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On Completion of Normed Vector Space and Contraction Mapping

Research Article

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Abstract: In this paper we will discuss in detail, the concept of completion of metric space and its application to completing normed vector space. Further if the necessity of a condition of a mapping being contraction is loosen, then the mapping may not have a fixed point.

Keywords: Contraction mapping principle, Fixed point theorem, Normed vector space, Complete Metric Space.

1. Introduction

The role of contraction mapping principle in the study of fixed point is very important. The role of Fixed point theorem assures the unique fixed point of contraction mapping of a complete metric space to itself and it may be used as numerical iterations. The development of many theories in different areas like metric space Banach space problems in nonlinear differential equations, system of algebraic equation has happened due to contraction mapping principle.

2. Preliminaries

Definition 2.1. *A Metric space (M, d) is a pair consisting non empty set M of elements and a notion of distance function* $d : M \times M \to R$ *satisfying following conditions*

(1). $d(x, y) = 0$ *and if* $d(x, y) = 0 \Leftrightarrow x = y$

(2). $d(x, y) = d(y, x)$

(3). $d(x, z) = d(x, y) + d(y, z)$*, which is also called triangle inequality.*

A sequence $\{x_n\}$ *in M is said to converge in M such that* $\lim_{n\to\infty} d(x_n, x) = 0$ *or* $\lim_{n\to\infty} x_n = x$*,* $\{x_n\}$ *will be called Cauchy's sequence if for all* $\epsilon > 0$*, there exists* $n_0 > 0$*, such that* $d(x_n, x_m) = \epsilon$*, for* $\forall\, n, m \geq n_0$*.*

Definition 2.2. *Complete Metric space: If every Cauchy's sequence in M converges to a point M then M is called as a Complete Metric space.*

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Definition 2.3. *Normed Vector Space (Banach Space): A mapping* $\parallel \cdot \parallel: M \rightarrow R$ *is called norm, where* M *is vector space over real or complex numbers, provided that following conditions hold:*

(1). $\parallel x \parallel = 0 \Leftrightarrow x = 0$*, for* $\forall\ x \in M$

(2). $\parallel \alpha x \parallel = |\alpha| \parallel x \parallel,\ \forall$ *scalar* $\alpha,\ \forall\ x \in M$

(3). $\parallel x + y \parallel \leq \parallel x \parallel + \parallel y \parallel,\ \forall\ x, y \in M$

Hence a vector space M *and norm* $\parallel \cdot \parallel$ *together the pair* $(M, \parallel \cdot \parallel)$ *is called normed vector space. This can also be defined as* $d\,(x, y) = \parallel x - y \parallel,\ \forall\ x, y \in M$*.*

Definition 2.4. *A Normed vector space which is complete metric space with respect to metric* d *defined above is called Banach Space.*

Example 2.5. $(R, |\cdot|)$ *is an example of Banach Space.*

Example 2.6. *The linear combination of a polynomial is a polynomial. The space of polynomial functions is a linear subspace of* $C([0, 1])$*. It is not closed but dense. Therefore the set* $\{f \in C\,([0, 1])|\ f\,(0) = 0\}$ *is a closed linear subspace of* $C([0, 1])$ *and is Banach space.*

3. Main Result

Theorem 3.1. *If (X , d) is a metric space then there exists a complete metric space* (X^*, d^*) *and mapping* $h : X \rightarrow X^*$ *such that*

(1). h is an isometry $d^*(h(x), h(y)) = d(x, y)$*.*

(2). h(x) is dense in X^**.*

Proof. Let $\{x_n\}$ and $\{y_n\}$ be the sequences in X and let C be the set of all Cauchy's sequences in X. Then $\{d(x_n, y_n)\}$ is the Cauchy's sequence in R. We define $d_c : C \times C \rightarrow R$ by $d_c\,(\{x_n\}, \{y_n\}) = 0 \nRightarrow \{x_n\} = \{y_n\}$. The mapping d_c is a pseudo-metric on C. Here $\{x_n\}R\,\{y_n\}$ iff $d\,(x_n, y_n) = 0$ as $n \rightarrow \infty$.
Obviously $d_c\,(\{x_n\}, \{y_n\}) = 0$ is an equivalence relation on C. Now the set of all equivalence classes C/R we denote by X^*. We define $d^*\,(R\,\{x_n\}, R\,\{y_n\}) = d_c\,(\{x_n\}, \{y_n\})$. This defines the metric on X^*. There is a natural mapping $X \rightarrow C$ given by $x \rightarrow \{x\}$, clearly, $d_c\,(\{x_n\}, \{y_n\}) = d(x, y)$. Therefore $x \rightarrow R\,(x)$, which is h is an isometry of X to X^* and clearly image $h(x)$ is dense in X^*. □

Theorem 3.2 (Contraction Mapping Principle)**.** *Let (M,d) be a complete metric space and* $T : M \rightarrow M$ *be a contraction mapping with Lipschitz constant* α*. Then* T *has unique fixed point* $x \in M$*.*

Proof. Let us consider the sequence $\{x_n\}$, $n \rightarrow \infty$, given by $x_0 = y$, $x_n = T\,(x_{n-1})$, $n \geq 1$. Where $y \in M$ is an arbitrary point in M. For $m < n$, we have triangle inequality,

$$d\,(x_m, x_n) \leq d\,(x_m, x_{m+1}) + d\,(x_{m+1}, x_{m+2}) + \cdots + d\,(x_{n-1}, x_n)$$

Since T is contraction, $d\,(x_p, x_{p+1}) = d\,(T(x_{p-1}), T(x_p)) \leq kd\,(x_{p-1}, x_p)$, for any integer ≥ 1. Successively we get, $d\,(x_p, x_{p+1}) = k^p d(x_0, x_1)$. Hence

$$d\,(x_m,\ x_n) \leq \left(k^m + k^{m+1} + \cdots + k^{n-1}\right) d(x_0, x_1) \leq \frac{k^m}{1-k} d(x_0, x_1),\ \text{whenever } m \leq n$$

Hence we can say that $\{x_n\}$ is a Cauchy's sequence. Since M is complete, this sequence has a limit say $x \in M$. Also since T is a continuous mapping, $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} (T(x_{n-1})) = T\left(\lim_{n\to\infty} x_{n-1}\right) = T(x)$. Thus x is a fixed point of T.

Uniqueness: If x and z are two fixed points of T then,

$$d(x, z) = d[T(x), T(z)] \leq kd(x, z)$$

But $k < 1$, we must have $x = z$. This follows the uniqueness of fixed point and hence the theorem. □

Theorem 3.3. *Let (M, d) be a complete metric space and let*

$$E = \{x \in M : d(z, x) < \in\}, \quad z \in M, \quad \epsilon > 0$$

Let $T : E \to M$ be a mapping such that $d(T(y), T(x)) = ad(x, y)$, $\forall\, x, y \in E$. With Lipschitz constant $a < 1$, also assume that $d(z, T(z)) < \epsilon(1 - a)$ then T has unique fixed point $x \in E$.

Proof. Let $x \in \overline{E}$, now as T is uniformly continuous, we can extend T to a mapping defined $\overline{E}$ which is a contraction mapping with a as a Lipschitz constant. Hence

$$d(z, T(x)) \leq d(z, T(z)) + d(T(z), T(x)) < \epsilon(1 - k) + k\epsilon = \epsilon$$

Therefore $T : \overline{E} \to E$. Hence by Theorem 3.2, contraction mapping principle and $\overline{E}$ is a complete metric space, T has unique fixed point in $\overline{E}$ which must be in E. □

Example 3.4. *Now let us define $d(x, y) = |x - y|$, $x, y \in M$. Where $M = \{x \in R : x \geq 1\}$ is a metric space. Let $T : M \to M$ defined as $T(x) = x + \frac{1}{x}$. Obviously $d[T(x), T(z)] = \frac{xy-1}{xy} |x - y| < |x - y| = d(x, y)$. Therefore there does not exists any constant $0 \leq \alpha < 1$ such that $d[T(x), T(y)] = \alpha d(x, y)$, $\forall\, x, y \in M$. Hence we claim that T has no fixed points in M.*

4. Conclusion

It is clear from the above discussion that if we replace the hypothesis of the contraction mapping principle of being contraction mapping of T by the condition $d[T(x), T(y)] \leq d(x, y)$. Then T may not (need not) have a fixed point.

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