Harmonic and Geometric-Arithmetic Indices of Boolean Function Graph $B(K_p, INC, \overline{K_q})$

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Abstract: For any graph $G$, let $V(G)$ and $E(G)$ denote the vertex set and edge set of $G$ respectively. The Harmonic index $H(G)$ of a graph $G$ is defined as the sum of the weights $\frac{1}{d(u)+d(v)}$ of all edges $uv$ of $G$ and the Geometric-Arithmetic index $GA(G)$ of $G$ is defined as the sum of the weights $\sqrt{\frac{d(u)\cdot d(v)}{d(u)+d(v)}}$, where $d(u)$ denotes the degree of a vertex $u$ in $G$. The Boolean function graph $B(K_p, INC, \overline{K_q})$ of $G$ is a graph with vertex set $V(G) \cup E(G)$ and two vertices in $B(K_p, INC, \overline{K_q})$ are adjacent if and only if they correspond to two adjacent vertices of $G$, two nonadjacent vertices of $G$ or to a vertex and an edge incident to it in $G$. For brevity, this graph is denoted by $B_4(G)$. In this paper, lower and upper bounds of $H(B_4(G))$ and $GA(B_4(G))$ are obtained. These indices are found for some particular graphs.

Keywords: Harmonic index, geometric-arithmetic index, Boolean Function Graph.

1. Introduction

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For any two vertices $u, v \in V(G)$, the distance between $u$ and $v$, denoted by $d_G(u, v)$, is defined as the number of edges in the shortest path connecting $u$ and $v$. The eccentricity of a vertex $v$, denoted by $e_G(v)$, is the largest distance of $v$ and any other vertex $u$ of $G$. The degree of a vertex $v$, denoted by $d_G(v)$ (or simply $d(u)$) is the number of vertices adjacent with the vertex $v$. For a graph $G$, the harmonic index $H(G)$ is defined as $H(G) = \sum_{u \in E(G)} \frac{2}{d(u)+d(v)}$. Favaron et al. [1] considered the relation between the harmonic index and the eigen values of graphs.

In 2012, Zhong [9] reintroduced this index as harmonic index and found the minimum and maximum values of the harmonic index for simple connected graphs and trees, and characterized the corresponding extremal graphs. Wu et al. [8] gave the minimum value of the Harmonic index among the graphs with the minimum degree at least two. In [7], Vukicevic et al. defined a new topological index geometric-arithmetic index of a graph $G$, denoted by $GA(G)$ and is defined by $GA(G) = \sum_{u \in E(G)} \frac{2\sqrt{d(u)\cdot d(v)}}{d(u)+d(v)}$. The geometric-arithmetic index has a number of interesting properties in [2]. The lower and upper bounds of the geometric-arithmetic index of a connected graph and characterization of graphs for which these bounds are best possible are found in [2]. Janakiraman et al., introduced the concepts of Boolean and Boolean function graphs [3–5]. The Boolean function graph $B(K_p, INC, \overline{K_q})$ of $G$ is a graph with vertex set $V(G) \cup E(G)$ and two vertices in $B(K_p, INC, \overline{K_q})$ are adjacent if and only if they correspond to two adjacent vertices of $G$, two nonadjacent vertices of $G$ or to a vertex and an edge incident to it in

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G. For brevity, this graph is denoted by $B_4(G)$. In this work, lower and upper bounds of $H(B_4(G))$ and $GA(B_4(G))$ are obtained. These indices are found for some particular graphs.

## 2. Prior Results

The following results are found in [3].

1. $K_p$ is an induced subgraph of $B_4(G)$ and subgraph of $B_4(G)$ induced by $q$ vertices is totally disconnected.

2. Number of vertices in $B_4(G)$ is $p + q$, since $B_4(G)$ contains vertices of both $G$ and the line graph $L(G)$ of $G$.

3. Number of edges in $B_4(G)$ is $\left(\frac{n(n-1)}{2}\right) + 2q$.

4. For every vertex $v \in V(G)$, $d_{B_4(G)}(v) = p - 1 + d_G(v)$.
   
   (a). If $G$ is complete, then $d_{B_4(G)}(v) = 2(p - 1)$.
   
   (b). If $G$ is totally disconnected, then $d_{B_4(G)}(v) = p - 1$.
   
   (c). If $G$ has at least one edge, then $2 \leq d_{B_4(G)}(v) \leq 2(p - 1)$ and $d_{B_4(G)}(v) = 1$ if and only if $G \cong 2K_1$.

5. For an edge $e \in E(G)$, $d_{B_4(G)}(e) = 2$.

6. $B_4(G)$ is always connected.

7. If $G$ is a graph with at least three vertices, then each vertex of $B_4(G)$ lies on a triangle and hence girth of $B_4(G)$ is 2.

8. If $G$ is a graph with at least four vertices and at least one edge, then $B_4(G)$ is bi-regular if and only if $G$ is regular and $B_4(G)$ is regular if and only if $G$ is totally disconnected.

9. If $G$ is a graph with at least three vertices, then $B_4(G)$ has no cut vertices.

10. If $G$ has at least one edge, then vertex connectivity of $B_4(G)$ = edge connectivity of $B_4(G) = 2$.

11. Let $G$ be a $(p, q)$ graph with at least one edge. If $p$ is odd, then $B_4(G)$ is Eulerian if and only if $G$ is Eulerian.

12. If $G$ is $r$-regular ($r \geq 1$ and is odd), then $B_4(G)$ is Eulerian.

13. For any graph $G$, $B_4(G)$ is geodetic if and only if $G$ is either $K_2$ or $nK_1$, $n \geq 2$.

14. If $G$ is a graph with at least four vertices, then $B_4(G)$ is $P_4$-free.

## 3. Harmonic and Geometric-Arithmetic Indices in $B_4(G)$

In this section, bounds of $H(B_4(G))$ and $GA(B_4(G))$ are obtained and these indices are found for some particular graphs.

**Theorem 3.1.** Let $G$ be a $(p, q)$ graph. Then $\frac{p^2 + 8q}{4p} \leq H(B_4(G)) \leq \frac{2(p + 1) + 8q}{2(p + 1)}$.

**Proof.** Let $v_1, v_2, \ldots, v_p$ be the vertices and $e_{ij} = (v_i, v_j)$, $i \neq j$ be the edges of $G$. Then $v_1, v_2, \ldots, v_p, e_{ij} \in V(B_4(G))$.

Let $F = \{(v_i, e_{ij}) : e_{ij} \in E(G) \}$ be incident with $v_i \in G, i = 1, 2, \ldots, n, i \neq j$. Then $F \subseteq E(B_4(G))$ and $|F| = 2q$.

$E(B_4(G)) = E(K_p) \cup F$. $d_{B_4}(v) = p - 1 + d_G(v)$ and $d_{B_4}(e_{ij}) = 2$. Number of edges in $B_4(G)$ is $\frac{p(p - 1)}{2} + 2q$. $H(B_4(G))$ is
equal to the sum of \( \frac{2}{d_{B_4}(v_i) + d_{B_4}(v_j)} \), where the sum is taken over all the edges in \( B_4(G) \). To compute \( H(B_4(G)) \), the sum is partitioned into two sums \( T_1 \) and \( T_2 \). The sum \( T_1 \) is taken over all edges \((u, v) \in E(K_p)\).

\[
T_1 = \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{2}{d_{B_4}(v_i) + d_{B_4}(v_j)}
= \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{2}{p - 1 + d_G(v_i) + p - 1 + d_G(v_j)}
= \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{2}{2p - 2 + d_G(v_i) + d_G(v_j)}
\]

The sum \( T_2 \) is taken over all the \( 2q \) edges in \( F \).

\[
T_2 = \sum_{(v_i, v_j) \in F} \frac{2}{d_{B_4}(v_i) + d_{B_4}(v_j)} = \sum_{(v_i, v_j) \in F} \frac{2}{p - 1 + d_G(v_i) + 2} = \sum_{(v_i, v_j) \in F} \frac{2}{p + 1 + d_G(v_i)}
\]

Therefore,

\[
H(B_4(G)) = T_1 + T_2
= \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{2}{2p - 2 + d_G(v_i) + d_G(v_j)} + \sum_{(v_i, v_j) \in F} \frac{2}{p + 1 + d_G(v_i)}
\]

But, \( 0 \leq d_G(v_i) \leq p - 1 \). Therefore

\[
H(B_4(G)) \geq \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{2}{2p - 2 + p - 1 + p - 1} + \sum_{(v_i, v_j) \in F} \frac{2}{p + 1 + p - 1}
= \frac{p(p - 1)}{2} \frac{2}{4(p - 1)} + \frac{2q}{p} = \frac{p^2 + 8q}{4p}
\]

Also

\[
H(B_4(G)) \leq \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{2}{2p - 2} + \sum_{(v_i, v_j) \in F} \frac{2}{p + 1}
= \frac{p(p - 1)}{2} \frac{1}{p - 1} + \frac{4q}{p + 1} - \frac{p(p + 1) + 8q}{2(p + 1)}
\]

Therefore, for any graph \( G \),

\[
\frac{p^2 + 8q}{4p} \leq H(B_4(G)) \leq \frac{p(p + 1) + 8q}{2(p + 1)}
\]

Lower bound is attained, if \( G \cong K_p \) and the upper bound is attained if \( G \) is totally disconnected.

Remark 3.2.

(1) If \( G \) is a tree on \( p \) vertices, then \( \frac{p^2 + 8(p - 1)}{4p} \leq H(B_4(G)) \leq \frac{p(p + 1) + 8(p - 1)}{2(p + 1)} \).

(2) If \( G \) is unicyclic on \( p \) vertices, then \( \frac{p^2 + 8}{4} \leq H(B_4(G)) \leq \frac{p(p + 2)}{2(p + 1)} \).

In the following, bounds of Geometric-Arithmetic index of \( B_4(G) \) are found.

Theorem 3.3. Let \( G \) be any \((p, q)\) graph with no isolated vertices. Then

\[
\frac{p^2}{4} + \frac{2q\sqrt{2}}{\sqrt{p}} \leq GA(B_4(G)) \leq (p - 1)^2 + \frac{8q\sqrt{p - 1}}{p + 2}.
\]


Let $v_1, v_2, \ldots, v_p$ be the vertices and $e_{ij} = (v_i, v_j), i \neq j$ be the edges of $G$. Then $v_1, v_2, \ldots, v_p, e_{ij} \in V(B_4(G))$. Let $F = \{(v_i, e_{ij}) : e_{ij} \in E(G) \text{ is incident with } v_i \in G, i = 1, 2, \ldots, n, i \neq j\}$. Then $F \subseteq E(B_4(G))$ and $|F| = 2q$.

Let $F = \{(v_i, e_{ij}) : e_{ij} \in E(G) \text{ is incident with } v_i \in G, i = 1, 2, \ldots, n, i \neq j\}$. Then $F \subseteq E(B_4(G))$ and $|F| = 2q$.

Let $F = \{(v_i, e_{ij}) : e_{ij} \in E(G) \text{ is incident with } v_i \in G, i = 1, 2, \ldots, n, i \neq j\}$. Then $F \subseteq E(B_4(G))$ and $|F| = 2q$.

Let $F = \{(v_i, e_{ij}) : e_{ij} \in E(G) \text{ is incident with } v_i \in G, i = 1, 2, \ldots, n, i \neq j\}$. Then $F \subseteq E(B_4(G))$ and $|F| = 2q$.
Theorem 3.7. Harmonic index of $B_4(P_n)$ is given by,

$$H(B_4(P_n)) = 4(n - 2) \left[ \frac{1}{2n + 1} + \frac{1}{n + 3} + \frac{n - 3}{8(n + 1)} \right] + \frac{5n + 2}{n(n + 2)}, \quad n \geq 3.$$  

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of $P_n$ with $v_1$ and $v_n$ as pendant vertices, and let $e_{i,i+1} = (v_i, v_{i+1})$, $i = 1, 2, \ldots, n-1$ be the edges of $P_n$. Then $v_1, v_2, \ldots, v_n, e_{i,i+1} \in B_4(P_n)$. $d_{B_4}(v_1) = d_{B_4}(v_n) = n$, $d_{B_4}(v_i) = n + 1$, $i = 2, 3, \ldots, n-1$ and $d_{B_4}(e_{i,i+1}) = 2$. Number of edges in $B_4(P_n) = \frac{n(n-1)}{2} + 2(n - 1)$. Let $F_1 = \bigcup_{i=2}^{n-1} \{ (v_1, v_i), (v_i, v_n) \}$, $F_2 = \{ (v_1, v_n) \}$, $F_3 = \{ (v_1, v_2), (v_n, e_{n-1,n}) \}$, $F_4 = \bigcup_{i=2}^{n-1} \bigcup_{j<i} \{ v_i, v_j \}$, $F_5 = \bigcup_{i=2}^{n-1} \{ (v_i, e_{i-1,i}), (v_i, e_{i,i+1}) \}$. Then $E(B_4(P_n)) = \bigcup_{i=1}^{5} F_i$. $H(B_4(P_n))$ is equal to the sum of $\frac{2}{d_{B_4}(v) + d_{B_4}(v)}$, where the summation is taken over all edges $xy \in E(B_4(P_n))$. The sum $T_i$ is taken over all edges $xy \in F_i$.

$$T_1 = \sum_{i=2}^{n-1} \frac{2}{d_{B_4}(v_1) + d_{B_4}(v_i)} + \sum_{i=2}^{n-1} \frac{2}{d_{B_4}(v_i) + d_{B_4}(v_n)}$$

$$= 2(n - 2) \frac{2}{2n + 1} = \frac{4(n - 2)}{2n + 1},$$

$$T_2 = \frac{2}{d_{B_4}(v_n) + d_{B_4}(v_1)} = \frac{2}{2n} = \frac{1}{n},$$

$$T_3 = \frac{2}{d_{B_4}(v_1) + d_{B_4}(e_{12})} + \frac{2}{d_{B_4}(v_n) + d_{B_4}(e_{i-1,i})} = \frac{4}{n + 2},$$

$$T_4 = \sum_{i=2}^{n-1} \left( \frac{2}{d_{B_4}(v_i) + d_{B_4}(e_{i-1,i})} + \frac{2}{d_{B_4}(v_i) + d_{B_4}(e_{i,i+1})} \right) = \frac{4(n - 2)}{n + 3},$$

$$T_5 = \sum_{i=2}^{n-1} \sum_{j<i} \frac{2}{d_{B_4}(v_i) + d_{B_4}(v_j)} = \frac{(n - 2)(n - 3)}{2(n + 1)},$$

$$H(B_4(P_n)) = \sum_{i=1}^{5} T_i = 4(n - 2) \left[ \frac{1}{2n + 1} + \frac{1}{n + 3} + \frac{n - 3}{8(n + 1)} \right] + \frac{5n + 2}{n(n + 2)}$$

Remark 3.8. Geometric Arithmetic index of $GA(B_4(P_n))$ is given by

$$GA(B_4(P_n)) = 1 + 4(n - 2)\sqrt{n + 1} \left[ \sqrt{\frac{n}{2n + 1}} + \sqrt{\frac{2}{n + 3}} \right] + 4\sqrt{\frac{2n}{n + 2}} \left( \frac{n - 2}{n - 3} \right) \frac{2}{2}$$

Theorem 3.9. Harmonic index of $B_4(K_n)$ is given by, $H(B_4(K_n)) = \frac{5n-4}{4}$.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of $K_n$, and let $e_{i,j} = (v_i, v_j)$, $i, j = 1,2,\ldots,n$, $i < j$ be the edges of $K_n$. Then $v_1, v_2, \ldots, v_n, e_{i,j} \in B_4(K_n)$. $d_{B_4}(v_1) = 2n - 2$, $d_{B_4}(e_{i,j}) = 2$. Number of edges in $B_4(K_n) = \frac{3n(n-1)}{2}$. Let $F_i = \{ (v_i, e_{ij}) : j = 1,2,\ldots,n, i \neq j \}$ and $F = \bigcup_{i=1}^{n} F_i$; $E(B_4(K_n)) = E(K_n) \cup F$. $H(B_4(K_n))$ is equal to the sum of $\frac{2}{d_{B_4}(v) + d_{B_4}(v)}$, where the summation is taken over all edges $xy \in E(B_4(K_n))$. The sum $T_i$ is taken over all edges $xy \in F_i$, $i = 1,2$.

$$T_1 = \sum_{i=1}^{n} \sum_{j<i,j} d_{B_4}(v_i) + d_{B_4}(v_j) \frac{2}{2} = \frac{n(n-1)}{2} \frac{2}{2n - 2} = \frac{n}{4},$$

$$T_2 = n(n-1) \frac{2}{2n - 2} = n - 1.$$  

Therefore $H(B_4(K_n)) = \frac{n}{4} + n - 1 = \frac{5n-4}{4}$.  

Remark 3.10. Geometric Arithmetic index of $B_4(K_n)$ is given by $GA(B_4(K_n)) = \frac{n+4}{n+1} \left[ n + 4\sqrt{n-1} \right]$.  

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Theorem 3.11. Harmonic index of $B_4(K_{1,n})$ is given by,

$$H(B_4(K_{1,n})) = n \left[ \frac{8(n+1)}{(3n+1)(n+3)} + \frac{1}{2} \right], \quad n \geq 3.$$ 

Proof. Let $v_1, v_2, \ldots, v_{n+1}$ be the vertices of $K_{1,n}$ with $v_1$ as the central vertex and let $e_{1,i} = (v_1, v_i), i = 2, 3, \ldots, n+1$ be the edges of $K_{1,n}$. Then $v_1, v_2, \ldots, v_{n+1}, e_{1,i} \in B_4(K_{1,n})$. $d_{B_4}(v_1) = 2n$, $d_{B_4}(v_i) = n + 1, i = 2, 3, \ldots, n+1$, $d_{B_4}(e_{1,i}) = 2$.

Number of edges in $B_4(K_{1,n}) = \binom{n+1}{2} + 1$. Let $F_1 = \bigcup_{i=2}^{n+1} \{v_1, v_i\}$; $F_2 = \bigcup_{i=2}^{n+1} \bigcup_{j<i} \{v_i, v_j\}$; $F_3 = \bigcup_{i=2}^{n+1} \{v_1, e_{1,i}\}$ and $F_4 = \bigcup_{i=2}^{n+1} \{v_i, e_{1,i}\}$, $F = \bigcup_{i=1}^{4} F_i$, $E(B_4(K_{1,n})) = F$. $H(B_4(K_{1,n}))$ is equal to the sum of $\frac{1}{d_{B_4}(u)+d_{B_4}(v)}$, where the summation is taken over all edges $xy \in E(B_4(K_{1,n}))$. The sum $T_i$ is taken over all edges $xy \in F_i, i = 1, 2, 3, 4$.

$$T_1 = \sum_{i=2}^{n+1} \frac{2}{d_{B_4}(v_i) + d_{B_4}(v_1)} = \frac{2}{2n + n + 1} = \frac{2n}{3n+1}$$

$$T_2 = \sum_{i=2}^{n+1} \sum_{j=2 \atop j<i}^{n+1} \frac{2}{d_{B_4}(v_i) + d_{B_4}(v_j)} = \frac{n(n-1)}{2} \cdot \frac{2}{2(n+1)} = \frac{n(n-1)}{2(n+1)}$$

$$T_3 = \sum_{i=2}^{n+1} \frac{2}{d_{B_4}(v_i) + d_{B_4}(e_{1,i})} = \frac{2n}{2n + 2} = \frac{n}{n+1}$$

$$T_4 = \sum_{i=2}^{n+1} \frac{2}{d_{B_4}(v_i) + d_{B_4}(e_{1,i})} = \frac{2n}{n+3}$$

$$H(B_4(K_{1,n})) = \sum_{i=1}^{4} T_i = \frac{2n}{3n+1} + \frac{n(n-1)}{2(n+1)} + \frac{n}{n+1} + \frac{2n}{n+3}$$

$$= n \left[ \frac{8(n+1)}{(3n+1)(n+3)} + \frac{1}{2} \right] \quad \square$$

Remark 3.12. Geometric Arithmetic index of $B_4(K_{1,n})$ is given by

$$GA(B_4(K_{1,n})) = \frac{2n \sqrt{2n(n+1)}}{3n+1} + \frac{n(n-1)\sqrt{(n+1)(n+1)}}{2(n+1)} + \frac{n \sqrt{2(2n)}}{n+1} + \frac{2n \sqrt{2(n+1)}}{n+3}$$

$$= \frac{2n \sqrt{2n(n+1)}}{3n+1} + \frac{n(n-1)}{2} + \frac{2n \sqrt{2(n+1)}}{n+3}$$

Theorem 3.13. Harmonic index of $B_4(K_{m,n})$: $(m, n \geq 2)$ is given by,

$$H(B_4(K_{m,n})) = \frac{m(m-1)}{2(m+2n-1)} + \frac{n(n-1)}{2(2m+n-1)} + \frac{mn}{(m+2n+1)(2m+n+1)}$$

$$+ \frac{2mn}{3m+3n-2}.$$ 

Proof. Let $[A, B]$ be the bipartition of $K_{m,n}$ and let $A = \{v_1, v_2, \ldots, v_m\}$ and $B = \{u_1, u_2, \ldots, u_n\}$. Let $e_{ij} = (v_i, u_j), i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$ be the edges of $K_{m,n}$. Then $v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_n, e_{ij} \in V(B_4(K_{m,n}))$ and

$$E(B_4(K_{m,n})) = E(K_{m+n}) \cup \bigcup_{i=1}^{m} \left( \bigcup_{j=1}^{n} \{v_i, e_{ij}\} \right) \cup \left( \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} \{u_i, e_{ji}\} \right)$$

and $E(K_{m+n}) = E(K_m) \cup E(K_n) \cup \{v_i, u_j\}, i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

$$d_{B_4}(v_i) = p - 1 + d_{K_{m,n}}(v_i) = m + n - 1 + n = m + 2n - 1$$

$$d_{B_4}(u_i) = p - 1 + d_{K_{m,n}}(u_i) = m + n - 1 + m = 2m + n - 1$$

$$d_{B_4}(e_{ij}) = 2$$
Let $F_1$ and $F_2$ be the set of edges in $K_m$ and $K_n$ induced by the vertices $v_1, v_2, \ldots, v_m$ and $u_1, u_2, \ldots, u_n$ respectively. $F_3 = \{(v_i, u_j), i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n\}$; $F_4 = \left( \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} (v_i, e_{ij}) \right)$ and $F_5 = \left( \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} (u_i, e_{ij}) \right)$. Therefore, $E(K_{m,n}) = \bigcup_{i=1}^{5} F_i$. $H(B_4(K_{m,n}))$ is equal to the sum of $2 \sum_{u \in u} d_B(u^+) + 2 \sum_{v \in v} d_B(v^+)$, where the summation is taken over all edges $xy \in E(B_4(K_{m,n}))$. The sum $T_i$ is taken over all edges $xy \in F_i$; $i = 1, 2, 3, 5$.

\begin{align*}
T_1 &= \frac{m(m-1)}{2} 
&= \frac{m(m-1)}{2(m+2n-1)} \quad T_2 = \frac{n(n-1)}{2} 
&= \frac{n(n-1)}{2(2m+n-1)} \quad T_3 = \frac{2mn}{d_{B_4}(u_i) + d_{B_4}(v_j)} 
&= \frac{2mn}{3m+3n-2} \quad T_4 = \frac{2}{d_{B_4}(v_i)} + \frac{2}{d_{B_4}(e_{ij})} 
&= \frac{2m + n - 1 + 2}{m + 2n + 1} \quad T_5 = \frac{2}{d_{B_4}(u_i) + d_{B_4}(e_{ij})} 
&= \frac{2m + n + 1}{2m + n + 1}
\end{align*}

Therefore,

\[H(B_4(K_{m,n})) = \sum_{i=1}^{5} T_i \]

\[= \frac{m(m-1)}{2(m+2n-1)} + \frac{n(n-1)}{2(2m+n-1)} + \frac{2mn}{3m+3n-2} + \frac{2mn}{m+2n+1} + \frac{2mn}{3m+3n-2}
\]

Remark 3.14.

(1) If $m = n$, then $H(B_4(K_{n,n})) = \frac{n(2n-1)}{3n-1} + \frac{4n^2}{3n^2+1}$.

(2) Geometric-Arithmetic index of $B_4(K_{m,n})$ is given by

\[GA(B_4(K_{m,n})) = \frac{m(m-1)}{2} + \frac{n(n-1)}{2} + \frac{2mn}{3m+3n-2} + \frac{2mn}{m+2n+1} + \frac{2mn}{3m+3n-2}
\]

(3) If $m = n$, then Geometric-Arithmetic index of $B_4(K_{n,n})$ is given by $GA(B_4(K_{n,n})) = n(2n-1) + \frac{4n^2}{3n^2+1}$.

Theorem 3.15. Harmonic index of $B_4(W_n)$ is given by,

\[H(B_4(W_n)) = (n-1) \left[ \frac{5}{3n} + \frac{n-2}{2(n+2)} + \frac{6}{n+4} \right], \quad n \geq 5
\]

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of $W_n$ with $v_1$ as the central vertex and let $e_{i,i+1} = (v_i, v_{i+1})$, $i = 2, 3, \ldots, n-1$, $e_{n,2} = (v_n, v_2)$ be the edges of $W_n$. Then $v_1, v_2, \ldots, v_n, e_{1,i}, e_{i,i+1} \in B_4(W_n)$, $d_{B_4}(v_1) = 2(n-1)$, $d_{B_4}(v_i) = n + 2$, $i = 2, 3, \ldots, n$, $d_{B_4}(e_{1,i}) = d_{B_4}(e_{i,i+1}) = d_{B_4}(e_{n,2}) = 2$. Number of edges in $B_4(W_n) = \frac{(n-1)(n+8)}{2}$. Let

\[F_1 = \bigcup_{i=2}^{n} \{(v_1, v_i)\}, \quad F_2 = \bigcup_{i=2}^{n} \bigcup_{j=2}^{n} \{(v_i, v_j)\}, \quad F_3 = \bigcup_{i=2}^{n} \{(v_1, e_{1,i})\}, \quad F_4 = \bigcup_{i=2}^{n} \{(v_i, e_{1,i})\}
\]

\[F_5 = \bigcup_{i=3}^{n-1} \{(v_1, e_{i-1,i}), (v_i, e_{i,i+1})\}
\]

\[F_6 = \{(v_2, e_{2,3}), (v_2, e_{n,2}), (v_n, e_{n,2}), (v_n, e_{n-1,n})\}
\]
$H(B_4(W_n))$ is equal to the sum of $\frac{2}{d_B(v_i) + d_B(v_j)}$, where the summation is taken over all edges $xy \in E(B_4(W_n))$. The sum $T_i$ is taken over all edges $xy \in F_i$, $i = 1, 2, \ldots, 6$.

$T_1 = \sum_{i=2}^n \frac{2}{d_B(v_i) + d_B(v_i)} = \sum_{i=2}^n \frac{2}{2(n-1) + n + 2} = \frac{2(n-1)}{3n}$

$T_2 = \sum_{i=2}^n \sum_{j<i} \frac{2}{d_B(v_i) + d_B(v_j)} = \sum_{i=2}^n \frac{2}{2(n-1) + n + 2} (\frac{1}{n-1} \frac{1}{n-2}) \frac{2}{2} = \frac{(n-1)(n-2)}{2(n+2)}$

$T_3 = \sum_{i=2}^n \frac{2}{d_B(v_i) + d_B(e_{ii})} = \frac{(n-1)}{2(n-1) + 2} = \frac{n-1}{n}$

$T_4 = \sum_{i=2}^n \frac{2}{d_B(v_i) + d_B(e_{ii})} = \frac{2(n-1)}{n+4}$

$T_5 = \sum_{i=3}^{n-1} \left( \frac{2}{d_B(v_i) + d_B(e_{i-1,i})} + \frac{2}{d_B(v_i) + d_B(e_{i,i+1})} \right) = 2(n-3) \frac{2}{n+2} = \frac{4(n-3)}{n+4}$

$T_6 = \frac{2}{d_B(v_2) + d_B(e_{21})} + \frac{2}{d_B(v_2) + d_B(e_{n2})} + \frac{2}{d_B(v_n) + d_B(e_{n-1,n})} + \frac{2}{d_B(v_n) + d_B(e_{n2})} = \frac{8}{n+4}$

Therefore

$H(B_4(W_n)) = \sum_{i=1}^6 T_i = \frac{2(n-1)}{3n} + \frac{(n-1)(n-2)}{2(n+2)} + \frac{n-1}{n} + \frac{2(n-1)}{n+4} + \frac{4(n-3)}{n+4} + \frac{8}{n+4}$

$$= (n-1) \left[ \frac{5}{3n} + \frac{n-2}{2(n+2)} + \frac{6}{n+4} \right]$$

\[ \square \]

**Remark 3.16.** Geometric Arithmetic index of $B_4(W_n)$ is given by

$$GA(B_4(W_n)) = \frac{2(n-1)^{3/2}}{n} \left[ 1 + \sqrt{2(n+2)} \right] + \frac{(n-1)(n-2)}{2} + \frac{6(n-1)\sqrt{2(n+2)}}{n+4}, \text{ } n \geq 5.$$