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An Epidemic Model with Immigration and Non-Monotonic Incidence Rate Under Treatment

Research Article

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Abstract: The present mathematical model deals with the study of SIRS epidemic model with immigration and under treatment function type. We start from formulation of model and analyze it. The equations depend on treatment function system. The system is established. If $R_0 < 1$ the DFE (Disease Free Equilibrium) is globally stable and if $R_0 > 1$ then the endemic equilibrium is obtained which is globally stable. An example also provides to justify the stability.

MSC: 92D30, 92D25, 34D20.

Keywords: Mathematical Model, Equilibrium Point, Stability Analysis, Saturated incidence.

1. Introduction

Mathematical models have become important tools to study and analyze the spread and control of infectious diseases. Recentlyl Badshah and Kumar talk [3], established a primary result of mathematical modeling. Most of the proposed mathematical models, those describe the transmission of infectious disease, have been derived from the classical susceptible infective recover SIR model, which is suggested originally by Kermack and Mckendrick [9], who also gave the result on simple mass action. In that model the susceptible individuals and then the infected individuals may recover and transfer to removal individuals at a specific rate. Number of mathematical models was developed to study and analyzed the spread of infectious diseases in order to prevent or minimize the transmission of them through quarantine and other measures. The incidence in an epidemiological model is the rate at which the susceptible become infectious. Cappaso and Serio [4] introduced a saturated incidence rate into epidemic model. Mena Lorca and Hethcote [13] also analyzed an SIRS model deterministic with the same saturation incidence. Ruan and Wang [19] studied an epidemic model with a specific nonlinear incident rate, Liu et al. [11, 12], Derrick and Ven den Driessche [5]; Hethcote and Ven Den Driessche [7] proposed a various epidemic models with non-monotonic incidence rate. Further in 2007 Xiao and Raun discussed non-monotonic incidence rate. Several different incidence rates have been proposed by many researchers see, for instance Anderson and May [1], Elteva and Matias [6], Hethcote and Driesech [7], Ruan and Wang [19], Liu et al. [11, 12] Derrick and Ven den Driessche [5], Alexander and Moghadas [2], Xiao and Raun. Recently Porwal, et al. [15–18] also presented their work in concerning field.

Mathematical epidemiology one of the oldest and richest areas in mathematical biology, has significantly enhanced our understanding of how pathogens emerge, evolve, and spread. Classical epidemiological models, the standard for predicting

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and managing the spread of infectious disease, assume that contacts between susceptible and infectious individuals depend on their relative frequency in the population. The behavioral factors that underpin contact rates are not generally addressed. There is, however, an emerging a class of models that addresses the feedbacks between infectious disease dynamics and the behavioral decisions driving host contact. Referred to as "Mathematical epidemiology," the approach explores the determinants of decisions about the number and type of contacts made by individuals, using insights and methods from mathematics. We show how the approach has the potential both to improve predictions of the course of infectious disease, and how to support mathematically model approaches to infectious disease.

In this paper, we investigate an Epidemic Model with Immigration and Non-Monotonic Incidence rate under Treatment, and modify the model of Kar and Batabyal [8] by considering the immigration rate and disease induced death rate. We present modified mathematical model, analyze the model and obtain disease free and endemic equilibrium point and also analyze for stability analysis and simulation result. Further we also give an example for verification of our results.

2. The Mathematical Model

2.1. Basic Model

Kar and Batayal [8] proposed an epidemic model with non-monotonic incidence rate under the treatment function by differential equations

$$\frac{dS}{dt} = a - dS - \frac{\lambda IS}{1+\alpha I^2} + \beta R, \tag{1}$$

$$\frac{dI}{dt} = \frac{\lambda IS}{1+\alpha I^2} - (d+m)I - T(I), \tag{2}$$

$$\frac{dR}{dt} = mI - (d+\beta)R + T(I), \tag{3}$$

Where

$S(t)$ = Number of Susceptibles,

$I(t)$ = Number of Infectives,

$R(t)$ = Number of Recovered Individuals,

a = Requirement Rate of Population,

d = Natural Death Rate of Population,

λ = Pr oportionality Constan t,

m = Natural Recovery Rate of the Infective Individuals,

β = The Rate at which Recovered Individuals loss Individuals. Loss Immunity and Return to Susceptible Class,

α = Parameter Measures of the Psychological or Inhibitory effect, and

$$T(I) = rI, \ \text{if } 0 \le I \le I_0,$$

$$= K_1, \ \text{if } I > I_0.$$

2.2. Model for Treatment Function with Immigration

The model (2) with immigration and disease induced death is given by:

$$\frac{dS}{dt} = a - dS - \frac{\lambda IS}{1+\alpha I^2} + \beta R + \mu, \tag{4}$$

$$\frac{dI}{dt} = \frac{\lambda IS}{1+\alpha I^2} - (d+m)I - T(I), \tag{5}$$

$$\frac{dR}{dt} = mI - (d+\beta)R + T(I), \tag{6}$$

$$T(I) = rI, \quad \text{if } 0 \le I \le I_0, \tag{7}$$

$$= K_1, \quad \text{if } I > I_0. \tag{8}$$

The parameters in (2) have same meanings as in the model (1).

Part I: SIR Model with $0 \le I \le I_0$. In this case the system (4)-(6) reduce to

$$\frac{dS}{dt} = a - dS - \frac{\lambda IS}{1+\alpha I^2} + \beta R + \mu, \tag{9}$$

$$\frac{dI}{dt} = \frac{\lambda IS}{1+\alpha I^2} - (d+m+r)I, \tag{10}$$

$$\frac{dR}{dt} = (m+r)I - (d+\beta)R. \tag{11}$$

The system of equations (9) to (11) always has the DFE $E_0\left(\frac{(\mu+a)}{d}, 0, 0\right)$ for any set of parameter values. Therefore by equation (10), we have

$$\frac{\lambda IS}{1+\alpha I^2} - (d+m+r)I = 0,$$

$$I\left(\frac{\lambda S}{1+\alpha I^2} - (d+m+r)\right) = 0,$$

$$I = 0.$$

Now by equation (11)

$$(m+r)I - (d+\beta)R = 0$$

$$R = 0 \qquad \{\because I = 0\}.$$

Now finally equation (9)

$$a - dS - \frac{\lambda IS}{1+\alpha I^2} + \beta R + \mu = 0,$$

$$S = \mu + \frac{a}{d}$$

the endemic equilibrium is the solution of the system of equations. Thus

$$a - dS - \frac{\lambda IS}{1+\alpha I^2} + \beta R + \mu = 0 \tag{12}$$

$$\frac{\lambda IS}{1+\alpha I^2} - (d+m+r)\,I = 0 \tag{13}$$

and

$$(m+r)\,I - (d+\beta)\,R = 0 \tag{14}$$

From equation (14)

$$(m+r)\,I - (d+\beta)\,R = 0$$

$$R = \frac{(m+r)\,I}{(d+\beta)}.$$

Equation (13)

$$\frac{\lambda IS}{1+\alpha I^2}-(d+m+r)\,I=0$$
$$S=\frac{(d+m+r)\left(1+\alpha I^2\right)}{\lambda}.$$

Now substituting R and S in the equation (12) we get

$$a-d\left[\frac{(d+m+r)\left(1+\alpha I^2\right)}{\lambda}\right]-\frac{\lambda I}{1+\alpha I^2}\left[\frac{(d+m+r)\left(1+\alpha I^2\right)}{\lambda}\right]+\beta\frac{(m+r)}{(d+\beta)}I+\mu=0,$$
$$a\lambda-d\,(d+m+r)\left(1+\alpha I^2\right)-\frac{\lambda I}{1+\alpha I^2}(d+m+r)\left(1+\alpha I^2\right)+\frac{\beta I\,(m+r)\,\lambda}{(d+\beta)}+\mu\lambda=0,$$
$$\alpha d\,(d+m+r)\,I^2+\lambda I\left[(d+m+r)-\frac{\beta\,(m+r)}{(d+\beta)}\right]+d\,(d+m+r)-a\lambda-\mu\lambda=0. \tag{15}$$

We define the basic reproduction number as

$$R_0=\frac{\lambda(\mu+a)}{d\,(d+m+r)}. \tag{16}$$

From the equation (15) we see that if $R_0\le 1$ there is no positive solution as in that case coefficient of I^2, I and constant terms are all positive but if$R_0>1$. Then by Descartes rule there exists a unique positive solution of (15) and consequently there exists a unique equilibrium $E^*\left(S^*,I^*,R^*\right)$ called endemic equilibrium. Here

$$R^*=\left\{\frac{(m+r)}{(d+\beta)}\right\},\quad S^*=\frac{(d+m+r)\left(1+\alpha I^2\right)}{\lambda}$$

with equation (15) give

$$\alpha d\,(d+m+r)\,I^2+\lambda\left\{(d+m+r)-\frac{\beta\,(m+r)}{(d+\beta)}\right\}I+d\,(d+m+r)-a\lambda-\mu\lambda=0,$$

$$I^*=\frac{-\lambda\left\{(d+m+r)-\frac{\beta(m+r)}{(d+\beta)}\right\}+\sqrt{\lambda^2\left\{(d+m+r)-\frac{\beta(m+r)}{(d+\beta)}\right\}^2-4\alpha d\,(d+m+r)\left[d\,(d+m+r)-\lambda\,(\mu+a)\right]}}{2\alpha d\,(d+m+r)}$$
$$I^*=\frac{\left[-\lambda\left\{(d+m+r)-\frac{\beta(m+r)}{(d+\beta)}\right\}+\sqrt{\Delta_1}\right]}{2\alpha d\,(d+m+r)} \tag{17}$$

where $\Delta_1=\lambda^2\left\{(d+m+r)-\frac{\beta(m+r)}{(d+\beta)}\right\}^2-4\alpha d^2\,(d+m+r)^2\,[1-R_0]$

$$\left\{\begin{array}{l}\because R_0=\frac{\lambda(\mu+a)}{d(d+m+r)}\\ \lambda\,(\mu+a)=R_0 d\,(d+m+R)\end{array}\right\}$$

Obviously $\Delta_1>0$,when $R_0>1$. To investigate the stability of the system, we prove that $S\,(t)+I\,(t)+R\,(t)=\frac{\mu+a}{d}$ is invariant manifold of system (9) to (11) which is attracting in the first octant. Let $N(t)=S\,(t)+I\,(t)+R\,(t)$, then

$$\frac{dN(t)}{dt}=\frac{dS(t)}{dt}+\frac{d}{dt}I(t)+\frac{d}{dt}R(t)$$
$$\frac{d}{dt}N\,(t)=a-dS-dI-dR+\mu,$$
$$\frac{dN}{dt}=\mu+a-dN(t),$$
$$\frac{dN}{dt}+dN(t)=\mu+a.$$

Integrating Factor (I.F.) $= e^{\int \mathrm{d.dt}} = e^{dt}$. Solution is given by

$$N(t)e^{dt} = \int \left(e^{dt}(\mu + a)\right) dt + A_1,$$

$$N(t) = \frac{\mu + a}{d} + A_1 e^{-dt}, \tag{18}$$

where

$$N(t_0) = A_1 e^{-dt_0} + \frac{\mu + a}{d},$$

$$N(t_0) - \frac{\mu + a}{d} = A_1 e^{-dt_0},$$

$$A_1 = \left[N(t_0) - \frac{\mu + a}{d}\right] e^{dt_0},$$

Putting this value in (18), we get

$$N(t) = \frac{\mu + a}{d} + \frac{1}{e^{dt}}\left[N(t_0) - \frac{\mu + a}{d}\right] e^{dt_0}$$

$$= \frac{\mu + a}{d} + \left[N(t_0) - \frac{\mu + a}{d}\right] e^{-d(t-t_0)}.$$

Thus $N(t) \to \frac{\mu+a}{d}, ast \to \infty$. So the limit set of system (9) and (11) is a plane $S + I + R = \frac{\mu+a}{d}$. Thus the reduced system is

$$\frac{dI}{dt} = \frac{\lambda I\left(\frac{\mu+a}{d} - I - R\right)}{1 + \alpha I^2} - (d + m + r)I = F_1(I, R), \quad \left\{\begin{array}{l} \because S + I + R = \frac{\mu+a}{d}, \\ S = \frac{\mu+a}{d} - I - R \end{array}\right\} \tag{19}$$

$$\frac{dR}{dt} = (m + r)I - (d + \beta) = F_2(I, R). \tag{20}$$

Now to test the local stability of the above system, we rescale the system by

$$x = \frac{\lambda I}{(d + \beta)}, \quad y = \frac{\lambda R}{(d + \beta)}, \quad T = (d + \beta)t,$$

and obtain

$$x = \frac{\lambda I}{(d + \beta)},$$

$$\frac{dx}{dt} = \frac{\lambda}{(d + \beta)} \cdot \frac{dI}{dT},$$

$$= \frac{\lambda}{(d + \beta)} \cdot \left[\frac{\lambda I\left(\frac{\mu+a}{d} - I - R\right)}{1 + \alpha I^2} - (d + m + r)I\right] \times \frac{1}{(d + \beta)},$$

$$= \left[\frac{\lambda I}{(d + \beta)(d + \beta)} \cdot \frac{\left(\frac{\lambda(\mu+a)}{d} - \lambda I - \lambda R\right)}{1 + \alpha I^2} - \frac{\lambda I}{(d + \beta)(d + \beta)} \cdot (d + m + r)\right],$$

$$= \frac{\lambda I}{(d + \beta)}\left[\frac{\lambda(\mu + a)}{(d + \beta)} - \frac{\lambda I}{(d + \beta)} - \frac{\lambda R}{(d + \beta)}\right] \times \frac{1}{1 + \alpha I^2} - \frac{\lambda I}{(d + \beta)} \cdot \frac{(d + m + r)}{(d + \beta)},$$

Hence

$$\frac{dx}{dt} = \frac{x[K - x - y]}{1 + vx^2} - ux, \tag{21}$$

Now

$$\frac{dy}{dt} = \frac{\lambda}{(d+\beta)} \cdot \frac{dR}{dT}, \\ = \frac{(m+r)}{(d+\beta)} \cdot \frac{\lambda I}{(d+\beta)} - \frac{\lambda R}{(d+\beta)}.$$

Hence

$$\frac{dy}{dt} = wx - y, \tag{22}$$

where

$$w = \frac{m+r}{(d+\beta)}, \quad K = \frac{(\mu+a)\lambda}{d(d+\beta)}, \quad u = \frac{d+m+r}{(d+\beta)}, \quad v = \frac{\alpha (d+\beta)^2}{\lambda^2}.$$

Here $E_0(0,0)$ is the DFE and unique equilibrium $(x*, y*)$ of the system (21) to (22) is the endemic equilibrium E^* of the model (9) to (11), $(x*, y*)$ exists if $u - K < 0$, and is given by

$$wx^* - y^* = 0, \quad \text{(by Equation (22))} \\ y^* = wx^*.$$

Now by equation (21)

$$\frac{x^*(K - x^* - y^*)}{1 + vx^{*2}} - ux^* = 0, \\ x^*\left[K - x^* - y^* - u - uvx^{*2}\right] = 0, \\ uvx^{*2} + (1+w)x^* + (u-K) = 0. \quad \because y^* = wx^*$$

Therefore

$$x* = \frac{-(1+w) + \sqrt{(1+w)^2 - 4uv(u-K)}}{2uv},$$

and

$$y^* = wx^* \tag{23}$$

The Jacobian matrix corresponding at $E_0(0,0)$ is $M_0 = \begin{bmatrix} K-u & 0 \\ w & -1 \end{bmatrix}$, (by differentiating equation (21) and (22) w.r.t..x and y and putting $x = 0$, $y = 0$). Obviously, if

(1). $K - u > 0, (0,0)$ is an unstable saddle point.

(2). $K = u, (0,0)$ is saddle node.

(3). $(K - u) < 0, (0,0)$ is a stable node.

Now when $(K - u) > 0$ i.e. $R_0 > 1$, we discuss the stability of endemic equilibrium (x^*, y^*). Jacobian matrix corresponding to (x^*, y^*) is

$$M_1 = \begin{bmatrix} \frac{x^*\left(vx^{*2} + 2vwx^{*2} - 2Kvx^* - 1\right)}{\left(1+vx^{*2}\right)^2} & \frac{-x^*}{1+vx^{*2}} \\ w & -1 \end{bmatrix},$$

(by differential equation (17) and (18) with respect to x and y). The sign of

$$\det(M_1) = \frac{x*\left\{1 + w + 2Kvx* - v(1+w)x*^2\right\}}{(1+vx*^2)^2}$$

is determined by the sign o

$$P_1 = -v(1+w)x^{*2} + 2Kvx^* + (1+w). \tag{24}$$

Also we have

$$uvx^{*2} + (1+w)x^* + (u-K) = 0 \tag{25}$$

Now multiplying (24) by (u) and (25) by $(1+w)$ and adding, we derive

$$uP_1 = \{uv(1+w) - uv(1+w)\}x^{*2} + \left\{2Kuv + (1+w)^2\right\}x^* + u(1+w) + (u-K)(1+w)$$
$$uP_1 = \left\{2Kuv + (1+w)^2\right\}x^* + (1+w) + \{u+u-K\},$$
$$uP_1 = \left\{2Kuv + (1+w)^2\right\}x^* + (1+w) + \{2u-K\},$$

Now by (25)

$$x^* = \left\{\frac{-(1+w)+\Delta_1}{2uv}\right\},$$

where

$$\Delta_1 = \sqrt{(1+w)^2 - 4uv(u-K)}.$$

Therefore

$$\begin{aligned} uP_1 &= \left\{2kuv + (1+w)^2\right\}\left\{-\frac{(1+w)+\Delta_1}{2uv}\right\} + (1+w)(2u-K) \\ &= \frac{-(1+w)\left(2Kuv + (1+w)^2\right)}{2uv} + \frac{\Delta_1\left(2Kuv + (1+w)^2\right)}{2uv} + (1+w)(2u-K) \\ &= \frac{-(1+w)}{2uv}\left\{(1+w)^2 + 2Kuv - 4u^2v + 2uvK\right\} + \frac{\Delta_1\left\{2Kuv + (1+w)^2\right\}}{2uv} \\ &= \frac{1}{2uv}\left[-(1+w)\Delta_1^2 + \Delta_1\left\{2Kuv + (1+w)^2\right\}\right]. \end{aligned}$$

Hence

$$\begin{aligned} P_1 &= \frac{-\Delta_1}{2u^2v}\left[(1+w)\Delta_1 - \left\{2Kuv + (1+w)^2\right\}\right] \\ &= \left\{(1+w) + \frac{2Kuv}{1+w}\right\}^2 - \Delta_1^2 \\ &= (1+w)^2 + \frac{4K^2u^2v^2}{(1+w)^2} - (1+w)^2 + 4u^2v - 4uvK, \\ &= \frac{4K^2u^2v^2}{(1+w)^2} + 4u^2v > 0. \end{aligned}$$

Therefore $P_1 > 0$.

$$P_2 = -v^2x^{*4} + (1+2w)vx^{*3} - 2(1+K)vx^{*2} - x^* - 1, \tag{26}$$
$$u^3vP_2 = -u^3v \cdot v^2x^{*4} + (1+2w)vx^{*3}u^3v - 2(1+K)vx^{*2}u^3v - x^*u^3v - u^3v,$$

Therefore

$$u^3vP_2 = -u^3v^3x^{*4} + (1+2w)u^3v^2x^{*3} - 2(1+K)u^3v^2x^{*2} - x^*u^3v - u^3v, \tag{27}$$

Now since

$$uvx^{*2} + (1+w)x^* + (u-K) = 0,$$

Therefore,

$$uvx^{*2} = -(1+w)x^* - (u-K).$$

Thus

$$x^{*2} = \frac{-(1+w)x^* - (u-K)}{uv}.$$

Put this value in (27), we get

$$\begin{aligned}
u^3vP_2 &= -u^3v^3\left[\frac{-(1+w)x^* - (u-K)}{uv}\right]^2 + (1+2w)u^3v^2\left[\frac{-(1+w)x^* - (u-K)}{uv}\right]x^* \\
&- 2(1+K)u^3v^2\left[\frac{-(1+w)x^* - (u-K)}{uv}\right] - x^*u^3v - u^3v \\
&= -u^3v^3\left[\frac{(1+w)^2x^{*2} + (u-K)^2 + 2(1+w)x^*(u-K)}{u^2v^2}\right] \\
&+ \frac{(1+2w)u^3v^2\left[-(1+w)x^{*2} - (u-K)x^*\right]}{uv} - 2(1+K)v^3v^2 \\
&\times\left[\frac{-(1+w)x^* - (u-K)}{uv}\right] - u^3vx^* - u^3v^3 \\
&= (1+w)^3 - 2u^2v - 2u^2vw + 2uvK + 2uvKw + (1+2w)(1+w)^2 \\
&- (u-K)u^2v - (u-K)u^2v\cdot 2w + 2u^2v + 2u^2vw + 2u^2vKw + u^2vKw - u^3v]\,x^* \\
&+ (u-K)(1+w)^2 - (u-K)^2uv + (1+w)(1+2w)(u-K) \\
&+ 2u^2v(u-K) + 2Ku^2v(u-K) - u^3v \\
&= \left[(1+w)^3 + 2uvK(1+w) + (1+2w)(1+w)^2 - 2u^2v - u^3v + Ku^2v - 2vwu^3 + 2wu^2vK\right. \\
&+ \left.2u^2v + 2u^2vK + 2u^2vKw - u^3v\right]x^* + (u-K)\left(1+w^2\right) - (u-K)^2uv + (1+w)(1+2w)(u-K) \\
&+ 2u^2v(u-K) + 2Ku^2v(u-K) - u^3v \\
&= \left[(1+w)^3 + 2uvK(1+w) + (1+2w)(1+w)^2 + Ku^2v + 2u^2vKw + 2u^2vK - 2u^3v + 2u^2vKw\right. \\
&- \left.2u^3vw\right]x* + (u-K)(1+w)^2 - uv(u-K)^2 + (1+w)(1+2w)(u-K) + 2u^2v(u-K)(1+K) - u^3v \\
&= \left[(1+w)\left\{(1+w)^2 + 2uvK + (1+2w)(1+w)\right\} + u^2vK(1+2w) + 2u^2v(1+w)(K-u)\right]x^* \\
&- \left[(K-u)\left\{(1+w)^2 + (1+w)(1+2w)u\right\} + +uv(K-u)^2 + 2u(K-u)(1+K) + u^2\right]
\end{aligned}$$

Therefore $u^3vP_2 = P_3x^* - P_4$, where

$$P_3 = (1+w)\left[(1+w)^2 + u(1+w)(1+2w) + 2uvK\right] + u^2vK(1+2w) + 2u^2v(1+w)(K-u).$$

Thus

$$P_4 = (k-u)\left[(1+w)^2 + u(1+w)(1+2w)\right] + uv\left[(K-u)^2 + 2u(K-u)(1+K) + u^2\right].$$

Hence P_3 and P_4 are positive for any set of parameters with $K > u$ so when (x^*, y^*) exists, the condition for the local stability of (x^*, y^*) becomes $x^* < \frac{P_4}{P_3}$.

Theorem 2.1.

(1). When the basic reproduction number $R_0 \leq 1$ there exist no positive equilibrium of system (21)-(22) and in that case the only $DFE\,(0,0)$ is stable node.

(2). When $R_0 > 1$, there exists a unique positive equilibrium of system (21)-(22) and in that case $(I(t), R(t)) \to (0,0)$ is a unstable saddle point. Also the condition for which the unique positive equilibrium will locally stable is $x^* < \frac{P_4}{P_3}$.

Global Stability: To investigate the global stability of the DFE it is sufficient to show that $(I(t), R(t)) \to (0,0)$ from here, it is clear that $S(t) \to \frac{a}{d}$ now from positivity of the solutions, $I(t)$ and $R(t)$ satisfy differential inequality given by

$$\frac{dI}{dt} \leq \left\{\frac{\lambda a}{d} - (d+m+r)\right\} I = \frac{dI}{dt}, \tag{28}$$
$$\frac{dR}{dt} \leq (m+r) I - (d-\beta) R = \frac{dr}{dt}, \tag{29}$$

Here $i(t), r(t)$ are linear and $i(t), r(t) \to (0,0)$ as $t \to \infty$if $\lambda \frac{a}{d} - (d+m+r) < 0$ i.e $R_0 < 1$. Since $I(t) \leq i(t)$ and $R(t) \leq r(t), I(t), R(t) \to (0,0)$ as $t \to \infty$ by simple comparison arguments. Hence disease free equilibrium is globally stable. Now to investigate whether system (19)-(20) admites limit cycle or not, we take Dulac Function $D(I,R) = \frac{1+\alpha I^2}{\lambda I}$. Then

$$\frac{\partial (DF_1)}{\partial I} + \frac{\partial (DF_2)}{\partial R} = -1 - \left\{\frac{2\alpha (d+m+r)}{\lambda}\right\} I - \left\{\frac{(d+\beta)\left(1+\alpha I^2\right)}{\lambda I}\right\} > 0.$$

Hence system (19)-(20) have no limit cycle in the positive quadrant. So we reach the Theorem 2.2.

Part II: SIR model with $I > I_0$.

3. Equilibrium States and Their Stability

In this case the model reduces to

$$\frac{dS}{dt} = a - ds - \frac{\lambda IS}{1+\alpha I^2} + \beta R + \mu, \tag{30}$$
$$\frac{dI}{dt} = \frac{\lambda IS}{1+\alpha I^2} - (d+m) I - K_1 \tag{31}$$
$$\frac{dR}{dt} = mI - (d+\beta) R + K_1, \tag{32}$$

Since $S + I + R = \mu + \frac{a}{d}$ is invariant manifold of the system (30)-(32), the model reduces to

$$\frac{dI}{dt} = \frac{\lambda I\left(\mu + \frac{a}{d} - I - R\right)}{1+\alpha I^2} - (d+m) I - K_1, \tag{33}$$
$$\frac{dR}{dt} = mI - (d+\beta) R + K_1. \tag{34}$$

Substituting,

$$x = \frac{\lambda I}{(d+\beta)},\ y = \frac{\lambda R}{(d+\beta)},\ T = (d+\beta) t,\ x = \frac{\lambda I}{(d+\beta)},$$

$$\begin{aligned}
\frac{dx}{dT} &= \frac{\lambda \frac{dI}{dT}}{(d+\beta)} \\
&= \frac{\lambda}{d+\beta} \cdot \frac{dI}{dt} \times \frac{dt}{dT}, \\
&= \frac{\lambda}{d+\beta} \left\{\frac{\lambda I\left(\mu + \frac{a}{d} - I - R\right)}{1+\alpha I^2} - (d+m) I - K_1\right\} \times \frac{1}{d+\beta}, \\
&= \frac{\lambda I}{(d+\beta)} \left\{\frac{\frac{(\mu+a)\lambda}{d(d+\beta)} - \frac{\lambda I}{(d+\beta)} - \frac{\lambda R}{(d+\beta)}}{\frac{1+\alpha I^2 \times (d+\beta)^2 \lambda^2}{(d+\beta)^2 \lambda^2}}\right\} - \frac{\lambda I (d+m)}{(d+\beta)(d+\beta)} - \frac{\lambda K_1}{(d+\beta)(d+\beta)}
\end{aligned}$$

Thus

$$\frac{dx}{dT} = \frac{x\{L - x - y\}}{1+Vx^2} - u_1 x - c. \tag{35}$$

Again

$$\begin{aligned} y &= \frac{\lambda R}{(d+\beta)}, T = (d+\beta)\, t, \\ y &= \frac{\lambda}{(d+\beta)} \cdot \frac{dR}{dT} \\ &= \frac{\lambda}{(d+\beta)} \left\{ \frac{mI - (d+\beta)\, R + K_1}{(d+\beta)} \right\}, \\ &= \frac{\lambda I}{(d+\beta)} \cdot \frac{m}{(d+\beta)} - \frac{\lambda R\,(d+\beta)}{(d+\beta)\,(d+\beta)} + \frac{\lambda K_1}{(d+\beta)^2} \\ \frac{dy}{dT} &= w_1 x - y + c, \end{aligned} \tag{36}$$

where

$$V_1 = v = \frac{\alpha\,(d+\beta)^2}{\lambda^2},\ C = \frac{\lambda K_1}{(d+\beta)^2},\ L = K = \frac{(\mu+a)\,\lambda}{d\,(d+\beta)},\ w_1 = \frac{m}{d+\beta},\ u_1 = \frac{d+m}{d+\beta},$$

Adding (35) and (36) we get

$$x\,(L - x - y) - u_1 x\left(1 + v_1 x^2\right) - c\left(1 + v_1 x^2\right) + c = 0.$$

Putting $y = w_1 x$, we derive Or

$$u_1 v x^3 + (1 + w_1 + cv)\, x^2 + (c + u_1 - K)\, x + c = 0. \tag{37}$$

If $u_1 + c > K$, (37), has no positive solution, but if $u_1 + c < K$, it has either two positive roots or no positive root. By theory of equation.

$$a_0 x^3 + 3a_1 x^2 + 3a_2 x + a_3 = 0, \tag{38}$$

has all of its roots real. If $G^2 + 4H^3 < 0$, and $H < 0$, where $H = a_0 a_2 - a_1^2$,

$$G = a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3,$$

Comparing equation (37) and (38), we have

$$a_0 = u_1 v,\ a_1 = \frac{(1 + w_1 + cv)}{3},\ a_2 = \frac{(u_1 + c - K)}{3},\ \text{and } a_3 = c,$$

Here

$$\begin{aligned} H &= a_0 a_2 - a_1^2, \\ &= u_1 v \left\{(u_1 + c - K)/3\right\} - \left\{(1 + w_1 + cv)/3\right\}^2 < 0,\ \text{ for } u_1 + c < K. \\ G^2 + 4H^3 &= \left(a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3\right)^2 + 4\left(a_0 a_2 - a_1^2\right)^3, \\ &= a_0^2\left(a_0^2 a_3^2 - 6a_0 a_1 a_2 a_3 + 4a_3 a_1^3 + 4a_0 a_2^3 - 3a_1^2 a_2^2\right), \end{aligned}$$

Therefore, for $G^2 + 4H^3 < 0$,

$$\left(a_0^2 a_3^2 + 4a_3 a_1^3 + 4a_0 a_2^3\right) < \left(3a_1^2 a_2^2 + 6a_0 a_1 a_2 a_3\right). \tag{39}$$

differentiate equations (35)-(36), with respect to x and y, the locally stability of the positive equilibrium (x, y) of the system (30)-(31), we consider the Jacobian matix

$$M_2(x, y) = \begin{bmatrix} \left\{ \frac{\left(1+vx^2\right)(K-x-y-x) - 2vx\left(Kx - x^2 - xy\right) - u_1}{1+vx^2} - u_1 \right\} & \frac{-x}{1+v} \\ w_1 & -1 \end{bmatrix}.$$

Now

$$\det(M_2) = \frac{-\left(1+vx^2\right)(K-2x-y)+2vx\left(Kx-x^2-xy\right)+u_1\left(1+vx^2\right)^2+w_1x-\left(1+vx^2\right)}{\left(1+vx^2\right)^2},$$
$$= \frac{-\left(1+vx^2\right)(K-2x-w_1x-c)+2vx\left(Kx-x^2-w_1x^2-cx\right)+u_1\left(1+vx^2\right)^2+w_1x-\left(1+vx^2\right)}{\left(1+vx^2\right)^2},$$

(for $y = w_1x+c$). Sign of $\det(M_2)$ is determined by

$$P_5 = u_1v^2x^4 + (Kv - vc + 2u_1v)\,x^2 + (2+2w_1)\,x + (c+u_1-K)\,. \tag{40}$$

Now subtracting vx time of (37) from (40), we have

$$P_5 = -v\,(1+w_1+cv)\,x^3 + v\,(2K+u_1-2c)\,x^2 + (2+2w_1-vc)\,x + (c+u_1-K)\,. \tag{41}$$

Again by $(1+w_1+cv)\times(37)+u_1\times(41)$, we derive

$$u_1P_5 = \xi_1x^2 + \xi_2x + \xi_3, \tag{42}$$

where $\xi_1 = \left\{u_1^2v + (1+w_1+cv)^2 + 2u_1v\,(K-c)\right\} > 0$, for $K > c$. Therefore, the sufficient condition for which $P_5 > 0$ is,

$$\xi_2^2 - 4\xi_1\xi_2 \le 0. \tag{43}$$

Now

$$Trace\,(M_2) = \frac{\left\{\left(1+vx^2\right)(K-2x-y)-2vx^2\,(K-x-y)\right\}-(u_1+1)}{\left(1+vx^2\right)^2},$$
$$\frac{\left\{\left(1+vx^2\right)(K-2x-w_1x-c)-2vx^2\,(K-x-w_1x-c)-(u_1+1)\left(1+vx^2\right)^2\right\}}{\left(1+vx^2\right)^2},$$

So the sign of $Trace\,(M_2)$ is determined by

$$P_6 = -\,(u_1+1)\,v^2x^4 + vw_1x^3 + (vc - Kv - 2vu_1 - 2v)\,x^2 - (2+w_1)\,x + (K-c-u_1-1)\,, \tag{44}$$

Now making an appeal to (37) and (44) with same calculations, we have after some algebraic calculation using (32) and (38) we get

$$u_1^2P_6 = -\,(u_1+1)\,u_1^2v^2x^4 + u_1^2vw_1x^3 + (vc-Kv-2vu_1-2v)\,u_1^2x^2 - (2+w_1)\,u_1^2x + (K-c-u_1-1)\,u_1^2$$

and

$$u_1vx^3 + (1+w_1+cv)\,x^2 + (c+u_1-K)\,x + c = 0,$$

i.e. or

$$x^3 = \frac{-\,(1+w_1+cv)\,x^2 - (c+u_1-K)\,x - c}{u_1v}.$$

Therefore

$$u_1^2 P_6 = -\left(u_1+1\right) u_1^2 v^2 \left\{ \frac{-\left(1+w_1+cv\right) x^2 - \left(c+u_1-K\right) x - c}{u_1 v} \right\} x + u_1^2 v w_1 \left\{ \frac{-\left(1+w_1+cv\right) x^2 - \left(c+u_1-K\right) x - c}{u_1 v} \right\}$$
$$+ \left(vc - Kv - 2vu_1 - 2v\right) u_1^2 x^2 - \left(2+w_1\right) u_1^2 x + \left(K-c-u_1-1\right) u_1^2,$$
$$= \left(u_1+1\right) u_1 v \left(1+w_1+cv\right) x^3 + \left(u_1+1\right) u_1 v \left(c+u_1-K\right) x^2 - cx\left(u_1+1\right) u_1 v$$
$$- u_1 w_1 \left(1+w_1+cv\right) x^2 - u_1 w_1 \left(c+u_1-K\right) x - cu_1 w_1 + \left(u_1^2 vc - Kvu_1^2 - 2vu_1^3 - 2u_1^2 v\right)$$
$$x^2 - \left(2u_1^2 + w_1 u_1^2\right) x + Ku_1^2 - cu_1^2 - u_1^3 - u_1^2,$$

Therefore

$$u_1^2 P_6 = -\left[\begin{array}{c} \left(1+w_1+cv\right)\left(u_1 w_1 + u_1 cv + u_1 + w_1 + cv + 1 + u_1 w_1\right) \\ +u_1 v \left\{ \left(K-c\right)\left(1+2u_1\right) + u_1^2 + u_1 \right\} \end{array} \right] x^2 .$$

For $K > u_1 + c$,

$$\left\{ u_1 \left(cvu_1 + cv - 2u_1 - w_1 u_1\right) - \left(u_1 + c - K\right)\left(2u_1 w_1 + u_1 cv + u_1 + w_1 + cv + 1\right) \right\} x$$
$$+ u_1 \left(K - c - u_1 - 1\right) - c\left(2u_1 w_1 + u_1 cv + u_1 + w_1 + cv + 1\right) \tag{45}$$

Thus

$$u_1^2 P_6 = \eta_1 x^2 + \eta_2 x + \eta_3, \tag{46}$$

where

$$\eta_1 = -\left[\begin{array}{c} \left(1+w_1+cv\right)\left(2u_1 w_1 + u_1 cv + u_1 + w_1 + cv + 1\right) \\ +u_1 v \left\{ \left(K-c\right)\left(1+2u_1\right) + u_1^2 + u_1 \right\} \end{array} \right] < 0,$$
$$\eta_2 = u_1 \left(cvu_1 + cv - 2u_1 - w_1 u_1\right) - \left(u_1 + c - K\right)\left(2u_1 w_1 + u_1 cv + u_1 + w_1 + cv + 1\right) \quad and$$
$$\eta_3 = u_1 \left(K - c - u_1 - 1\right) - c\left(2u_1 w_1 + u_1 cv + u_1 + w_1 + cv + 1\right),$$

Therefore the sufficient condition for which $P_6 < 0$ is,

$$\eta_2^2 - 4\eta_1 \eta_3 \leq 0 \tag{47}$$

4. Numerical Simulations and Conclusion

Case 1: When the treatment rate is ∞ to the infective so that $\theta \leq I \leq I_0$ we choose the parameters as follows $a = 3$, $d = 0.1$, $\lambda = 0.4$, $m = 0.01$, $r = 0.2$, $\mu = 0.3$. Hence the basic reproduction number $R_0 = 42.58 > 1$ when $a = 15$, $d = 2.5$, $\lambda = 0.4$, $m = 10$, $r = 0.5$, $\mu = 0.3$ we have $R_0 = 0.1883 < 1$. In this case disease dies out. Consider $a = 3$, $d = 0.1$, $\beta = 0.1$, $\lambda = 0.3$, $m = 0.01$, $r = 0.2$, $\mu = 0.2$, $\alpha = 1$. By rescaling the system we see that $(u - K) < 0$ and hence there exists unique positive equilibrium point $(x*, y*)$, where $x^* = 6.8855$ and $y^* = 7.2298$. For the parameters $\frac{P_4}{P_3} = 13.485$ and so the sufficient condition for stability $x^* < \frac{P_4}{P_3}$ is satisfied. Hence the point is locally stable.

Case 2: When $I > I_0$ we have the parameters $a = 2.6$, $d = 0.06$, $\beta = 0.16$, $\lambda = 0.4$, $m = 0.01$, $K = 0.7$, $\alpha = 2.0$, $\mu = 0.3$. Here $S + I + R = \frac{\mu + a}{d} = 48.33$ is invariant manifold so the system is

$$\frac{dI}{dt} = \frac{0.41}{1 + \alpha I^2}\left(48.33 - I - R\right) - 0.07I - 0.7,$$
$$\frac{dR}{dt} = \left(0.01\right) I - \left(0.76\right) R + 0.7$$

The rescaling system

$$\frac{dx}{dT} = x\left(87.88 - x - y\right) - 0.318x - 5.76$$
$$\frac{dy}{dt} = 0.045x - y + 5.76$$

In this paper, we see that the basic reproduction number plays an important role to control the diseases, if $R_0 < 1$ then DFE is globally stable and if $R_0 > 1$ then the endemic equilibrium is globally stable. Also with the help of immigration rate the treatment function gave a better result.

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