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# Second-Order Boundary Value Problem With Set of the Associated Green's Function May Have Zeros

**Research Article**

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**Abstract:** We consider the nonlinear second-order boundary value problem

$$u'' + \kappa^2 u = f(t, u(t)), \quad t \in (0, T), T > 0,$$
$$u(0) = u(T), u'(0) = u'(T),$$

where $0 < \kappa < \frac{\pi}{T}$, $f : [0, T] \times [0, \infty) \to [0, \infty]$ is continuous. We use the sets of the associated Green's function may have zeros at some interior points. In particular, we study the problems where the associated Green's function may have zeros. The proof is based on the fixed point theorem in cones.

**MSC:** 34B15, 34B18, 34B27.

**Keywords:** Second-order, Positive solutions, Sets of associated Green's function zeros, Fixed point theorem in cones.



## 1. Introduction

We consider the nonlinear second-order boundary value problem

$$u'' + \kappa^2 u = f(t, u(t)), \quad t \in \Omega, \tag{1}$$

$$u(0) = u(T), u'(0) = u'(T), \tag{2}$$

where $0 < \kappa < \frac{\pi}{T}$, $f : [0, T] \times [0, \infty) \to [0, \infty]$ is continuous. It's well-known that when the Green's function is positive, we can always find its positive minimum A and maximum B. Define a cone as follows:

$$K := \left\{ u \in X | u(t) \geq 0, \min_{t \in [0,T]} u(t) \geq \frac{A}{B} \parallel u \parallel \right\}. \tag{3}$$

Then, Krasnosel'skii's fixed point theorem can be used to prove the existence and multiplicity of positive solutions; see, [1, 2, 5, 7–10, 12–15] and references therein. Boundary value problem with the associated Green's function may have zeros has been studied in [4, 11, 14] and references therein. In a recent paper [4], Graef, Kong and Wang establish the existence of nonnegative solutions in the case where the associated Green's function may have zeros. However, if $\kappa = 1/2$, then the

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Green's function is zero at $t = s$. The minimum value of the Green's function is zero and the above cone cannot be used to apply Krasnosel'skii's theorem. Graef et al used a new cone of form

$$\bar{K} := \left\{ u \in X | u(t) \geq 0, \min_{t\in[0,2\pi]} \int_0^{2\pi} u(t) \geq \frac{\bar{A}}{\bar{B}} \parallel u \parallel \right\}, \tag{4}$$

where $\bar{A}$ is defined by $\bar{A} = \min_{t\in[0,2\pi]} \int_0^{2\pi} G(t,s)ds > 0$ and $\bar{B} = \min_{t\in[0,2\pi]} \mid G(t,s) \mid$. Under a sub-linear condition on $f$ and also under a super-linear condition on $f$ provided that $f$ is convex. Webb [14] used fixed point theory and a new open set to improve the main results of [1]. By bifurcation techniques Ma and Zhong [11] obtained the existence of positive solutions of integral equations in $C[0,1]$ where the kernel may vanish at the interior points of $[0,1] \times [0,1]$. All of these earlier results use the cone (4) cannot be used to apply Krasnosel'skii's theorem. In the present paper we apply Krasnosel'skii's theorem in a cone (3) to study the problem where the associated Green's function may have zeros. The important tool define and use the set of the associated Green's function may have zeros at some interior points and by these set of zeros element in the associated Green's function determine the minimum and maximum value of Green's function. It is our purpose to prove the existence of positive solutions of (1), (2). Assuming that:

(H1) $0 < \kappa < \frac{\pi}{T}$

(H2) $f : [0,T] \times [0,\infty) \to [0,\infty]$ is continuous.

The proof of the main results is based upon an application of the following fixed point theorem in cones [3, 9].

**Theorem 1.1** ([3, 9]). *Let $E$ be a Banach space, and let $K \subset E$ be a cone. Assume $\Omega_1$ , $\Omega_2$ are open bounded subsets of $E$ with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$. And let $T : K \cap (\bar{\Omega}_2 \backslash \Omega_1) \to K$ be a completely continuous operator such that:*

*(1). If $\parallel Tu \parallel \leq \parallel u \parallel, u \in K \cap \partial\Omega_1$ and $\parallel Tu \parallel \geq \parallel u \parallel, u \in K \cap \partial\Omega_2$, or*

*(2). If $\parallel Tu \parallel \geq \parallel u \parallel, u \in K \cap \partial\Omega_1$ and $\parallel Tu \parallel \leq \parallel u \parallel, u \in K \cap \partial\Omega_2$,*

*Then $T$ has a fixed point in $K \cap (\bar{\Omega}_2 \backslash \Omega_1)$.*

## 2. Main result

In this section, we present and prove our main result. Consider $y \in C[0,1]$, the problem

$$u'' + \kappa^2 u = y(t), \quad t \in (0,1), \tag{5}$$

$$u(0) = u(T), \ u'(0) = u'(T), \tag{6}$$

where $0 < \kappa < \frac{\pi}{T}$ is a constant. It is well known that then the Green's function for (5) and (6) is given by

$$G(t,s) = \frac{1}{2\kappa(1-\cos kT)} \begin{cases} \sin\kappa(t-s) + \sin\kappa(T+s-t), & 0 \leq s \leq t \leq T, \\ \sin\kappa(s-t) + \sin\kappa(T+t-s), & 0 \leq t \leq s \leq T, \end{cases}$$

We can verify that $G$ is strictly positive, in fact, let $\hat{G}(x) = \frac{\sin\kappa(x)+\sin\kappa(T-x)}{2\kappa(1-\cos kT)}, x \in [0,T]$. It easy to check that $\hat{G}(x)$ is increasing on $[0, \frac{T}{2}]$ and nondecreasing on $[\frac{T}{2}, T]$, and $G(t,s) = \hat{G}(\mid t-s \mid)$

$$0 < \hat{G}(0) = \frac{\sin\kappa T}{2\kappa(1-\cos kT)} \leq G(t,s) \leq \hat{G}(\frac{T}{2}) = \frac{\sin\kappa\frac{T}{2}}{2\kappa(1-\cos kT)} = \frac{1}{2\kappa\sin\kappa\frac{T}{2}}.$$

Green's function have zeros in $s = t$, and $\kappa = \frac{\pi}{T}$, see [1]. Now, define set of the associated Green's function may have zeros at $t = s$ by

$$\Gamma = \{(t, s) \in [0, T] \times [0, T] | G(t, s) = 0, \ as \ s = t, \kappa \in (0, \frac{\pi}{T}]\}.$$

Then, define

$$J = [0, T] \backslash \Gamma. \tag{7}$$

Then, easy can find its positive minimum $A$ and maximum $B$ by

$$0 < A = \min_{t,s \in [0,T] \backslash \Gamma} G(t, s), \qquad B = \max_{t,s \in [0,T]} | G(t, s) |, \quad \sigma = \frac{A}{B}. \tag{8}$$

Our main result are:

We now state our main results in this work. Analogous results for the Dirichlet/Neumann boundary value problems were established in [2].

**Theorem 2.1.** *Assume (H1) and (H2) holds. And* $f(t, u(t)) \neq 0$ *on any subinterval of [0, T].*

*(a). If* $\lim_{u \to 0_+} \max_{t \in [0,T]} \frac{f(t,u)}{u} = 0$ *and* $\lim_{u \to +\infty} \min_{t \in [0,T]} \frac{f(t,u)}{u} = \infty$*, then (1), (2) has a positive solution; or*

*(b). If* $\lim_{u \to 0_+} \min_{t \in [0,T]} \frac{f(t,u)}{u} = \infty$ *and* $\lim_{u \to +\infty} \max_{t \in [0,T]} \frac{f(t,u)}{u} = 0$*, then (1), (2) has a positive solution.*

*Proof.* Superlinear case.

$$\lim_{u \to 0_+} \max_{t \in [0,T]} \frac{f(t,u)}{u} = 0 \ \ and \ \ \lim_{u \to +\infty} \min_{t \in [0,T]} \frac{f(t,u)}{u} = \infty.$$

It is clear that the problem (1) and (2) has a solution $u = u(t)$ if and only if $u$ solves the operator equation

$$u(t) = \int_0^T G(t, s) f(s, u(s)) ds := Tu(t), \qquad u \in C[0, T].$$

Denote

$$K = \{u \in C[0, T] | u(t) \geq 0, \min_{t \in J} u(t) \geq \sigma \parallel u \parallel\}$$

where $J$, $\sigma$ defined in (7), (8) respectively above and $\parallel u \parallel = \max_{t \in [0,T]} | u(t |$ . ☐

**Lemma 2.2.** *Assume (H1) and (H2) holds. Then* $T(K) \subset K$ *and the map* $T : K \to K$ *is completely continuous.*

*Proof.* If $u \in K$ then

$$\begin{aligned}
\min_{t \in J} Tu(t) &= \min_{t \in J} \int_0^T G(t, s) f(s, u(s)) ds, \\
&\geq A \int_0^T f(s, u(s)) ds, \\
&= B \ \sigma \int_0^T f(s, u(s)) ds \geq \sigma \sup \parallel Tu \parallel .
\end{aligned}$$

Thus, $T(K) \subset K$. It easy to verify that $T$ is completely continuous.

Now, since $\lim_{u \to 0_+} \max_{t \in [0,T]} \frac{f(t,u)}{u} = 0$, we may choose $H_1 > 0$, such that $f(t, u) \leq \ \varepsilon \ u$ , for $t \in [0, 1], 0 < u \leq H_1$, where $\varepsilon > 0$ satisfies $B \ \varepsilon \ T \leq 1$. Thus, if $u \in K$ and $\parallel u \parallel = H_1$, then

$$\begin{aligned}
Tu(t) &\leq \max_{t \in [0,T]} \int_0^T G(t, s) f(s, u(s)) ds, \\
&\leq B \int_0^T f(s, u(s)) ds \leq \parallel u \parallel,
\end{aligned}$$

Now if we let $\Omega_1 := \{u \in K :\| u \|< H_1\}$. Then we have $\| Tu \|\leq\| u \|$, $u \in K \cap \partial\Omega_1$. Further, since $\lim_{u\to+\infty} \min_{t\in[0,T]} \frac{f(t,u)}{u} = \infty$, there exist $\hat{H}_2 > 0$, such that for $t \in [0, T]$, and $u \geq \hat{H}_2, f(t, u) \geq \mu u$ where $\mu > 0$ and $A\mu \int_J ds \geq 1$. Take $H_2 = \{2H_1, \frac{\hat{H}_2}{\sigma}\}$, $\Omega_2 := \{u \in K :\| u \|< H_2\}$. Then $u \in K, \| u \|= H_2$ implies $\min_{t\in J} u(t) \geq \sigma \| u \|\geq \hat{H}_2$, for $\hat{t} \in J$

$$\begin{aligned}
Tu(\hat{t}) &= \int_0^T G(\hat{t}, s)f(s, u(s)ds, \\
&\geq A \int_J f(s, u(s)ds \\
&\geq A \int_J \mu \| u \| ds \\
&\geq\| u \| .
\end{aligned}$$

Hence, $\| Tu \|\geq\| u \|$ for $u \in K \cap \partial\Omega_2$. Therefore, by the first part of the Theorem 1.1 and Lemma 2.2, it follows that $T$ has a fixed point in $K \cap \bar{\Omega}_2\backslash\Omega_1$ such that $H_1 \leq\| u \|\leq H_2$ and $u(t) > 0$ for $t \in [0, 1]$. This completes the superlinear part of the theorem. Sublinear case.

$$\lim_{u\to0_+} \min_{t\in[0,T]} \frac{f(t,u)}{u} = \infty \text{ and } \lim_{u\to+\infty} \max_{t\in[0,T]} \frac{f(t,u)}{u} = 0.$$

We first choose $H_1 > 0$ such that $f(t, u) \geq \hat{\eta} u$ for $0 < u \leq H_1$, where $\hat{\eta}A \int_J ds \geq 1$ ($A$ and $J$ define in above of proof). Then for $u \in K$, $\| u \|= H_1$ and $\hat{t} \in J$ we have

$$\begin{aligned}
Tu(\hat{t}) &= \int_0^T G(\hat{t}, s)f(s, u(s))ds, \\
&\geq A \int_J f(s, u(s)ds, \\
&\geq A\hat{\eta} \int_J \| u \| ds, \\
&\geq\| u \| .
\end{aligned}$$

Let $\Omega_1 := \{u \in K :\| u \|< H_1\}$ such that $\| Tu \|\geq\| u \| \quad for\ u \in K \cap \partial\Omega_1$. Now, since $\lim_{u\to+\infty} \max_{t\in[0,T]} \frac{f(t,u)}{u} = 0$, there exist $\hat{H}_2 > 0$ so that $f(t, u) \leq \lambda u$ for $u \geq \hat{H}_2$ where $\lambda > 0$ satisfies $\lambda BT \leq 1$.

We consider two cases:

**Case (i):** $f$ is bounded, $f(t, u) \leq N$ for all $u \in (0, \infty)$. In this case choose $H_2 := \max\{2H_1, NBT\}$ so that for $u \in K$ with $\| u \|= H_2$ we have

$$Tu(t) = \int_0^T G(t, s)f(s, u(s))ds \leq NBT \leq H_2.$$

and therefore $\| Tu \|\leq\| u \|$ .

**Case (ii):** f is unbounded. Then choose $H_2 > \max\{2H_1, \hat{H}_2\}$ such that

$$f(t, u) \leq f(t, H_2) \quad for \ \ 0 < u \leq H_2.$$

Then for $u \in K$ and $\| u \|= H_2$, we have

$$\begin{aligned}
Tu(t) &= \int_0^T G(t, s)f(s, u(s))ds \\
&\leq \lambda B \int_0^T f(s, u(s))ds \\
&\leq \lambda B \int_0^T f(s, H_2)ds \\
&\leq \lambda TBH_2 \\
&\leq H_2 =\| u \| .
\end{aligned}$$

Therefore in either case we may put $\Omega_2 := \{u \in K :\| u \|< H_2\}$, and for $u \in K \cap \partial\Omega_2$ we have $\| Tu \| \leq \| u \|$. By the second part of the Theorem 1.1, it follows that the problem (1), (2) has a positive solution,And this completes the proof of the theorem. □

**Example 2.3.** *Let us consider the periodic boundary value problem (1), (2) with*

*(1).* $\kappa = \frac{1}{2}$ *and* $T = 2\pi$ *then the set of Green's function may have zeros given by*

$$\Gamma = \left\{(t, s) \in [0, T] \times [0, T] | G(t, s) = 0,\ as\ s = t, \kappa = \frac{1}{2}, T = 2\pi\right\}.$$

*Then, define*

$$J = [0, T]\backslash\Gamma. \tag{9}$$

*Then, easy can find its positive minimum A and maximum B by*

$$0 < A = \min_{t,s\in[0,2\pi]\backslash\Gamma} G(t, s), \quad B = \max_{t,s\in[0,2\pi]} | G(t, s) |, \sigma = \frac{A}{B}. \tag{10}$$

*or*

*(2).* $\kappa = \frac{1}{4}$ *and* $T = 4\pi$ *then the set of Green's function may have zeros given by*

$$\Gamma_1 = \left\{(t, s) \in [0, T] \times [0, T] | G(t, s) = 0,\ as\ s = t, \kappa = \frac{1}{4}, T = 4\pi\right\}.$$

*Then, define*

$$J_1 = [0, T]\backslash\Gamma_1. \tag{11}$$

*Then, easy can find its positive minimum A and maximum B by*

$$0 < A = \min_{t,s\in[0,4\pi]\backslash\Gamma_1} G(t, s), \quad B = \max_{t,s\in[0,4\pi]} | G(t, s) |,\ \sigma = \frac{A}{B}. \tag{12}$$

*Now let* $f(t, u) = u^{\alpha}(t)$, $\alpha \in (0, 1) \cup (1, \infty)$. *We see that (H1)(*$\kappa = \frac{1}{2}, \frac{1}{4}$*) hold and (H2) hold. Moreover, it is easy to see that*

*(a).* $\lim_{u\to 0_+} \max_{t\in[0,T]} \frac{f(t,u)}{u} = 0$ *and* $\lim_{u\to+\infty} \min_{t\in[0,T]} \frac{f(t,u)}{u} = \infty$, *if* $\alpha \in (1, \infty)$.

*(b).* $\lim_{u\to 0_+} \min_{t\in[0,T]} \frac{f(t,u)}{u} = \infty$ *and* $\lim_{u\to+\infty} \max_{t\in[0,T]} \frac{f(t,u)}{u} = 0$, *if* $\alpha \in (0, 1)$.

*Then the conclusion follows from Theorem 2.1 (a) and (b).*

## References


[1] F.M.Atici and G.Sh.Guseinov, *On the existence of positive solutions for nonlinear differential equations with periodic boundary conditions*, J. Comput. Appl. Math., 132(2001), 341-356.

[2] L.H.Erbe and H.Wang, *On the existence of positive solutions of ordinary differential equations*, Proc. Amer. Math. Soc., 120(1994), 743-748.

[3] D.Guo and V.Lakshmikantham, *Nonlinear Problem in Abstract Cones*, Academic Press, San Diego, (1998).

[4] J.R.Graef, L.Kong and H.Wang, *A periodic boundary value problem with vanishing Green's function,Applied Mathematics Letters*, 21(2), (2008), 176-180.

[5] J.R.Graef, L.Kong and H.Wang, *Existence, multiplicity, and dependence on a parameter for a periodic boundary value problem*, J. Differential Equations, 245(2008), 1185-1197.

[6] D.Jiang, *On the existence of positive solutions to second order periodic BVPs*, Acta Math. Sci., 18(1998), 31-35.

[7] D.Jiang, J.Chu, D.ORegan and R.Agarwal, *Multiple positive solutions to superlinear periodic boundary value problems with repulsive singular forces*, J. Math. Anal. Appl., 286(2003), 563-576.

[8] D.Jiang, J.Chu and M.Zhang, *Multiplicity of positive periodic solutions to superlinear repulsive singular equations*, J. Differential Equations, 211(2005), 282302.

[9] M.A.Krasnoselskii, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, (1964).

[10] X.Li and Z.Zhang, *Periodic solutions for second-order differential equations with a singular nonlinearity*, Nonlinear Anal., 69(2008), 38663876.

[11] R.Ma and C.Zhong, *Existence of positive solutions for integral equations with vanishing kernels*, Communications in Applied Analysis, 15(2-4)(2011), 529538.

[12] D.ORegan and H.Wang, *Positive periodic solutions of systems of second order ordinary differential equations*, Positivity, 10(2006), 285-298.

[13] P.Torres, *Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem*, J. Differential Equations, 190(2003), 643-662.

[14] J.R.L.Webb, *Boundary value problems with vanishing Greens function*, Commun. Appl. Anal., 13(4)(2009), 587-595.

[15] Z.Zhang and J.Wang, *On existence and multiplicity of positive solutions to periodic boundary value problems for singular nonlinear second order differential equations*, J. Math. Anal. Appl., 281(2003), 99-107.