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Oscillatory Behavior of Fractional Differential Equation with Damping

Research Article

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Abstract: In this paper, we establish some oscillation criteria for a fractional differential equation with damping term using Riccati transformation technique. Here we use the Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$. We give some examples to illustrate our main results.

MSC: 34C10, 34K11, 34A08.

Keywords: Oscillation, Riemann-Liouville fractional derivative, Riccati technique.

1. Introduction

Fractional differential equations are generalizations of classical differential equations of integer order and they have been widely applied in many areas of engineering, physics, mechanics, nonlinear control and so on, See [4, 6–8]. Lot of works have been done on the oscillation of integer order differential equations [1–3]. Recently many articles have discussed the oscillation of fractional differential equations [5, 10, 12]. In [11], Yang et al. have discussed oscillation of the following nonlinear fractional differential equation

$$\left(D_{0+}^{1+\alpha} y\right)(t) + p(t)\left(D_{0+}^{\alpha} y\right)(t) + q(t) f(y(t)) = g(t)$$

for $t > 0$ and $0 < \alpha < 1$. In [9], Tunc et al. considered the oscillation of the following equation

$$\left(D_{0+}^{1+\alpha} y\right)(t) + p(t)\left(D_{0+}^{\alpha} y\right)(t) + q(t) f(G(t)) = 0$$

for $t \geq t_0 > 0$ and $0 < \alpha < 1$. In this paper, we consider the following fractional differential equation with damping term of the form

$$\left[r(t)\psi(x(t)) D_{0+}^{\alpha} x(t)\right]' + p(t)\psi(x(t)) D_{0+}^{\alpha} x(t) + F\left(t, \int_0^t (t-s)^{-\alpha} x(s)ds\right) = 0,\ t \geq t_0 > 0, \tag{1}$$

where $0 < \alpha < 1$, $D_{0+}^{\alpha}(x(t))$ denotes the Riemann-Liouville fractional derivative of order α of $x(t)$ and is defined by

$$D_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} x(s)ds,\ t > 0,$$

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where $\Gamma(.)$ is the gamma function (refer [6]). By a solution of equation (1), we mean a function $x(t) \in C(\mathbb{R}_+, \mathbb{R})$ such that $\int\limits_0^t (t-s)^{-\alpha} x(s) ds \in C^1(\mathbb{R}_+, \mathbb{R})$, $r(t)\psi(x(t)) D_{0+}^{\alpha} x(t) \in C^1(\mathbb{R}_+, \mathbb{R})$ and satisfies (1) on $(0, \infty)$. We restricted to those solutions of equation (1) satisfying $\sup\{|x(t)| : t \geq T\} > 0$ for any $T > 0$. A nontrivial solution of equation (1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. Equation (1) is called oscillatory if all of its solutions are oscillatory. Throughout this paper, the following conditions are assumed to hold:

(H_1) $r(t) \in C^1([t_0, \infty), \mathbb{R}_+)$ such that $r(t) \leq K$ for some $K > 0$.

(H_2) $p(t) \in C([t_0, \infty), \mathbb{R})$ such that $p(t) < 0$.

(H_3) $\psi \in C(\mathbb{R}, \mathbb{R})$, $0 < \psi(x) \leq m$ for some positive constant m and for all $x \neq 0$.

(H_4) $F(t, G) \in C^1([t_0, \infty) \times \mathbb{R}, \mathbb{R}_+)$ such that $\frac{F(t,G)}{G} \geq q(t)$, where $q(t) \in C^1([t_0, \infty), \mathbb{R}_+)$, for $G \neq 0$ and $t \geq t_0$.

In this paper, we establish some oscillation criteria with examples.

2. Main Results

We need the following lemma to prove our main results.

Lemma 2.1 ([9]). *Let $x(t)$ be a solution of equation (1) and*

$$G(t) = \int_0^t (t-s)^{-\alpha} x(s) ds, \tag{2}$$

then

$$G'(t) = \Gamma(1-\alpha) D_{0+}^{\alpha} x(t). \tag{3}$$

Theorem 2.2. *If*

$$t \overset{lim}{\to} \infty \left\{ \frac{\Gamma(1-\alpha)}{M} \int_{t_0}^t \frac{ds}{r(s)} \right\} = \infty \tag{4}$$

and

$$t \overset{lim}{\to} \infty \int_{t_0}^t \left[q(s) - \frac{M p^2(s)}{4\Gamma(1-\alpha) r(s)} \right] ds = \infty, \tag{5}$$

then every solution of equation (1) is oscillatory.

Proof. Suppose that equation (1) has a nonoscillatory solution $x(t)$ on $[t_0, \infty)$. Without loss of generality, we assume that $x(t)$ is an eventually positive solution of equation (1). Then $x(t) > 0$ and $G(t) > 0$ on $[t_1, \infty)$ for $t_1 > t_0$. Define

$$w(t) = -\frac{r(t)\psi(x(t)) D_{0+}^{\alpha} x(t)}{G(t)}$$

for $t \geq t_1$. Then we get

$$\begin{aligned} w'(t) &= -\left[\frac{(r(t)\psi(x(t)) D_{0+}^{\alpha} x(t))' \, G(t) - (r(t)\psi(x(t)) D_{0+}^{\alpha} x(t)) \, G'(t)}{G^2(t)} \right] \\ &= p(t) \frac{\psi(x(t)) D_{0+}^{\alpha} x(t)}{G(t)} + \frac{F(t,G)}{G(t)} + \frac{\Gamma(1-\alpha) r(t) \psi(x(t)) (D_{0+}^{\alpha} x(t))^2}{G^2(t)} \\ &\geq -p(t) w(t) + q(t) + \frac{\Gamma(1-\alpha)}{M r(t)} w^2(t). \end{aligned} \tag{6}$$

Integrating both sides from t_2 to t, we have

$$\begin{aligned} w(t) &\geq w(t_1) + \int_{t_1}^{t} \left[\Gamma(1-\alpha)\frac{w^2(s)}{Mr(s)} - p(s)w(s) + q(s)\right] ds \\ &= w(t_1) + \Gamma(1-\alpha)\int_{t_1}^{t} \left(\frac{w(s)}{(Mr(s))^{1/2}} - \frac{M^{1/2}p(s)}{2r^{1/2}\Gamma(1-\alpha)}\right)^2 ds + \int_{t_0}^{t} \left[q(s) - \frac{Mp^2(s)}{4\Gamma(1-\alpha)r(s)}\right] ds. \end{aligned} \tag{7}$$

By equation (5), there exists $t_2 \geq t_1$, such that

$$w(t) > \Gamma(1-\alpha)\left(\frac{w(s)}{(Mr(s))^{1/2}} - \frac{M^{1/2}p(s)}{2r^{1/2}\Gamma(1-\alpha)}\right)^2 ds \text{ for } t \geq t_2.$$

Set

$$L_1(t) = \Gamma(1-\alpha)\left(\frac{w(s)}{(Mr(s))^{1/2}} - \frac{M^{1/2}p(s)}{2r^{1/2}\Gamma(1-\alpha)}\right)^2 ds. \tag{8}$$

Then $w(t) > L_1(t) > 0$ for $t \geq t_2$. From (H_2) and equation (8), we get

$$\begin{aligned} L_1'(t) &= \Gamma(1-\alpha)\left(\frac{w(s)}{(Mr(s))^{1/2}} - \frac{M^{1/2}p(s)}{2r^{1/2}\Gamma(1-\alpha)}\right)^2 \\ &> \Gamma(1-\alpha)\frac{w^2(t)}{Mr(t)} > \Gamma(1-\alpha)\frac{L_1^2(t)}{Mr(t)}. \end{aligned}$$

That is,

$$\frac{\Gamma(1-\alpha)}{Mr(t)} < \frac{L_1'(t)}{L_1^2(t)} \text{ for } t \geq t_2.$$

Integrating both sides from t_2 to t, and letting $t \to \infty$, we have

$$\frac{\Gamma(1-\alpha)}{M}\int_{t_2}^{t}\frac{ds}{r(s)} < \frac{1}{L_1(t_2)} - \frac{1}{L_1(t)} < \frac{1}{L_1(t_2)}.$$

So

$$t \overset{lim}{\to} \infty \left\{\frac{\Gamma(1-\alpha)}{M}\int_{t_2}^{t}\frac{ds}{r(s)}\right\} < \frac{1}{L_1(t_2)},$$

which is a contradiction to equation (4). Hence the proof is complete. □

Theorem 2.3. *Assume that there exist a positive function $g \in C^1([t_0,\infty)$ such that*

$$t \overset{lim}{\to} \infty \left\{\left(\frac{\Gamma(1-\alpha)}{MK}\right)^{1/2}\int_{t_0}^{t}\frac{1}{g(s)}ds\right\} = \infty \tag{9}$$

and

$$t \overset{lim}{\to} \infty \left\{\frac{MK}{4\Gamma(1-\alpha)}\int_{t_0}^{t}\left\{p^2(s)g(s) + \frac{(g'(s))^2}{g(s)} - 2p(s)g'(s) - \frac{4\Gamma(1-\alpha)}{MK}g(s)q(s)\right\}ds + \frac{1}{2}\frac{Mr(t)}{\Gamma(1-\alpha)}g'(t)\right\} = \infty, \tag{10}$$

then every solution of equation (1) *is oscillatory.*

Proof. Suppose that equation (1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)$. Without loss of generality, we assume that $x(t)$ is an eventually positive solution of equation (1). Then $x(t) > 0$ and $G(t) > 0$ on $[t_1,\infty)$ for $t_1 > t_0$. Define

$$w(t) = -g(t)\frac{r(t)\psi(x(t))D_{0+}^{\alpha}x(t)}{G(t)}$$

for $t \geq t_1$. Then from (H_3) and (H_4), we obtain

$$\begin{aligned} w'(t) &= -g'(t)\frac{r(t)\psi(x(t))D_{0+}^{\alpha}x(t)}{G(t)} - g(t)\left[\frac{r(t)\psi(x(t))D_{0+}^{\alpha}x(t)}{G(t)}\right]' \\ &= \frac{g'(t)}{g(t)}w(t) + g(t)p(t)\frac{r(t)\psi(x(t))D_{0+}^{\alpha}x(t)}{G(t)} + g(t)\frac{F(t,G)}{G(t)} + \Gamma(1-\alpha)g(t)\frac{w^2(t)}{r(t)\psi(x(t))} \\ &\geq \frac{1}{g(t)}\left[\frac{\Gamma(1-\alpha)}{Mr(t)}w^2(t) - g(t)p(t)w(t) + g'(t)w(t)\right] + g(t)q(t). \end{aligned} \tag{11}$$

Set

$$L_2(t) = \left(\frac{\Gamma(1-\alpha)}{Mr(t)}\right)^{1/2} w(t) + \frac{1}{2}\left(\frac{Mr(t)}{\Gamma(1-\alpha)}\right)^{1/2} g'(t). \tag{12}$$

From equations (11) and (12), we get

$$\begin{aligned} w'(t) \geq\ & \frac{1}{g(t)}\Bigg\{\left(L_2(t) - \frac{1}{2}\left(\frac{Mr(t)}{\Gamma(1-\alpha)}\right)^{1/2} p(t)g(t)\right)^2 - \left(\frac{1}{2}\left(\frac{Mr(t)}{\Gamma(1-\alpha)}\right)^{1/2} p(t)g(t)\right)^2 - \frac{Mr(t)}{4\Gamma(1-\alpha)}(g'(t))^2 \\ & + \frac{Mr(t)}{2\Gamma(1-\alpha)}p(t)g(t)g'(t)\Bigg\} + g(t)q(t) \\ \geq\ & \frac{1}{g(t)}\Bigg\{\left(L_2(t) - \frac{1}{2}\left(\frac{Mr(t)}{\Gamma(1-\alpha)}\right)^{1/2} p(t)g(t)\right)^2 - \frac{Mr(t)}{4\Gamma(1-\alpha)}\Bigg\{p^2(t)g(t) + \frac{(g'(t))^2}{g(t)} - 2p(t)g'(t) - \frac{4\Gamma(1-\alpha)}{Mr(t)}g(t)q(t)\Bigg\}. \end{aligned}$$

Integrating both sides from t_1 to t, we obtain

$$\begin{aligned} w(t) \geq\ & w(t_1) + \int_{t_1}^{t} \frac{1}{g(s)}\left(L_2(s) - \frac{1}{2}\left(\frac{Mr(s)}{\Gamma(1-\alpha)}\right)^{1/2} p(s)g(s)\right)^2 ds \\ & - \frac{M}{4\Gamma(1-\alpha)}\int_{t_1}^{t} r(s)\Bigg\{p^2(s)g(s) + \frac{(g'(s))^2}{g(s)} - 2p(s)g'(s) - \frac{4\Gamma(1-\alpha)}{Mr(s)}g(s)q(s)\Bigg\}ds. \end{aligned} \tag{13}$$

From equations (12) and (13), we have

$$\begin{aligned} L_2(t) \geq\ & \left(\frac{\Gamma(1-\alpha)}{MK}\right)^{1/2} w(t_1) + \left(\frac{\Gamma(1-\alpha)}{MK}\right)^{1/2}\int_{t_1}^{t} \frac{1}{g(s)}\left(L_2(s) - \frac{1}{2}\left(\frac{MK}{\Gamma(1-\alpha)}\right)^{1/2} p(s)g(s)\right)^2 ds \\ & - \frac{MK}{4\Gamma(1-\alpha)}\int_{t_1}^{t}\Bigg\{p^2(s)g(s) + \frac{(g'(s))^2}{g(s)} - 2p(s)g'(s) - \frac{4\Gamma(1-\alpha)}{MK}g(s)q(s)\Bigg\}ds + \frac{1}{2}\left(\frac{Mr(t)}{\Gamma(1-\alpha)}\right)^{1/2} g'(t). \end{aligned} \tag{14}$$

By (10) and (H_1), there exists $t_2 \geq t_1$, such that

$$L_2(t) > \left(\frac{\Gamma(1-\alpha)}{MK}\right)^{1/2}\int_{t_2}^{t} \frac{1}{g(s)}\left(L_2(s) - \frac{1}{2}\left(\frac{MK}{\Gamma(1-\alpha)}\right)^{1/2} p(s)g(s)\right)^2 ds.$$

Let

$$Q(t) = \left(\frac{\Gamma(1-\alpha)}{MK}\right)^{1/2}\int_{t_2}^{t} \frac{1}{g(s)}\left(L_2(s) - \frac{1}{2}\left(\frac{MK}{\Gamma(1-\alpha)}\right)^{1/2} p(s)g(s)\right)^2 ds. \tag{15}$$

From (H_2), we have $L_2(t) > Q(t) > 0$. Then

$$\begin{aligned} Q'(t) &\geq \left(\frac{\Gamma(1-\alpha)}{MK}\right)^{1/2}\frac{1}{g(t)}\left(Q(t)(t) - \frac{1}{2}\left(\frac{MK}{\Gamma(1-\alpha)}\right)^{1/2} p(t)g(t)\right)^2 \\ &> \left(\frac{\Gamma(1-\alpha)}{MK}\right)^{1/2}\frac{1}{g(t)}Q^2(t). \end{aligned} \tag{16}$$

That is,

$$\left(\frac{\Gamma(1-\alpha)}{MK}\right)^{1/2}\frac{1}{g(t)} < \frac{Q'(t)}{Q^2(t)}.$$

Integrating both sides from t_2 to t, we obtain

$$\left(\frac{\Gamma(1-\alpha)}{MK}\right)^{1/2}\int_{t_2}^{t}\frac{1}{g(s)} < \frac{1}{Q(t_2)} - \frac{1}{Q(t)} < \frac{1}{Q(t_2)}.$$

Letting $t \to \infty$, we get a contradiction to equation (9). Hence the proof is complete. $\square$

3. Examples

In this section, we give two examples to illustrate our main results.

Example 3.1. *Consider the fractional differential equation*

$$\left[e^{-x^2(t)}D^{1/2}_{0+}x(t)\right]' - \frac{2e^{-x^2(t)}}{t^3}D^{1/2}_{0+}x(t) + \left(2+\frac{1}{t^2}\right)\int_0^t (t-s)^{-\alpha}x(s)ds = 0 \tag{17}$$

for $t \geq 1$. *Here* $\alpha = 1/2,\ r(t) = 1,\ \psi(x(t)) = e^{-x^2} \leq 1 = M,\ p(t) = -\frac{2}{t^3},\ F(t,G) = \left(2+\frac{1}{t^2}\right)\int\limits_0^t (t-s)^{-\alpha}x(s)ds,\ q(t) = 2+\frac{1}{t^2}$. *Then*

$$t \overset{lim}{\to} \infty \left\{\frac{\Gamma(1-\alpha)}{M}\int_{t_0}^t \frac{ds}{r(s)}\right\} = t \overset{lim}{\to} \infty \left\{\sqrt{\pi}\int_1^t ds\right\} = \infty$$

and

$$\begin{aligned} t \overset{lim}{\to} \infty \int_{t_0}^t \left[q(s) - \frac{Mp^2(s)}{4\Gamma(1-\alpha)r(s)}\right] ds \quad &= t \overset{lim}{\to} \infty \int_1^t \left[\left(2+\frac{1}{s^2}\right) - \frac{1}{2\sqrt{\pi}s^2}\right] ds \\ &= t \overset{lim}{\to} \infty \left[t\left(2 - \frac{1}{t^2} + \frac{1}{2\sqrt{\pi}t^3}\right) - \left(1+\frac{1}{2\sqrt{\pi}}\right)\right] = \infty. \end{aligned}$$

Thus by Theorem 2.1, equation (17) *is oscillatory.*

Example 3.2. *Consider the fractional differential equation*

$$\left[\frac{1}{2+x^2(t)}D^{1/2}_{0+}x(t)\right]' - \frac{1}{3t(1+x^2(t))}D^{1/2}_{0+}x(t) + \left(5t + \exp^{\int_0^t (t-s)^{-\alpha}x(s)ds}\right)\int_0^t (t-s)^{-\alpha}x(s)ds = 0 \tag{18}$$

for $t \geq 1$. *Here* $\alpha = 1/2,\ r(t) = 1 = K,\ \psi(x(t)) = \frac{1}{2+x^2} \leq \frac{1}{2} = M,\ p(t) = -\frac{2}{3t}$,

$$\frac{F(t,G)}{G(t)} = 5t + \exp^{\int_0^t (t-s)^{-\alpha}x(s)ds} \geq 5t = q(t).$$

If we take $g(t) = 2t$, *then it is easy to verify that*

$$t \overset{lim}{\to} \infty \left\{\frac{\Gamma(1-\alpha)}{MK}\int_{t_0}^t \frac{ds}{g(s)}\right\} = t \overset{lim}{\to} \infty \left\{\sqrt{2}\pi\int_1^t \frac{1}{2s}ds\right\} = \infty$$

and

$$\begin{aligned} &t \overset{lim}{\to} \infty \left\{\frac{MK}{4\Gamma(1-\alpha)}\int_{t_0}^t \left\{p^2(s)g(s) + \frac{(g'(s))^2}{g(s)} - 2p(s)g'(s) - \frac{4\Gamma(1-\alpha)}{MK}g(s)q(s)\right\}ds + \frac{1}{2}\frac{Mr(t)}{\Gamma(1-\alpha)}g'(t)\right\} \\ &= t \overset{lim}{\to} \infty \left\{\frac{1}{2\sqrt{\pi}}\left[\frac{1}{4}\int_1^t \left(\frac{32}{9s} - 80\sqrt{\pi}s^2\right)ds + 1\right]\right\} = \infty \end{aligned}$$

Thus by Theorem 2.2, equation (18) *is oscillatory.*

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