. Main Results

Now, we give our main results in this paper.

Theorem 3.1. *Let (X, d) be a complete cone metric spaces, P be a solid cone and q be a c-distance on X. In addition $T : X \to X$ be an one to one, continuous function and $f : X \to X$ be a mappings satisfies the contractive condition*

$$q\left(Tfx, Tfy\right) \leq kq\left(Tx, Ty\right) + lq\left(Tx, Tfx\right) + mq\left(Ty, Tfy\right) + nq\left(Tx, Tfy\right) + rq(Ty, Tfx) \tag{1}$$

for all $x, y \in X$. Where $k, l, m, n, r \in [0, 1)$ are constants such that $k + l + m + n + r < 1$. Then f has an unique fixed point $x^* \in X$. And for any $x \in X$, iterative sequence $\{fx_n\}$ converges to the fixed point. If $u = fu$. Then $q\left(Tu, Tu\right) = \theta$.

Proof. Choose $x_0 \in X.X$. Set $x_1 = fx_0, x_2 = fx_1 = f^2x_0, \ldots, x_{n+1} = fx_n = f^{n+1}x_0$. Then, from (1), we have

$$\begin{aligned} q(Tx_n, Tx_{n+1}) &= q(Tfx_{n-1}, Tfx_n) \\ &\leq kq(Tx_{n-1}, Tx_n) + lq(Tx_{n-1}, Tfx_{n-1}) + mq(Tx_n, Tfx_{2n}) + nq(Tx_{n-1}, Tfx_{2n}) + rq(Tx_n, Tfx_{n-1}) \\ &\leq kq(Tx_{n-1}, Tx_n) + lq(Tx_{n-1}, Tx_n) + mq(Tx_n, Tx_{n+1}) + nq(Tx_{n-1}, Tx_{n+1}) + rq(Tx_n, Tx_n) \end{aligned}$$

$$\begin{aligned} q\left(Tx_{2n}, Tx_{2n+1}\right) &\leq \left(k + l + n\right) q(Tx_{2n-1}, Tx_{2n}) + (m + n)q(Tx_{2n}, Tx_{2n+1}) \\ \Rightarrow ([1 - (m + n)]q(Tx_{2n}, Tx_{2n+1}) &\leq \left(k + l + n\right) q(Tx_{2n-1}, Tx_{2n}) \\ \Rightarrow q(Tx_{2n}, Tx_{2n+1}) &\leq \frac{k + l + n}{1 - (m + n)} q(Tx_{2n-1}, Tx_{2n}) \\ &\leq hq(Tx_{2n-1}, Tx_{2n}). \end{aligned} \tag{2}$$

Where $\frac{k+l+n}{1-(m+n)} = h < 1$. Let $m > n \geq 1$, we have

$$\begin{aligned} q(Tx_n, Tx_m) &\leq q\left(Tx_n, Tx_{n+1}\right) + q\left(Tx_{n+1}, Tx_{n+2}\right) + \cdots + q(Tx_{n-1}, Tx_n) \\ &\leq \left(h^n + h^{n+1} + \cdots + h^{n-1}\right) q(Tx_0, Tx_1) \\ &\leq \frac{h^n}{1 - h} q\left(Tx_0, Tx_1\right) \to \infty, \quad h \to \infty. \end{aligned}$$

Thus, Lemma 2.11 (3), which implies that, $\{Tx_n\}$ is a Cauchy sequence in X. Since X is complete cone metric space, then there exist $v \in X$ such that

$$Tx_n \to v \text{ as } n \to \infty \tag{3}$$

Since T is subsequently convergent, $\{x_n\}$ has a convergent subsequence. So, there are $x^* \in X$ and $\{x_{ni}\}$ such that

$$x_{ni} \to x^* \text{ as } i \to \infty. \tag{4}$$

Since T is continuous, then by (3), we obtain

$$Tx_i = Tx^*. \tag{5}$$

Now from (3) and (5), we conclude that

$$Tx^* = v \tag{6}$$

By Definition 2.7. (Q_3), we have

$$q(Tx_{2n}, Tx^*) \leq \frac{h^{2n}}{1-h} q\left(Tx_0, Tx_1\right) \tag{7}$$

On the other hand and using (5), we have

$$\begin{aligned} q(Tx_n, Tfx^*) &\leq q(Tfx_{2n-1}, Tfx^*) \\ &\leq kq(Tx_{2n-1}, Tx^*) \\ &\leq k\frac{h^{2n-1}}{1-h} q\left(Tx_0, Tx_1\right) \\ &= \frac{h^{2n}}{1-h} q\left(Tx_0, Tx_1\right). \end{aligned} \tag{8}$$

By Lemma 2.11 (1), from (7) and (8), we have

$$Tx^* = Tfx^* \tag{9}$$

Since T is one to one, then $x^* = fx^*$. Thus x^* is a fixed point of f. Moreover, suppose that, $u = fu$ and then we have

$$\begin{aligned} q(Tu, Tu) &= q(Tfu, Tfu) \\ &\leq kq\left(Tu, Tu\right) + lq(Tu, Tfu) + mq(Tu, Tfu) + nq(Tu, Tfu) + rq(Tu + Tfu) \\ &= (k + l + m + n + r)q(Tu, Tu) \end{aligned}$$

Since $k + l + m + n + r < 1$, Lemma 2.6 (1), shows that $q\left(Tu, Tu\right) = \theta$. Finally suppose that, if y^* is another common fixed point of f. Then we have

$$\begin{aligned} q(Tx^*, Ty^*) &= q(Tfx^*, Tfy^*) \\ &\leq kq\left(Tx^*, Ty^*\right) + lq(Tx^*, Tfx^*) + mq(Ty^*, Tfy^*) + nq(Tx^*, Tfy^*) + lq(Ty^*, Tfx^*) \\ &= kq\left(x^*, y^*\right) + lq\left(x^*, x^*\right) + mq\left(y^*, y^*\right) + nq\left(x^*, y^*\right) + rq\left(y^*, x^*\right) \\ &= (k + n + r)q(Tx^*, Ty^*) \\ &\leq (k + l + m + n + r)q(Tx^*, Ty^*). \end{aligned}$$

Since $k + l + m + n + r < 1$, Lemma2.6 (1), shows that $q\left(Tx^*, Ty^*\right) = \theta$. Also we have $q\left(Tx^*, Tx^*\right) = \theta$. Thus, Lemma 2.11 (1), $Tx^* = Ty^*$. Since T is one to one, then $x^* = y^*$ so, x^* is the unique fixed point of f. □

Remark 3.2.

(1). If we put $n = r = 0$ in Theorem 3.1, we get the result of Theorem 3.1 of Fadail et al. [26].

(2). If we put $l = m = n = r = 0$ in Theorem 3.1, we get the result of Corollary 3.2 of Fadail et al. [26].

(3). We put $l = n = r = 0$ and $m = l$ in Theorem 3.1, we get the result of Corollary 3.3 of Fadail et al. [26].

If we take $T = 1$ in the above theorem, we get the following corollary.

Corollary 3.3. *Let (X, d) be a complete cone metric spaces, P be a solid cone and q be a c-distance on X. Let $f : X \to X$ be a mappings satisfies the contractive condition*

$$q\left(fx, fy\right) \leq kq\left(x, y\right) + lq\left(x, fx\right) + mq\left(y, fy\right) + nq\left(x, fy\right) + rq(y, fx)$$

for all $x, y \in X$, where $k, l, m, n, r \in [0, 1)$ are constants such that $k + l + m + n + r < 1$. Then f has an unique fixed point $x^* \in X$. And for any $x \in X$, iterative sequence $\{fx_n\}$ converges to the fixed point. If $u = fu$. Then $q\left(u, u\right) = \theta$.

References

[1] S.Banach, *Sur les operations dans les ensembles abstraits et leur applications aux equations integrals*, Fundamental Mathematicae, 3(7)(1922), 133-181.

[2] Huang and Zhang, *Cone metric spaces and fixed point theorems of Contractive Mappings*, J. Math. Anal. Appl., 332(2007), 1468-1476.

[3] B.E.Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc., 226(1977), 257-290.

[4] A.Beiranvand, O.Moradi and H.Pazandeh, *Two fixed point for special mapping*, arxiv:0903. 1504 v1 math FA, (2009).

[5] R.Kannan, *Some results on fixed point*, Bull. Calc. Math. Soc., 60(1968), 71-76.

[6] R.Kannan, *Some results on fixed point-II*, Amer. Bull. Calc. Math. Monthly, 76(1969), 405-408.

[7] S.K.Chatterjee, *Fixed point theorems*, C. R. Acad. Bulgare Sci., 25(1972), 727-730.

[8] L.B.M.Cric, *A generalization of Banach's contraction principle*, proceeding of the American Mathematical Society, 45(1975), 267-273.

[9] G.Jungck, *Commuting maps anf fixed points*, American Mathematic Monthly, 83(1976), 261-263.

[10] G.Jungck, *Common fixed points for non continuous non self maps on non metric spaces*, For East Journal of Mathematical Sciences, 4(1966), 199-215.

[11] O.Kada, T.Suzuki and W.Takahashi, *Non convex minimization theorems and fixed point theorems in complete metric spaces*, Mathematica Japonica, 44(1996), 381-391.

[12] Y.J.Cho, R.Saadati and S.Wang, *Common fixed point theorems on generalized distance in ordered cone metric spaces*, Comput. Math. Appl., 61(2011), 1254-1260.

[13] W.Situnavarat, Y.J.Cho and P.Kumam, *Common fixed point theorems for c-distance in ordered cone metric spaces*, Comput. Math. Appl., 62(2011), 1969-1978.

[14] S.Wang and B.Guo, *Distance in cone metric spaces and common fixed point theorems*, Applied Mathematics Letters, 24(2011), 17-1739.

[15] G.E.Hardy and T.D.Rogers, *A generalization of a fixed point theorem of Reich*, Canad. Math. Bull., 16(1973), 201-206.

[16] Z.M.Fadail, A.G.B.Ahmad and Z.Golubovic, *Fixed point theorems of single valued mapping for c-distance in cone metric spaces*, Abstract and Applied Analysis, 2012(2012), 1-11.

[17] Z.M.Fadail, A.G.B.Ahmad and L.Paunovic, *New Fixed point results of single valued mapping for c-distance in cone metric spaces*, Abstract and Applied Analysis, 2012(2012), 1-12.

[18] Z.M.Fadail and A.G.B.Ahmad, *Coupled Fixed point theorems of single valued mapping for c-distance in cone metric spaces*, Journal of Applied Mathematics, 2012(2012), 1-20.

[19] Z.M.Fadail and A.G.B.Ahmad, *Common coupled Fixed point theorems of single valued mapping for c-distance in cone metric spaces*, Abstract and Applied Analysis, 2012(2012), 1-24.

[20] Z.M.Fadail, A.G.B.Ahmad and S.Radenovic, *Common Fixed point and fixed point results under c- distance in cone metric spaces*, Applied Mathematics and Information Sciences Letters, 1(2)(2013), 47-52.

[21] Z.M.Fadail and A.G.B.Ahmad, *New coupled Coincidence point and common coipled Fixed point results in cone metric spaces with c-distance*, For East Journal of Mathematical Sciences, 77(1)(2013), 65-84.

[22] S.M.Abusalim and M.S.M.Noorani, *Generalized distance in cone metric spaces and tripled coincidence point and common tripled fixed point theorems*, For East Journal of Mathematical Sciences, 91(1)(2014), 65-87.

[23] S.K.Tiwari, R.Verma and A.K.Dubey, *Cone metric spaces and Fixed point theorems of contractive mapping for c-*

distance, International Journal of Mathematics and its Applications, 3(1)(2015)83-88.

[24] M.Dordevic, D.Dordevic, Z.Kadelburg, S.Rdenovic and D.Spasic, *Fixed point results under c-distance in tvs cone metric spaces*, Fixed point Theory and Applications, 2011(2011).

[25] A.K.Dubey and Urmila Mishra, *Some fixed point results single valued mapping for c-distance in Tvs-cone metric spaces*, Filomat, 30(11)(2016), 2925-2934.

[26] Z.M.Fadail and S.M.Abusalim, *T-Reich Contraction and Fixed point results in cone metric spaces with c-distance*, Int. J. of Math. Anal., 11(8)(2017), 397-405.

[27] W.Shatanawi, E.Karapinar and H.Aydi, *Coupled coincidence points in partially ordered cone metrc spaces with c-distance*, Journal of Applied Mathematics, 2012(2012), 1-5.

[28] M.Filipovic, L.Paunovic, S.Radenovic and M.Rajovic, *Remarks on cone metric spaces and fixed point theorems of T-Kannan and T-Chaterjea contractive mappings*, Mathematical and Computer Modeling, 54(2011), 1467-1472.

[29] Z.Kadelburg, S.Radenvic and V.Rakolevic, *Remarks on Quasi-contraction in cone Metric spaces*, Applied Mathematics Letters, 22(11)(2009), 1674-1679.