ψ\(^{∗}\)\(α\)-Closed Sets in Bitopological Spaces

Research Article

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Abstract: In this paper we introduce ψ\(^{∗}\)\(α\)-closed sets in bitopological spaces and obtain the relationship between the other existing closed sets. Also we study the notion of \((i,j)\)-ψ\(^{∗}\)\(α\)-closure operator and some of its properties. As applications we introduce \((i,j)\)-ψ\(^{∗}\)\(α\)\(T_2\)-space, \((i,j)\)-ψ\(^{∗}\)\(α\)\(T_\alpha\)-space and study some of their properties.

Keywords: ψ\(^{∗}\)\(α\)-closed set, ψ\(^{∗}\)\(α\)-open set, ψg-open set, \(τ_i\)-open set.

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1. Introduction

Levine [10] introduced the concepts of generalized closed sets in topological spaces and studied their basic properties. Several authors have introduced and investigated various types generalized closed sets in topological spaces. Only a few class of generalized closed sets form a topology. The class of ψ\(^{∗}\)\(α\)-closed sets in topological spaces is one among them and it was introduced by Balamani and Parvathi [1]. The study of bitopological spaces was initiated by Kelly [7] and thereafter topological concepts have been generalized to bitopological setting. Fukutake [5] introduced g-closed sets in bitopological spaces. In this paper we introduce a new class of sets in bitopological spaces called \((i,j)\)-ψ\(^{∗}\)\(α\)-closed sets and study their basic properties. Also we define \((i,j)\)-ψ\(^{∗}\)\(α\)-closure of a set and prove that the closure operator \((i,j)\)-ψ\(^{∗}\)\(α\)-closure is the Kuratowski closure operator on \((X, τ_1, τ_2)\).

2. Preliminaries

The interior, closure and complement of a subset A of a space \((X, τ)\) are denoted by \(int(A)\), \(cl(A)\) and \(A^c\) respectively. Throughout this paper \((X, τ_1, τ_2)\) represents bitopological space on which no separation axioms are assumed, unless otherwise mentioned.

Definition 2.1. A subset A of a topological space \((X, τ)\) is called

1. g-closed set [10] if \(cl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \((X, τ)\).
2. sg-closed [3] if \(scl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open in \((X, τ)\).
3. ψ\(^{-}\)closed set [13] if \(scl(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is sg-open in \((X, τ)\).

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(4). \( \psi g \)-closed set [11] if \( \psi cl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \((X, \tau)\).

(5). \( \psi \alpha \)-closed set [1] if \( \alpha cl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \psi \)-open in \((X, \tau)\).

**Definition 2.2.** For \( i, j = 1, 2 \) and \( i \neq j \), a subset \( A \) of a bitopological space \((X, \tau_1, \tau_2)\) is called

(1). \((i, j)\)-\( g \)-closed [5] if \( \tau_j-cl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_i \)-open in \( X \).

(2). \((i, j)\)-\( gp \)-closed [4] if \( \tau_j-pcl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_i \)-open in \( X \).

(3). \((i, j)\)-\( gp-r \)-closed [6] if \( \tau_j-pcl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_i \)-regular open in \( X \).

(4). \((i, j)\)-\( \omega \)-closed [6] if \( \tau_j-cl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_i \)-semi open in \( X \).

(5). \((i, j)\)-\( g^{*} \)-closed [12] if \( \tau_j-cl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_i \)-\( g \)-open in \( X \).

(6). \((i, j)\)-\( ga \)-closed [8] if \( \tau_j-acl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_i \)-\( \alpha \)-open in \( X \).

(7). \((i, j)\)-\( \alpha g \)-closed [4] if \( \tau_j-acl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_i \)-\( \alpha \)-open in \( X \).

(8). \((i, j)\)-\( \alpha g \)-closed [9] if \( \tau_j-acl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_i \)-\( \# gs \)-open in \( X \).

(9). \((i, j)\)-\( \psi g \)-closed [11] if \( \tau_j-\psi cl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_i \)-\( \psi \)-open in \( X \).

**Definition 2.3.** A topological space \((X, \tau)\) is said to be a

(1). \( \psi \alpha T_e \)-space if every \( \psi \alpha \)-closed subset of \((X, \tau)\) is closed in \((X, \tau)\) [2].

(2). \( \psi \alpha T_\alpha \)-space if every \( \psi \alpha \)-closed subset of \((X, \tau)\) is \( \alpha \)-closed in \((X, \tau)\) [2].

3. \((i, j)\)-\( \psi \alpha \)-Closed Sets

**Definition 3.1.** A subset \( A \) of a bitopological space \((X, \tau_1, \tau_2)\) is called \((i, j)\)-\( \psi \alpha \)-closed if \( \tau_j-acl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \tau_i \)-\( \psi \)-open in \((X, \tau_1, \tau_2)\), where \( i, j = 1, 2 \) and \( i \neq j \). The family of all \((i, j)\)-\( \psi \alpha \)-closed sets in \((X, \tau_1, \tau_2)\) is denoted by \( \psi \alpha \tau C(i, j) \).

**Remark 3.2.** By setting \( \tau_1 = \tau_j \) in Definition 3.1, an \((i, j)\)-\( \psi \alpha \)-closed set reduces to a \( \psi \alpha \)-closed set.

**Example 3.3.** Let \( X = \{a, b, c\} \), \( \tau_1 = \{\phi, \{a\}, X\} \) and \( \tau_2 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\} \). Then \( \phi \), \( \{c\} \), \( \{a, c\} \), \( \{b, c\} \), \( X \) are \((1, 2)\)-\( \psi \alpha \)-closed.

**Proposition 3.4.** Every \( \tau_j \)-closed (resp. \( \tau_j \)-\( \alpha \)-closed) set in \((X, \tau_1, \tau_2)\) is \((i, j)\)-\( \psi \alpha \)-closed but not conversely.

**Proof.** Let \( A \) be \( \tau_j \)-closed (resp. \( \tau_j \)-\( \alpha \)-closed) in \((X, \tau_1, \tau_2)\) such that \( A \subseteq U \), where \( U \) is \( \tau_i \)-\( \psi \)-open. Since \( A \) is \( \tau_j \)-closed (resp. \( \tau_j \)-\( \alpha \)-closed) \( \tau_j-cl(A) \) (resp. \( \tau_j-acl(A) \)) = \( A \subseteq U \). But \( \tau_j-acl(A) \subseteq \tau_j - cl(A) \). Therefore \( \tau_j-acl(A) \subseteq U \). Hence \( A \) is an \((i, j)\)-\( \psi \alpha \)-closed set in \((X, \tau_1, \tau_2)\).

**Example 3.5.** Let \( X = \{a, b, c\} \), \( \tau_1 = \{\phi, \{a\}, X\} \) and \( \tau_2 = \{\phi, \{a\}, \{a, b\}, X\} \). The subset \( \{b\} \) is \((1, 2)\)-\( \psi \alpha \)-closed but not \( \tau_2 \)-\( \alpha \)-closed.

**Example 3.6.** Let \( X = \{a, b, c\} \), \( \tau_1 = \{\phi, \{a\}, X\} \) and \( \tau_2 = \{\phi, \{a, b\}, X\} \). The subset \( \{b, c\} \) is \((1, 2)\)-\( \psi \alpha \)-closed but not \( \tau_2 \)-\( \alpha \)-closed.
Proposition 3.7. Every \((i,j)\)-\(\psi^*\)\(\alpha\)-closed set in \((X, \tau_1, \tau_2)\) is \((i,j)\)-\(gp\)-closed but not conversely.

Proof. Let \(A \subseteq U\) and \(U\) be \(\tau_\gamma\)-open in \((X, \tau_1, \tau_2)\). Since every \(\tau_\gamma\)-open set is \(\tau_\gamma\)-\(gp\)-open and \(A\) is \((i,j)\)-\(\psi^*\)\(\alpha\)-closed in \((X, \tau_1, \tau_2)\), \(\tau_\gamma\)-\(acl\)(\(A\)) \(\subseteq U\). We know that \(\tau_\gamma\)-\(pcl\)(\(A\)) \(\subseteq \tau_\gamma\)-\(acl\)(\(A\)) \(\subseteq U\). Therefore \(A\) is \((i,j)\)-\(gp\)-closed.

Example 3.8. Let \(X = \{a, b, c, d\}\), \(\tau_1 = \{\emptyset, \{a\}, X\}\) and \(\tau_2 = \{\emptyset, \{c\}, \{a,b\}, \{a, b, c\}, \{a, b, d\}, X\}\). The subset \(\{a, c, d\}\) is \((1,2)\)-\(g\)-\(pcl\)-closed but not \((1,2)\)-\(\psi^*\)\(\alpha\)-closed.

Proposition 3.9. Every \((i,j)\)-\(\psi^*\)\(\alpha\)-closed set in \((X, \tau_1, \tau_2)\) is \((i,j)\)-\(gpr\)-closed but not conversely.

Proof. Let \(A \subseteq U\) and \(U\) be \(\tau_\gamma\)-regular open in \((X, \tau_1, \tau_2)\). Since every \(\tau_\gamma\)-regular open set is \(\tau_\gamma\)-\(g\)-\(pr\)-open and \(A\) is \((i,j)\)-\(\psi^*\)\(\alpha\)-closed in \((X, \tau_1, \tau_2)\), \(\tau_\gamma\)-\(acl\)(\(A\)) \(\subseteq U\). We know that \(\tau_\gamma\)-\(pcl\)(\(A\)) \(\subseteq \tau_\gamma\)-\(acl\)(\(A\)) \(\subseteq U\). Therefore \(A\) is \((i,j)\)-\(gpr\)-closed.

Example 3.10. Let \(X = \{a, b, c, d\}\), \(\tau_1 = \{\emptyset, \{a\}, X\}\) and \(\tau_2 = \{\emptyset, \{c\}, \{a,b\}, \{a, b, c\}, \{a, b, d\}, X\}\). The subset \(\{a, d\}\) is \((1,2)\)-\(gpr\)-closed but not \((1,2)\)-\(\psi^*\)\(\alpha\)-closed.

Proposition 3.11. Every \((i,j)\)-\(\psi^*\)\(\alpha\)-closed set in \((X, \tau_1, \tau_2)\) is \((i,j)\)-\(g\)-\(pr\)-\(cl\)-closed but not conversely.

Proof. Let \(A \subseteq U\) and \(U\) be \(\tau_\gamma\)-\(gs\)-open in \((X, \tau_1, \tau_2)\). Since every \(\tau_\gamma\)-\(gs\)-open set is \(\tau_\gamma\)-\(g\)-\(pr\)-open and \(A\) is \((i,j)\)-\(\psi^*\)\(\alpha\)-closed in \((X, \tau_1, \tau_2)\), \(\tau_\gamma\)-\(acl\)(\(A\)) \(\subseteq U\). Therefore \(A\) is \((i,j)\)-\(g\)-\(pr\)-\(cl\)-closed.

Example 3.12. Let \(X = \{a, b, c, d\}\), \(\tau_1 = \{\emptyset, \{a\}, \{a,b\}, X\}\) and \(\tau_2 = \{\emptyset, \{c\}, \{a,b\}, \{a, b, d\}, X\}\). The subset \(\{b, c\}\) is \((1,2)\)-\(g\)-\(pr\)-\(cl\)-closed but not \((1,2)\)-\(\psi^*\)\(\alpha\)-closed.

Proposition 3.13. Every \((i,j)\)-\(\psi^*\)\(\alpha\)-closed set in \((X, \tau_1, \tau_2)\) is \((i,j)\)-\(g\)-\(al\)-closed but not conversely.

Proof. Let \(A \subseteq U\) and \(U\) be \(\tau_\gamma\)-\(al\)-open in \((X, \tau_1, \tau_2)\). Since every \(\tau_\gamma\)-\(al\)-open set is \(\tau_\gamma\)-\(g\)-\(al\)-open and \(A\) is \((i,j)\)-\(\psi^*\)\(\alpha\)-closed in \((X, \tau_1, \tau_2)\), \(\tau_\gamma\)-\(acl\)(\(A\)) \(\subseteq U\). Therefore \(A\) is \((i,j)\)-\(g\)-\(al\)-closed.

Example 3.14. Let \(X = \{a, b, c\}\), \(\tau_1 = \{\emptyset, \{a\}, \{b, c\}, X\}\) and \(\tau_2 = \{\emptyset, \{a\}, \{a,b\}, \{a, c\}, X\}\). The subsets \(\{a, b\}\) and \(\{a, c\}\) are \((1,2)\)-\(g\)-\(al\)-closed but not \((1,2)\)-\(\psi^*\)\(\alpha\)-closed.

Proposition 3.15. Every \((i,j)\)-\(\psi^*\)\(\alpha\)-closed set in \((X, \tau_1, \tau_2)\) is \((i,j)\)-\(g\)-\(al\)-\(g\)-closed but not conversely.

Proof. Let \(A \subseteq U\) and \(U\) be \(\tau_\gamma\)-open in \((X, \tau_1, \tau_2)\). Since every \(\tau_\gamma\)-open set is \(\tau_\gamma\)-\(g\)-\(al\)-open and \(A\) is \((i,j)\)-\(\psi^*\)\(\alpha\)-closed in \((X, \tau_1, \tau_2)\), \(\tau_\gamma\)-\(acl\)(\(A\)) \(\subseteq U\). Therefore \(A\) is \((i,j)\)-\(g\)-\(al\)-\(g\)-closed.

Example 3.16. Let \(X = \{a, b, c\}\), \(\tau_1 = \{\emptyset, \{a\}, \{b, c\}, X\}\) and \(\tau_2 = \{\emptyset, \{a\}, X\}\). The subsets \(\{a, b\}\) and \(\{a, c\}\) are \((1,2)\)-\(g\)-\(al\)-\(g\)-closed but not \((1,2)\)-\(\psi^*\)\(\alpha\)-closed.

Proposition 3.17. Every \((i,j)\)-\(\psi^*\)\(\alpha\)-closed set in \((X, \tau_1, \tau_2)\) is \((i,j)\)-\(g\)-\(al\)-closed but not conversely.

Proof. Let \(A \subseteq U\) and \(U\) be \(\tau_\gamma\)-open in \((X, \tau_1, \tau_2)\). Since every \(\tau_\gamma\)-open set is \(\tau_\gamma\)-\(g\)-\(al\)-open and \(A\) is \((i,j)\)-\(\psi^*\)\(\alpha\)-closed in \((X, \tau_1, \tau_2)\), \(\tau_\gamma\)-\(acl\)(\(A\)) \(\subseteq U\). We know that \(\tau_\gamma\)-\(pcl\)(\(A\)) \(\subseteq \tau_\gamma\)-\(acl\)(\(A\)) \(\subseteq U\) and so \(\tau_\gamma\)-\(pcl\)(\(A\)) \(\subseteq U\). Therefore \(A\) is \((i,j)\)-\(g\)-\(al\)-closed.

Example 3.18. Let \(X = \{a, b, c\}\), \(\tau_1 = \{\emptyset, \{a\}, X\}\) and \(\tau_2 = \{\emptyset, \{a,b\}, X\}\). The subsets \(\{b\}\), \(\{c\}\), \(\{a,b\}\) and \(\{a,c\}\) are \((1,2)\)-\(g\)-\(al\)-closed but not \((1,2)\)-\(\psi^*\)\(\alpha\)-closed.

Remark 3.19. The following example show that \((i,j)\)-\(\psi^*\)\(\alpha\)-closed set is independent of \((i,j)\)-\(g\)-\(cl\)-closed set, \((i,j)\)-\(g^*\)-closed set and \((i,j)\)-\(\omega\)-closed set.
Example 3.20. Let $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, \{c\}, \{a, b\}, \{a, b, c\}, X\}$ and $\tau_2 = \{\phi, \{a\}, X\}$. The subset $\{a, c, d\}$ is $(1, 2)$-$g$-closed, $(1, 2)$-$g^*$-closed and $(1, 2)$-$\omega$-closed but not $(1, 2)$-$\psi^\alpha$-closed. The subset $\{b\}$ is $(1, 2)$-$\psi^\alpha$-closed but not $(1, 2)$-$g$-closed, not $(1, 2)$-$g^*$-closed and not $(1, 2)$-$\omega$-closed.

Theorem 3.21. If $A$ is $\tau_j$-$\psi^g$-open and $(i, j)$-$\psi^\alpha$-closed in $(X, \tau_1, \tau_2)$ then $A$ is $\tau_j$-$\alpha$-closed.

Proof. Let $A$ be $\tau_j$-$\psi^g$-open and $(i, j)$-$\psi^\alpha$-closed. Since $A \subseteq A$, then $\tau_j$-$\alpha$cl$(A) \subseteq A$. Therefore $\tau_j$-$\alpha$cl$(A) = A$. Consequently $A$ is $\tau_j$-$\alpha$-closed.

Theorem 3.22. If $A$ is $(i, j)$-$\psi^\alpha$-closed and $\tau_j$-$\psi^g$-open and $F$ is $\tau_j$-$\alpha$-closed in $(X, \tau_1, \tau_2)$ then $A \cap F$ is $\tau_j$-$\alpha$-closed.

Proof. Since $A$ is $(i, j)$-$\psi^\alpha$-closed and $\tau_j$-$\psi^g$-open in $(X, \tau_1, \tau_2)$, $A$ is $\tau_j$-$\alpha$-closed (by Theorem 3.21). Since $F$ is $\tau_j$-$\alpha$-closed, $A \cap F$ is $\tau_j$-$\alpha$-closed in $(X, \tau_1, \tau_2)$.

Theorem 3.23. Union of two $(i, j)$-$\psi^\alpha$-closed sets is $(i, j)$-$\psi^\alpha$-closed.

Proof. Let $A$ and $B$ are $(i, j)$-$\psi^\alpha$-closed sets and $U$ be any $\psi^g$-open set in $(X, \tau_1)$ containing $A$ and $B$. Then $\tau_j$-$\alpha$cl$(A) \subseteq U$, $\tau_j$-$\alpha$cl$(B) \subseteq U$, $\tau_j$-$\alpha$cl$(A \cup B) = \tau_j$-$\alpha$cl$(A) \cup \tau_j$-$\alpha$cl$(B) \subseteq U$. Hence $A \cup B$ is $(i, j)$-$\psi^\alpha$-closed.

Remark 3.24. The intersection of two $(i, j)$-$\psi^\alpha$-closed sets need not be $(i, j)$-$\psi^\alpha$-closed set as seen from the following example.

Example 3.25. Let $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, \{c\}, \{a, b\}, \{a, b, c\}, X\}$. The subsets $A = \{a, b, d\}$ and $B = \{b, c, d\}$ are $(1, 2)$-$\psi^\alpha$-closed but their intersection $A \cap B = \{b, d\}$ is not $(1, 2)$-$\psi^\alpha$-closed.

Theorem 3.26. If a subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is $(i, j)$-$\psi^\alpha$-closed then $\tau_j$-$\alpha$cl$(A) - A$ contains no nonempty $\tau_j$-$\psi^g$-closed set.

Proof. Let $A$ be an $(i, j)$-$\psi^\alpha$-closed set and $S$ be a $\tau_j$-$\psi^g$-closed set such that $S \subseteq \tau_j$-$\alpha$cl$(A) - A$. Therefore $A \subseteq F^c$ and $F \subseteq \tau_j$-$\alpha$cl$(A)$. Since $F^c$ is $\tau_j$-$\psi^g$-open and $A$ is $(i, j)$-$\psi^\alpha$-closed, $\tau_j$-$\alpha$cl$(A) \subseteq F^c$. Thus $F \subseteq [\tau_j$-$\alpha$cl$(A)]^c$. Hence $F \subseteq [\tau_j$-$\alpha$cl$(A)] \cap [\tau_j$-$\alpha$cl$(A)]^c = \phi$. Therefore $F = \phi$. Hence $\tau_j$-$\alpha$cl$(A) - A$ contains no nonempty $\tau_j$-$\psi^g$-closed set.

Remark 3.27. The converse of the above theorem is not as true as seen from the following example.

Example 3.28. Let $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, \{a, b, c\}, X\}$ and $\tau_2 = \{\phi, \{c\}, \{a, b\}, \{a, b, c\}, X\}$. $\psi^g$O$(X, \tau_1) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, c, d\}, \{a, b, d\}, \{a, c, d\}, \{a, c, d\}, \{a, b, d\}, \{a, c, d\}, X\}$. $\psi^\alpha$(1, 2) = $\{\phi, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, c, d\}, \{a, b, d\}, \{a, c, d\}, \{a, b, d\}, \{a, c, d\}, \{a, b, d\}, \{a, c, d\}, \{a, b, d\}, \{a, c, d\}, X\}$. If $A = \{a, b\}$, $\tau_j$-$\alpha$cl$(A) - A = \{a, b\} - \{a, b\} = \{d\}$. But $\{a, b\}$ is not $(1, 2)$-$\psi^\alpha$-closed.

Theorem 3.29. Let $A$ be an $(i, j)$-$\psi^\alpha$-closed set in $(X, \tau_1, \tau_2)$. Then $A$ is $\tau_j$-$\alpha$-closed if and only if $\tau_j$-$\alpha$cl$(A) - A$ is $\tau_j$-$\psi^g$-closed in $(X, \tau_1, \tau_2)$.

Proof. Suppose that $A$ is $(i, j)$-$\psi^\alpha$-closed. Let $A$ be $\tau_j$-$\alpha$-closed. Then $\tau_j$-$\alpha$cl$(A) = A$. Therefore $\tau_j$-$\alpha$cl$(A) - A = \phi$ is $\tau_j$-$\psi^g$-closed in $(X, \tau_1, \tau_2)$.

Conversely, suppose that $A$ is $(i, j)$-$\psi^\alpha$-closed and $\tau_j$-$\alpha$cl$(A) - A$ is $\tau_j$-$\psi^g$-closed. Since $A$ is $(i, j)$-$\psi^\alpha$-closed, $\tau_j$-$\alpha$cl$(A) - A$ contains no nonempty $\tau_j$-$\psi^g$-closed set (by Theorem 3.26). Since $\tau_j$-$\alpha$cl$(A) - A$ is $\tau_j$-$\psi^g$-closed, $\tau_j$-$\alpha$cl$(A) - A = \phi$. Then $\tau_j$-$\alpha$cl$(A) = A$. Hence $A$ is $\tau_j$-$\alpha$-closed.

Theorem 3.30. Let $A$ and $B$ be subsets of $(X, \tau_1, \tau_2)$ such that $A \subseteq B \subseteq \tau_j$-$\alpha$cl$(A)$. If $A$ is $(i, j)$-$\psi^\alpha$-closed then $B$ is $(i, j)$-$\psi^\alpha$-closed.
Proof. Let A and B be subsets such that $A \subseteq B \subseteq \tau_j - \text{acl}(A)$. Suppose that A is $(i, j)$-$\psi^*\alpha$-closed. Let $B \subseteq U$ and U be $\tau_i$-$\psi^g$-open in $(X, \tau_1, \tau_2)$. Then $A \subseteq U$. Since A is $(i, j)$-$\psi^*\alpha$-closed, $\tau_j - \text{acl}(A) \subseteq U$. Since $B \subseteq \tau_j - \text{acl}(A)$, $\tau_j - \text{acl}(B) \subseteq \tau_j - \text{acl}[\tau_j - \text{acl}(A)] = \tau_j - \text{acl}(A) \subseteq U$. Therefore B is $(i, j)$-$\psi^*\alpha$-closed. \qed

Theorem 3.31. Let $B \subseteq A \subseteq X$ and suppose that B is $(i, j)$-$\psi^*\alpha$-closed in $(X, \tau_1, \tau_2)$, then B is $(i, j)$-$\psi^*\alpha$-closed relative to A. The converse is true if A is $\tau_i$-open and $(i, j)$-$\psi^*\alpha$-closed in $(X, \tau_1, \tau_2)$.

Proof. Let B be $(i, j)$-$\psi^*\alpha$-closed in $(X, \tau_1, \tau_2)$. Let $B \subseteq U$ and U be $\tau_i$-$\psi^g$-open in A. Since U is $\tau_i$-$\psi^g$-open in A, $U = V \cap A$, where V is $\tau_i$-$\psi^g$-open in $(X, \tau_1, \tau_2)$. Hence $B \subseteq U \subseteq V$. Since B is $(i, j)$-$\psi^*\alpha$-closed in $(X, \tau_1, \tau_2)$, $\tau_j - \text{acl}(B) \subseteq V \cap A$. Hence $\tau_j - \text{acl}(B) \subseteq \tau_j - \text{acl}(A)$. Therefore $\tau_j - \text{acl}(B) \subseteq A \cap U \subseteq U$. Hence $\tau_j - \text{acl}(B) \subseteq A \cap \text{U-closure}(A)$. \qed

Remark 3.32. In general $\psi^*\alpha C(\tau_i, \tau_j) \neq \psi^*\alpha C(\tau_1, \tau_2)$ which can be seen from the following example.

Example 3.33. Let $X = \{a, b, c\}$ with the topologies $\tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b, c\}, \{a, b, c\}\}$. Then $\psi^*\alpha C(\tau_1, \tau_2) = \{\phi, \{a\}, \{b, c\}, \{a, b, c\}, X\}$ and $\psi^*\alpha C(\tau_1, \tau_1) = \{\phi, \{a\}, \{b, c\}, \{a, b, c\}, X\}$. This shows that $\psi^*\alpha C(\tau_1, \tau_1) \neq \psi^*\alpha C(\tau_1, \tau_2)$.

Theorem 3.34. If $\tau_1 \subseteq \tau_2$ in $(X, \tau_1, \tau_2)$ then $\psi^*\alpha C(2, 1) \subseteq \psi^*\alpha C(1, 2)$.

Proof. Let $A \in \psi^*\alpha C(2, 1)$. Let $U \in \psi^g O(\tau_1, \tau_1)$ such that $A \subseteq U$. Since $\psi^g O(\tau_1, \tau_1) \subseteq \psi^g O(\tau_1, \tau_2)$, $U \in \psi^g O(\tau_1, \tau_2)$. Since A is $(2, 1)$-$\psi^*\alpha$-open, $\tau_1 - \text{acl}(A) \subseteq U$. Since $\tau_1 \subseteq \tau_2$, $\tau_2 - \text{acl}(A) \subseteq \tau_1 - \text{acl}(A)$. Thus $\tau_2 - \text{acl}(A) \subseteq U$. Hence A is $(1, 2)$-$\psi^*\alpha$-closed. That is, $A \in \psi^*\alpha C(1, 2)$.

The converse of the above theorem need not be true as seen from the following example:

Example 3.35. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, \{a, b\}, \{a, c\}\}$ and $\tau_2 = \{\phi, \{a\}, \{b, c\}, X\}$. Then $\psi^*\alpha C(2, 1) \subseteq \psi^*\alpha C(1, 2)$ but $\tau_1 \nsupseteq \tau_2$.

Definition 3.36. A set A of a bitopological space $(X, \tau_1, \tau_2)$ is called $(i, j)$-$\psi$ star alpha open (briefly, $(i, j)$-$\psi^*\alpha$-open) if its complement is $(i, j)$-$\psi^*\alpha$-closed in $(X, \tau_1, \tau_2)$. The set of all $(i, j)$-$\psi^*\alpha$-open sets in $(X, \tau_1, \tau_2)$ is denoted by $\psi^*\alpha O(i, j)$.

Example 3.37. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{b, a\}, X\}$. Then $\phi, \{a\}, \{b, a\}, \{a, b\}$ are $(1, 2)$-$\psi^*\alpha$-open.

Definition 3.38. An $(i, j)$-$\psi$ star alpha interior of a subset A (briefly, $(i, j)$-$\psi^*\alpha$ int(A)) in $(X, \tau_1, \tau_2)$ is defined as follows.

$$(i, j) - \psi^*\alpha \text{ int}(A) = \cup\{F \subseteq X : F \subseteq A \text{ and } F \text{ is } (i, j)-\psi^*\alpha\text{-open in } (X, \tau_1, \tau_2)\}.$$ 

Proposition 3.39.

(1). Every $\tau_j$-open set in $(X, \tau_1, \tau_2)$ is $(i, j)$-$\psi^*\alpha$-open.

(2). Every $\tau_j$-$\alpha$-open set in $(X, \tau_1, \tau_2)$ is $(i, j)$-$\psi^*\alpha$-open.

(3). Every $(i, j)$-$\psi^*\alpha$-open set in $(X, \tau_1, \tau_2)$ is $(i, j)$-$\psi^g$-open.
(4). Every \((i, j)\)-\(\psi^*\alpha\)-open set in \((X, \tau_1, \tau_2)\) is \((i, j)\)-gpr-open.

(5). Every \((i, j)\)-\(\psi^*\alpha\)-open set in \((X, \tau_1, \tau_2)\) is \((i, j)\)-\(g\alpha\)-open.

(6). Every \((i, j)\)-\(\psi^*\alpha\)-open set in \((X, \tau_1, \tau_2)\) is \((i, j)\)-ga-open.

(7). Every \((i, j)\)-\(\psi^*\alpha\)-open set in \((X, \tau_1, \tau_2)\) is \((i, j)\)-ag-open.

(8). Every \((i, j)\)-\(\psi^*\alpha\)-open set in \((X, \tau_1, \tau_2)\) is \((i, j)\)-\(\psi\)g-open.

The converses of the statements in the above proposition are not true in general as seen from the Examples 3.5, 3.6, 3.8, 3.10, 3.12, 3.14, 3.16 and 3.18.

**Theorem 3.40.** A subset \(A\) of a bitopological space \((X, \tau_1, \tau_2)\) is \((i, j)\)-\(\psi^*\alpha\)-open if and only if \(F \subseteq \tau_j - \text{aint}(A)\) whenever \(F \subseteq A\) and \(F\) is \(\tau_i\)-\(\psi\)g-closed in \((X, \tau_1, \tau_2)\).

**Proof.** Suppose that \(A\) is \((i, j)\)-\(\psi^*\alpha\)-open. Let \(F \subseteq A\) and \(F\) be \(\tau_i\)-\(\psi\)g-closed. Then \(A^c \subseteq F^c\) and \(F^c\) is \(\tau_i\)-\(\psi\)g-closed. Since \(A^c\) is \((i, j)\)-\(\psi^*\alpha\)-closed, \(\tau_j - \text{acl}(A^c) \subseteq F^c\). Hence \(F \subseteq \tau_j - \text{aint}(A)\) if \(F \subseteq A\) and \(F\) is \(\tau_i\)-\(\psi\)g-closed in \((X, \tau_1, \tau_2)\).

Conversely, suppose that \(F \subseteq \tau_j - \text{aint}(A)\) whenever \(F \subseteq A\) and \(F\) is \(\tau_i\)-\(\psi\)g-closed in \((X, \tau_1, \tau_2)\). Let \(U\) be \(\tau_i\)-\(\psi\)g-closed in \((X, \tau_1, \tau_2)\) and \(A^c \subseteq U\). Then \(U^c\) is \(\tau_j\)-\(\psi\)g-closed and \(U^c \subseteq A\). Hence by assumption \(U^c \subseteq \tau_j - \text{aint}(A)\). Therefore \([\tau_j - \text{aint}(A)]^c \subseteq U\). That is \(\tau_j - \text{acl}(A^c) \subseteq U\). Therefore \(A^c\) is \((i, j)\)-\(\psi^*\alpha\)-closed. Hence \(A\) is \((i, j)\)-\(\psi^*\alpha\)-open.

**Theorem 3.41.** If a subset \(A\) is \((i, j)\)-\(\psi^*\alpha\)-closed in \((X, \tau_1, \tau_2)\) then \(\tau_j\cdot\text{acl}(A^c)\cdot A\) is \((i, j)\)-\(\psi^*\alpha\)-open.

**Proof.** Suppose that \(A\) is \((i, j)\)-\(\psi^*\alpha\)-closed in \((X, \tau_1, \tau_2)\). Let \(F \subseteq \tau_j - \text{acl}(A)\) and \(F\) be \(\tau_i\)-\(\psi\)g-closed. Since \(A\) is \((i, j)\)-\(\psi^*\alpha\)-closed, \(\tau_j\cdot\text{acl}(A^c)\cdot A\) does not contain nonempty \(\tau_i\)-\(\psi\)g-closed sets (by Theorem 3.26). Hence \(F = \emptyset\). Thus \(F \subseteq \tau_j - \text{aint}(\tau_j - \text{acl}(A) - A)\). Hence \(\tau_j\cdot\text{acl}(A)\cdot A\) is \((i, j)\)-\(\psi^*\alpha\)-open.

**Theorem 3.42.** If a set \(A\) is \((i, j)\)-\(\psi^*\alpha\)-open in \((X, \tau_1, \tau_2)\) then \(G = X\) whenever \(G\) is \(\tau_i\)-\(\psi\)g-open and \(\tau_j - \text{aint}(A) \cup A^c \subseteq G\).

**Proof.** Suppose that \(A\) is \((i, j)\)-\(\psi^*\alpha\)-open in \((X, \tau_1, \tau_2)\), \(G\) is \(\tau_i\)-\(\psi\)g-open and \(\tau_j - \text{aint}(A) \cup A^c \subseteq G\). Then \(G^c \subseteq \{\tau_j - \text{aint}(A) \cup A^c\}^c = \tau_j - \text{acl}(A^c) \equiv A^c\). Since \(A^c\) is \((i, j)\)-\(\psi^*\alpha\)-closed, \(\tau_j - \text{acl}(A^c) \equiv A^c\) contains no nonempty \(\tau_i\)-\(\psi\)g-closed set in \((X, \tau_1, \tau_2)\) (by Theorem 3.26). Therefore \(G^c = \emptyset\). Hence \(G = X\).

**Remark 3.43.** The converse of the above theorem is not true in general as seen from the following example.

**Example 3.44.** Let \(X = \{a, b, c\}, \tau_1 = \{\phi, \{a\}, X\}\) and \(\tau_2 = \{\phi, \{a, b\}, X\}\). Let \(A = \{c\}\) and \(G = X\). Then \(G\) is \(\tau_1\)-\(\psi\)g-open, \(\tau_2 - \text{aint}(A) \cup A^c = \phi \cup \{a, b\} \subseteq G\), but \(A = \{c\}\) is not \((1, 2)\)-\(\psi^*\alpha\)-open.

**Theorem 3.45.** Let \((X, \tau_1, \tau_2)\) be a bitopological space. If \(x \in X\), then singleton \(\{x\}\) is either \(\tau_i\)-\(\psi\)g-closed or \((i, j)\)-\(\psi^*\alpha\)-open.

**Proof.** Let \(x \in X\) and suppose that \(\{x\}\) is not \(\tau_i\)-\(\psi\)g-closed. Then \(X - \{x\}\) is not \(\tau_i\)-\(\psi\)g-open. Consequently, \(X\) is the only \(\tau_i\)-\(\psi\)g-open set containing the set \(X - \{x\}\). Therefore \(X - \{x\}\) is \((i, j)\)-\(\psi^*\alpha\)-closed. Hence \(\{x\}\) is \((i, j)\)-\(\psi^*\alpha\)-open.

4. \((i, j)\)-\(\psi^*\alpha\)-closure

**Definition 4.1.** An \((i, j)\)-\(\psi^*\alpha\)-closure of a subset \(A\) (briefly, \((i, j)\)-\(\psi^*\alpha\text{cl}(A)\) of \((X, \tau_1, \tau_2)\) is defined as \((i, j) - \psi^*\text{acl}(A) = \cap\{F \subseteq X : A \subseteq F \text{ and } F \text{ is } (i, j)\text{-}\psi^*\alpha\text{-closed in } (X, \tau_1, \tau_2)\}\).

**Proposition 4.2.** Let \(E\) and \(F\) be any two subsets of \((X, \tau_1, \tau_2)\). Then the following results hold.
(a). \((i,j) - \psi^\alpha acl(\phi) = \phi\) and \((i,j) - \psi^\alpha acl(X) = X\).

(b). If \(E \subseteq F\), then \((i,j) - \psi^\alpha acl(E) \subseteq (i,j) - \psi^\alpha acl(F)\).

(c). \(E \subseteq (i,j) - \psi^\alpha acl(E) \subseteq \tau_j - cl(E)\).

(d). If \(A\) is \((i,j) - \psi^\alpha\) closed in \((X, \tau_1, \tau_2)\) then \((i,j) - \psi^\alpha acl(E) = E\).

(e). \((i,j) - \psi^\alpha acl(E \cap F) \subseteq (i,j) - \psi^\alpha acl(E) \cap (i,j) - \psi^\alpha acl(F)\).

(f). \((i,j) - \psi^\alpha acl(E \cup F) = (i,j) - \psi^\alpha acl(E) \cup (i,j) - \psi^\alpha acl(F)\).

(g). \((i,j) - \psi^\alpha acl((i,j) - \psi^\alpha acl(E)) = (i,j) - \psi^\alpha acl(E)\).

Proof.

(a). Since \(\phi\) and \(X\) are \((i,j) - \psi^\alpha\) closed in \((X, \tau_1, \tau_2)\), the results follows.

(b). Let \(E \subseteq F\). Then by the definition of \((i,j) - \psi^\alpha\) closure, \((i,j) - \psi^\alpha acl(E) \subseteq (i,j) - \psi^\alpha acl(F)\).

(c). From the definition of \((i,j) - \psi^\alpha\) closure, it follows that \(E \subseteq (i,j) - \psi^\alpha acl(E)\). By Proposition 3.4 every \(\tau_j\)-closed set is \((i,j) - \psi^\alpha\) closed. Therefore \(E \subseteq (i,j) - \psi^\alpha acl(E) \subseteq \tau_j - cl(E)\).

(d). Follows from (c) and by the definition of \((i,j) - \psi^\alpha\) closure.

(e). Since \(E \cap F \subseteq E\) and \(E \cap F \subseteq F\), by (b) \((i,j) - \psi^\alpha acl(E \cap F) \subseteq (i,j) - \psi^\alpha acl(E) \cap (i,j) - \psi^\alpha acl(F)\).

Hence \((i,j) - \psi^\alpha acl(E \cap F) \subseteq (i,j) - \psi^\alpha acl(E) \cap (i,j) - \psi^\alpha acl(F)\).

(f). Since \(E \subseteq E \cup F\) and \(F \subseteq E \cup F\), by (b) \((i,j) - \psi^\alpha acl(E) \subseteq (i,j) - \psi^\alpha acl(E \cup F)\) and \((i,j) - \psi^\alpha acl(F) \subseteq (i,j) - \psi^\alpha acl(E \cup F)\).

To prove the reverse inclusion, let \(x \in (i,j) - \psi^\alpha acl(E \cup F)\) and suppose that \(x \notin (i,j) - \psi^\alpha acl(E) \cup (i,j) - \psi^\alpha acl(F)\).

Then \(x \notin (i,j) - \psi^\alpha acl(E)\) and \(x \notin (i,j) - \psi^\alpha acl(F)\). Therefore there exist \((i,j) - \psi^\alpha\) closed sets \(U\) and \(V\) such that \(E \subseteq U\), \(F \subseteq V\), \(x \notin U\) and \(x \notin V\). Hence we have \(E \cup F \subseteq U \cup V\) and \(x \notin U \cup V\). By Theorem 3.23, \(U \cup V\) is a \((i,j) - \psi^\alpha\) closed set and hence \(x \notin (i,j) - \psi^\alpha acl(E \cup F)\), which is a contradiction. Hence \((i,j) - \psi^\alpha acl(E \cup F) \subseteq (i,j) - \psi^\alpha acl(E) \cup (i,j) - \psi^\alpha acl(F)\).

Therefore \((i,j) - \psi^\alpha acl(E \cup F) = (i,j) - \psi^\alpha acl(E) \cup (i,j) - \psi^\alpha acl(F)\).

(g). Follows from the definition of \((i,j) - \psi^\alpha\) closure.

\[\square\]

**Theorem 4.3.** The closure operator \((i,j) - \psi^\alpha\) closure is a Kuratowski closure operator on \((X, \tau_1, \tau_2)\).

Proof. From \((i,j) - \psi^\alpha acl(\phi) = \phi\), \(A \subseteq (i,j) - \psi^\alpha acl(A)\), \((i,j) - \psi^\alpha acl(E \cup F) = (i,j) - \psi^\alpha acl(E) \cup (i,j) - \psi^\alpha acl(F)\) and \((i,j) - \psi^\alpha acl((i,j) - \psi^\alpha acl(E)) = (i,j) - \psi^\alpha acl(E)\), we can say that \((i,j) - \psi^\alpha\)-a is a Kuratowski closure operator on \((X, \tau_1, \tau_2)\).

\[\square\]

**Definition 4.4.** A bitopological space \((X, \tau_1, \tau_2)\) is called an

1. \((i,j) - \psi^\alpha T_c\)-space if every \((i,j) - \psi^\alpha\) closed subset of \((X, \tau_1, \tau_2)\) is \(\tau_j\)-closed in \((X, \tau_1, \tau_2)\).
2. \((i,j) - \psi^\alpha T_\alpha\)-space if every \((i,j) - \psi^\alpha\) closed subset of \((X, \tau_1, \tau_2)\) is \(\tau_j\)-\(\alpha\)-closed in \((X, \tau_1, \tau_2)\).

**Proposition 4.5.** Every \((i,j) - \psi^\alpha T_c\)-space is an \((i,j) - \psi^\alpha T_\alpha\)-space but not conversely.

Proof. Assume that \((X, \tau_1, \tau_2)\) is an \((i,j) - \psi^\alpha T_c\)-space. Let \(A\) be an \((i,j) - \psi^\alpha\) closed set in \((X, \tau_1, \tau_2)\). Then \(A\) is \(\tau_j\)-closed. Since every \(\tau_j\)-closed set is \(\tau_j\)-\(\alpha\)-closed, \(A\) is \(\tau_j\)-\(\alpha\)-closed in \((X, \tau_1, \tau_2)\). Thus \((X, \tau_1, \tau_2)\) is an \((i,j) - \psi^\alpha T_\alpha\)-space. \[\square\]
Example 4.6. Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, X\}$. Then $(X, \tau_1, \tau_2)$ is an $(i,j) - \psi^*\alpha T_\alpha$-space but not an $(i,j) - \psi^*\alpha T_\alpha$-space, since the subsets $\{b\}$ and $\{c\}$ are $(1,2) - \psi^*\alpha$-closed but not $\tau_2$-closed in $(X, \tau_1, \tau_2)$.

Theorem 4.7. For a space $(X, \tau_1, \tau_2)$ the following statements are equivalent.

(1) $(X, \tau_1, \tau_2)$ is an $(i,j) - \psi^*\alpha T_\alpha$-space.

(2) For each $x \in X$, $\{x\}$ is either $\tau_i\psi\alpha$-closed or $\tau_j\alpha$-open.

Proof. (1) $\Rightarrow$ (2) Let $x \in X$ and $\{x\}$ be not a $\tau_i\psi\alpha$-closed set in $(X, \tau_1, \tau_2)$. Then $X - \{x\}$ is not $\tau_i\psi\alpha$-open. Hence $X$ is the only $\tau_i\psi\alpha$-open set containing $X - \{x\}$. This implies that $X - \{x\}$ is an $(i,j) - \psi^*\alpha$-closed set of $(X, \tau_1, \tau_2)$. Since $X$ is an $(i,j) - \psi^*\alpha T_\alpha$-space, $X - \{x\}$ is a $\tau_j\alpha$-closed set in $(X, \tau_1, \tau_2)$ or equivalently $\{x\}$ is $\tau_j\alpha$-open in $(X, \tau_1, \tau_2)$.

(2) $\Rightarrow$ (1) Let $A$ be an $(i,j) - \psi^*\alpha$-closed set in $(X, \tau_1, \tau_2)$ and $x \in \tau_j\alpha\text{-}\text{cl}(A)$. We show that $x \in A$. By (2), $\{x\}$ is either $\tau_i\psi\alpha$-closed or $\tau_j\alpha$-open.

Case 1: Assume that $\{x\}$ is $\tau_j\alpha$-open. Then $X - \{x\}$ is $\tau_j\alpha$-closed. If $x \notin A$, then $A \subseteq X - \{x\}$. Since $x \in \tau_j - \text{cl}(A)$, $x \in [X - \{x\}]$, which is a contradiction. Hence $x \in A$.

Case 2: Assume that $\{x\}$ is $\tau_i\psi\alpha$-closed and $x \notin A$. Then $\tau_j\alpha\text{-}\text{cl}(A)$-A contains a $\tau_j\psi\alpha$-closed set $\{x\}$. This contradicts Theorem 3.26. Therefore $x \in A$. ⪞

References

[1] N.Balamani and A.Parvathi, Between $\alpha$-closed sets and $\overline{g\alpha}$-closed sets, International Journal of Mathematical Archive, 7(6)(2016), 1-10.