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Harmonic and Geometric-Arithmetic Indices of Boolean Function Graph $B(K_p, INC, \overline{K_q})$

Research Article

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Abstract: For any graph G, let $V(G)$ and $E(G)$ denote the vertex set and edge set of G respectively. The Harmonic index $H(G)$ of a graph G is defined as the sum of the weights $\frac{2}{d(u)+d(v)}$ of all edges uv of G and the Geometric-Arithmetic index $GA(G)$ of G is defined as the sum of the weights $\frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)}$, where $d(u)$ denotes the degree of a vertex u in G. The Boolean function graph $B(K_p, INC, \overline{K_q})$ of G is a graph with vertex set $V(G) \cup E(G)$ and two vertices in $B(K_p, INC, \overline{K_q})$ are adjacent if and only if they correspond to two adjacent vertices of G, two nonadjacent vertices of G or to a vertex and an edge incident to it in G, For brevity, this graph is denoted by $B_4(G)$. In this paper, lower and upper bounds of $H(B_4(G))$ and $GA(B_4(G))$ are obtained. These indices are found for some particular graphs.

Keywords: Harmonic index, geometric-arithmetic index, Boolean Function Graph.

1. Introduction

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For any two vertices $u, v \in V(G)$, the distance between u and v, denoted by $d_G(u, v)$, is defined as the number of edges in the shortest path connecting u and v. The eccentricity of a vertex v, denoted by $e_G(v)$, is the largest distance of v and any other vertex u of G. The degree of a vertex v, denoted by $d_G(v)$ (or simply $d(u)$) is the number of vertices adjacent with the vertex v. For a graph G, the harmonic index $H(G)$ is defined as $H(G) = \sum\limits_{uv \in E(G)} \frac{2}{d(u)+d(v)}$. Favaron et al. [1] considered the relation between the harmonic index and the eigen values of graphs.

In 2012, Zhong [9] reintroduced this index as harmonic index and found the minimum and maximum values of the harmonic index for simple connected graphs and trees, and characterized the corresponding extremal graphs. Wu et al. [8] gave the minimum value of the Harmonic index among the graphs with the minimum degree at least two. In [7], Vukicevic et al. defined a new topological index geometric-arithmetic index of a graph G, denoted by $GA(G)$ and is defined by $GA(G) = \sum\limits_{uv \in E(G)} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)}$.

The geometric-arithmetic index has a number of interesting properties in [2]. The lower and upper bounds of the geometric-arithmetic index of a connected graph and characterization of graphs for which these bounds are best possible are found in [2]. Janakiraman et al., introduced the concepts of Boolean and Boolean function graphs [3–5]. The Boolean function graph $B(K_p, INC, \overline{K_q})$ of G is a graph with vertex set $V(G) \cup E(G)$ and two vertices in $B(K_p, INC, \overline{K_q})$ are adjacent if and only if they correspond to two adjacent vertices of G, two nonadjacent vertices of G or to a vertex and an edge incident to it in

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G, For brevity, this graph is denoted by $B_4(G)$. In this work, lower and upper bounds of $H(B_4(G))$ and $GA(B_4(G))$ are obtained. These indices are found for some particular graphs.

2. Prior Results

The following results are found in [3].

(1). K_p is an induced subgraph of $B_4(G)$ and subgraph of $B_4(G)$ induced by q vertices is totally disconnected.

(2). Number of vertices in $B_4(G)$ is $p+q$, since $B_4(G)$ contains vertices of both G and the line graph $L(G)$ of G.

(3). Number of edges in $B_4(G)$ is $\left(\frac{p(p-1)}{2}\right)+2q$.

(4). For every vertex $v \in V(G)$, $d_{B4(G)}(v) = p-1+d_G(v)$.

(a). If G is complete, then $d_{B4(G)}(v) = 2(p-1)$.

(b). If G is totally disconnected, then $d_{B4(G)}(v) = p-1$.

(c). If G has atleast one edge, then $2 \leq d_{B4(G)}(v) \leq 2(p-1)$ and $d_{B4(G)}(v) = 1$ if and only if $G \cong 2K_1$.

(5). For an edge $e \in E(G)$, $d_{B4(G)}(e) = 2$.

(6). $B_4(G)$ is always connected.

(7). If G is a graph with atleast three vertices, then each vertex of $B_4(G)$ lies on a triangle and hence girth of $B_4(G)$ is 2.

(8). If G is a graph with atleast four vertices and atleast one edge, then $B_4(G)$ is bi-regular if and only if G is regular and $B_4(G)$ is regular if and only if G is totally disconnected.

(9). If G is a graph with atleast three vertices, then $B_4(G)$ has no cut vertices.

(10). If G has atleast one edge, then vertex connectivity of $B_4(G)$ =edge connectivity of $B_4(G) = 2$.

(11). Let G be a (p, q) graph with atleast one edge. If p is odd, then $B_4(G)$ is Eulerian if and only ifG is Eulerian.

(12). If G is r-regular ($r \geq 1$ and is odd), then $B_4(G)$ is Eulerian.

(13). For any graph G, $B_4(G)$ is geodetic if and only if G is either K_2 or nK_1, $n \geq 2$.

(14). If G is a graph with atleast four vertices, then $B_4(G)$ is P_4-free.

3. Harmonic and Geometric-Arithmetic Indices in $B_4(G)$

In this section, bounds of $H(B_4(G))$ and $GA(B_4(G))$ are obtained and these indices are found for some particular graphs.

Theorem 3.1. *Let G be a (p, q) graph. Then* $\frac{p^2+8q}{4p} \leq H(B_4(G)) \leq \frac{p(p+1)+8q}{2(p+1)}$.

Proof. Let $v_1, v_2, \ldots, v_p$ be the vertices and $e_{ij} = (v_i, v_j)$, $i \neq j$ be the edges of G. Then $v_1, v_2, \ldots, v_p$, $e_{ij} \in V(B_4(G))$. Let $F = \{(v_i, e_{ij}) : e_{ij} \in E(G) \quad \text{is incident with } v_i \in G, i = 1, 2, \ldots, n,\ i \neq j\}$. Then $F \subseteq E(B_4(G))$ and $|F| = 2q$. $E(B_4(G)) = E(K_p) \cup F$. $d_{B4}(v) = p-1+d_G(v)$ and $d_{B4}(e_{ij}) = 2$. Number of edges in $B_4(G)$ is $\frac{p(p-1)}{2}+2q$. $H(B_4(G))$ is equal to the sum of $\frac{2}{d_{B_4}(u)+d_{B_4}(v)}$, where the sum is taken over all the edges in $B_4(G)$. To compute $H(B_4(G))$, the sum is partitioned into two sums T_1 and T_2. The sum T_1 is taken over all edges (u,v) in $E(K_p)$.

$$\begin{aligned} T_1 &= \sum_{i=1}^{p}\sum_{\substack{j=1,\\ i<j}}^{p} \frac{2}{d_{B4}(v_i)+d_{B4}(v_j)} \\ &= \sum_{i=1}^{p}\sum_{\substack{j=1,\\ i<j}}^{p} \frac{2}{p-1+d_G(v_i)+p-1+d_G(v_j)} \\ &= \sum_{i=1}^{p}\sum_{\substack{j=1,\\ i<j}}^{p} \frac{2}{2p-2+d_G(v_i)+d_G(v_j)} \end{aligned}$$

The sum T_2 is taken over all the $2q$ edges in F.

$$T_2 = \sum_{(v_i,e_{ij})\in F} \frac{2}{d_{B4}(v_i)+d_{B4}(e_{ij})} = \sum_{(v_i,e_{ij})\in F} \frac{2}{p-1+d_G(v_i)+2} = \sum_{(v_i,e_{ij})\in F} \frac{2}{p+1+d_G(v_i)}$$

Therefore,

$$\begin{aligned} H(B_4(G)) &= T_1 + T_2 \\ &= \sum_{i=1}^{p}\sum_{\substack{j=1,\\ i<j}}^{p} \frac{2}{2p-2+d_G(v_i)+d_G(v_j)} + \sum_{(v_i,e_{ij})\in F} \frac{2}{p+1+d_G(v_i)} \end{aligned}$$

But, $0 \le d_G(v_i) \le p-1$. Therefore

$$\begin{aligned} H(B_4(G)) &\ge \sum_{i=1}^{p}\sum_{\substack{j=1,\\ i<j}}^{p} \frac{2}{2p-2+p-1+p-1} + \sum_{(v_i,e_{ij})\in F} \frac{2}{p+1+p-1} \\ &= \frac{p(p-1)}{2}\frac{2}{4(p-1)} + \frac{2q}{p} = \frac{p^2+8q}{4p} \end{aligned}$$

Also

$$\begin{aligned} H(B_4(G)) &\le \sum_{i=1}^{p}\sum_{\substack{j=1,\\ i<j}}^{p} \frac{2}{2p-2} + \sum_{(v_i,e_{ij})\in F} \frac{2}{p+1} \\ &= \frac{p(p-1)}{2}\frac{1}{p-1} + \frac{4q}{p+1} = \frac{p(p+1)+8q}{2(p+1)} \end{aligned}$$

Therefore, for any graph G,

$$\frac{p^2+8q}{4p} \le H(B_4(G)) \le \frac{p(p+1)+8q}{2(p+1)}$$

Lower bound is attained, if $G \cong K_p$ and the upper bound is attained if G is totally disconnected. □

Remark 3.2.

(1). If G is a tree on p vertices, then $\frac{p^2+8(p-1)}{4p} \le H(B_4(G)) \le \frac{p(p+1)+8(p-1)}{2(p+1)}$.

(2). If G is unicyclic on p vertices, then $\frac{p+8}{4} \le H(B_4(G)) \le \frac{p(p+q)}{2(p+1)}$.

In the following, bounds of Geometric-Arithmetic index of $B_4(G)$ are found.

Theorem 3.3. *Let G be any (p, q) graph with no isolated vertices. Then*

$$\frac{p^2}{4} + \frac{2q\sqrt{2}}{\sqrt{p}} \le GA(B_4(G)) \le (p-1)^2 + \frac{8q\sqrt{p-1}}{p+2}.$$

Proof. Let $v_1, v_2, \ldots, v_p$ be the vertices and $e_{ij} = (v_i, v_j)$, $i \neq j$ be the edges of G. Then $v_1, v_2, \ldots, v_p$, $e_{ij} \in V(B_4(G))$. Let $F = \{(v_i, e_{ij}) : e_{ij} \in E(G)$ is incident with $v_i \in G$, $i = 1, 2, \ldots, n$, $i \neq j\}$. Then $F \subseteq E(B_4(G))$ and $|F| = 2q$. $E(B_4(G)) = E(K_p) \cup F$. $d_{B4}(v) = p-1+d_G(v)$ and $d_{B4}(e_{ij}) = 2$. $GA(B_4(G))$ is equal to the sum of $\frac{2\sqrt{d_{B4}(u)d_{B4}(v)}}{d_{B4}(u)+d_{B4}(v)}$, where the sum is taken over all the edges in $B_4(G)$.

$$\begin{aligned} GA(B_4(G)) &= \sum_{i=1}^{p}\sum_{\substack{j=1,\\ i<j}}^{p} \frac{2\sqrt{d_{B4}(v_i)d_{B4}(v_j)}}{d_{B4}(v_i)+d_{B4}(v_j)} + \sum_{(v_i,e_{ij})\in F} \frac{2\sqrt{d_{B4}(v_i)d_{B4}(e_{ij})}}{d_{B4}(v_i)+d_{B4}(e_{ij})} \\ &= \sum_{i=1}^{p}\sum_{\substack{j=1,\\ i<j}}^{p} \frac{2\sqrt{(p-1+d_G(v_i))(p-1+d_G(v_j))}}{p-1+d_G(v_i)+p-1+d_G(v_j)} + \sum_{(v_i,e_{ij})\in F} \frac{2\sqrt{2(p-1+d_G(v_i))}}{p-1+d_G(v_i)+2} \end{aligned}$$

But, $1 \leq d_G(v_i) \leq p-1$.

$$GA(B_4(G)) \leq \frac{p(p-1)}{2}\left(\frac{2(p-1)}{2p}\right) + \frac{8q\sqrt{p-1}}{p+2} = (p-1)^2 + \frac{8q\sqrt{p-1}}{p+2}$$

Also,

$$GA(B_4(G)) \geq \frac{p(p-1)}{2}\left(\frac{\sqrt{2p}}{2p-2+p-1+p-1}\right) + 2q\left(\frac{2\sqrt{2p}}{2p}\right) = \frac{p^2}{4} + \frac{2\sqrt{2}q}{\sqrt{q}}$$

Therefore,

$$\frac{p^2}{4} + \frac{2q\sqrt{2}}{\sqrt{p}} \leq GA(B_4(G)) \leq (p-1)^2 + \frac{8q\sqrt{p-1}}{p+2}.$$

□

Remark 3.4.

(1). If G is totally disconnected, then $GA(B_4(G)) = \frac{p(p-1)}{2}$.

(2). If G is unicyclic, then $\frac{p^2}{4} + 2\sqrt{2p} \leq GA(B_4(G)) \leq (p-1)^2 + \frac{8p\sqrt{p-1}}{p+2}$.

In the following, Harmonic and Geometric-Arithmetic indices of Boolean function graphs of cycle, path, complete graph, star and wheel are obtained.

Theorem 3.5. *Harmonic index of* $B_4(C_n)$ *is given by,* $H(B_4(C_n)) = \frac{n(n^2+10n+5)}{2(n+1)(n+3)}$, $n \geq 3$.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of C_n and let $e_{i,i+1} = (v_i, v_{i+1})$, $i = 1, 2, \ldots, n-1$, $e_{n1} = (v_n, v_1)$ be the edges of C_n. Then $v_1, v_2, \ldots, v_n, e_{i,i+1}, e_{n1} \in B_4(C_n)$. $d_{B4}(v_i) = n+1$ and $d_{B4}(e_{i,i+1}) = d_{B4}(e_{n,1}) = 2$. Let $F = \{(v_i, e_{i-1,i}), (v_i, e_{i,i+1}), i = 2, 3, \ldots, n\}$, where $e_{01} = e_{n1}$ and $e_{i,i+1} = e_{n1}$. $|F| = 2n$. Number of edges in $B_4(C_n) = \frac{n(n-1)}{2} + 2n$ and $E(B_4(C_n)) = E((K_n)) \cup F$.

$$\begin{aligned} H(B_4(C_n)) &= \sum_{i=1}^{n}\sum_{\substack{j=1,\\ i<j}}^{n} \frac{2}{d_{B4}(v_i)+d_{B4}(v_j)} + \sum_{(v_i,e_{ij})\in F} \frac{2}{d_{B4}(v_i)+d_{B4}(e_{ij})} \\ &= \frac{n(n-1)}{2(n+1)} + \frac{4n}{n+3} = \frac{n(n^2+10n+5)}{2(n+1)(n+3)} \end{aligned}$$

□

Remark 3.6. *Geometric index of* $B_4(C_n)$ *is given by*

$$\begin{aligned} GA(B_4(C_n)) &= \sum_{i=1}^{n}\sum_{\substack{j=1,\\ i<j}}^{n} \frac{2\sqrt{d_{B4}(v_i)d_{B4}(v_j)}}{d_{B4}(v_i)+d_{B4}(v_j)} + \sum_{(v_i,e_{ij})\in F} \frac{2\sqrt{d_{B4}(v_i)d_{B4}(e_{ij})}}{d_{B4}(v_i)+d_{B4}(e_{ij})} \\ &= \frac{n(n-1)}{2} + \frac{4n\sqrt{2(n+1)}}{n+3} \end{aligned}$$

Theorem 3.7. *Harmonic index of $B_4(P_n)$ is given by,*

$$H(B_4(P_n)) = 4(n-2)\left[\frac{1}{2n+1}+\frac{1}{n+3}+\frac{n-3}{8(n+1)}\right]+\frac{5n+2}{n(n+2)}, \quad n \geq 3.$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of P_n with v_1 and v_n as pendant vertices, and let $e_{i,i+1} = (v_i, v_{i+1})$, $i = 1, 2, \ldots, n-1$ be the edges of P_n. Then $v_1, v_2, \ldots, v_n, e_{i,i+1} \in B_4(P_n)$. $d_{B4}(v_1) = d_{B4}(v_n) = n$, $d_{B4}(v_i) = n+1$, $i = 2, 3, \ldots, n-1$ and $d_{B4}(e_{i,i+1}) = 2$. Number of edges in $B_4(P_n) = \frac{n(n-1)}{2} + 2(n-1)$. Let $F_1 = \bigcup_{i=2}^{n-1}\{(v_1, v_i), (v_n, v_i)\}$, $F_2 = \{(v_1, v_n)\}$, $F_3 = \{(v_1, e_{12}), (v_n, e_{n-1,n})\}$, $F_4 = \bigcup_{i=2}^{n-1}\bigcup_{\substack{j=2,\\ i<j}}^{n-1}\{v_i, v_j)\}$, $F_5 = \bigcup_{i=2}^{n-1}\{(v_i, e_{i-1,i}), (v_i, e_{i,i+1})\}$. Then $E(B_4(P_n)) = \bigcup_{i=1}^{5} F_i$. $H(B_4(P_n))$ is equal to the sum of $\frac{2}{d_{B_4}(u)+d_{B_4}(v)}$, where the summation is taken over all edges $xy \in E(B_4(P_n))$. The sum T_i is taken over all edges $xy \in F_i$.

$$\begin{aligned}
T_1 &= \sum_{i=2}^{n-1}\frac{2}{d_{B4}(v_1)+d_{B4}(v_i)} + \sum_{i=2}^{n-1}\frac{2}{d_{B4}(v_n)+d_{B4}(v_i)}\\
&= 2(n-2)\frac{2}{2n+1} = \frac{4(n-2)}{2n+1}\\
T_2 &= \frac{2}{d_{B4}(v_n)+d_{B4}(v_i)} = \frac{2}{2n} = \frac{1}{n}\\
T_3 &= \frac{2}{d_{B4}(v_1)+d_{B4}(e_{12})} + \frac{2}{d_{B4}(v_n)+d_{B4}(e_{n-1,n})} = \frac{4}{n+2}\\
T_4 &= \sum_{i=2}^{n-1}\left(\frac{2}{d_{B4}(v_i)+d_{B4}(e_{i-1,i})}+\frac{2}{d_{B4}(v_i)+d_{B4}(e_{i,i+1})}\right) = \frac{4(n-2)}{n+3}\\
T_5 &= \sum_{i=2}^{n-1}\sum_{\substack{j=2,\\ i<j}}^{n-1}\frac{2}{d_{B4}(v_i)+d_{B4}(v_j)} = \frac{(n-2)(n-3)}{2(n+1)}\\
H(B_4(P_n)) &= \sum_{i=1}^{5} T_i = 4(n-2)\left[\frac{1}{2n+1}+\frac{1}{n+3}+\frac{n-3}{8(n+1)}\right]+\frac{5n+2}{n(n+2)}
\end{aligned}$$

□

Remark 3.8. *Geometric Arithmetic index of $GA(B_4(P_n))$ is given by*

$$GA(B_4(P_n)) = 1 + 4(n-2)\sqrt{n+1}\left[\frac{\sqrt{n}}{2n+1}+\frac{\sqrt{2}}{n+3}\right]+\frac{4\sqrt{2n}}{n+2}+\frac{(n-2)(n-3)}{2}$$

Theorem 3.9. *Harmonic index of $B_4(K_n)$ is given by, $H(B_4(K_n)) = \frac{5n-4}{4}$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of K_n, and let $e_{i,j} = (v_i, v_j)$, $i, j = 1, 2, \ldots, n$, $i < j$ be the edges of K_n. Then $v_1, v_2, \ldots, v_n, e_{i,j} \in B_4(K_n)$. $d_{B4}(v_i) = 2n-2$, $d_{B4}(e_{i,j}) = 2$. Number of edges in $B_4(K_n) = \frac{3n(n-1)}{2}$. Let $F_i = \{(v_i, e_{ij}) : j = 1, 2, \ldots, n, i \neq j\}$ and $F = \bigcup_{i=1}^{n} F_i$; $E(B4(K_n)) = E(K_n) \cup F$. $H(B_4(K_n))$ is equal to the sum of $\frac{2}{d_{B_4}(u)+d_{B_4}(v)}$, where the summation is taken over all edges $xy \in E(B_4(K_n))$. The sum T_i is taken over all edges $xy \in F_i$, $i = 1, 2$.

$$\begin{aligned}
T_1 &= \sum_{i=1}^{n}\sum_{\substack{j=1\\ i<j}}^{n}\frac{2}{d_{B4}(v_i)+d_{B4}(v_j)}\\
&= \frac{n(n-1)}{2}\frac{2}{2(2n-2)} = \frac{n}{4}\\
T_2 &= n(n-1)\frac{2}{2n-2+2} = n-1
\end{aligned}$$

Therefore $H(B_4(K_n)) = \frac{n}{4} + n - 1 = \frac{5n-4}{4}$. □

Remark 3.10. *Geometric Arithmetic index of $B_4(K_n)$ is given by $GA(B_4(K_n)) = \frac{n-1}{2}\left[n + 4\sqrt{n-1}\right]$.*

Theorem 3.11. *Harmonic index of $B_4(K_{1,n})$ is given by,*

$$H(B_4(K_{1,n})) = n\left[\frac{8(n+1)}{(3n+1)(n+3)} + \frac{1}{2}\right],\ n \geq 3.$$

Proof. Let $v_1, v_2, \ldots, v_{n+1}$ be the vertices of $K_{1,n}$ with v_1 as the central vertex. and let $e_{1,i} = (v_1, v_i)$, $i = 2, 3, \ldots, n+1$ be the edges of $K_{1,n}$. Then $v_1, v_2, \ldots, v_{n+1}, e_{1,i} \in B_4(K_{1,n})$. $d_{B4}(v_1) = 2n$, $d_{B4}(v_i) = n+1$, $i = 2, 3, \ldots, n+1$, $d_{B4}(e_{1,i}) = 2$. Number of edges in $B_4(K_{1,n}) = \frac{n(n+5)}{2}$. Let $F_1 = \bigcup_{i=2}^{n+1}\{(v_1, v_i)\}$; $F_2 = \bigcup_{i=2}^{n+1}\bigcup_{\substack{j=2\\ i<j}}^{n+1}\{(v_i, v_j)\}$; $F_3 = \bigcup_{i=2}^{n+1}\{(v_1, e_{1,i})\}$ and $F_4 = \bigcup_{i=2}^{n+1}\{(v_i, e_{1,i})\}$, $F = \bigcup_{i=1}^{4} F_i$, $E(B_4(K_{1,n})) = F$. $H(B_4(K_{1,n}))$ is equal to the sum of $\frac{2}{d_{B_4}(u)+d_{B_4}(v)}$, where the summation is taken over all edges $xy \in E(B_4(K_{1,n}))$. The sum T_i is taken over all edges $xy \in F_i$, $i = 1, 2, 3, 4$.

$$T_1 = \sum_{i=2}^{n+1} \frac{2}{d_{B4}(v_1) + d_{B4}(v_i)} = \sum_{i=2}^{n+1} \frac{2}{2n+n+1} = \frac{2n}{3n+1}$$

$$T_2 = \sum_{i=2}^{n+1}\sum_{\substack{j=2,\\ i<j}}^{n+1} \frac{2}{d_{B4}(v_i) + d_{B4}(v_j)} = \frac{n(n-1)}{2}\,\frac{2}{2(n+1)} = \frac{n(n-1)}{2(n+1)}$$

$$T_3 = \sum_{i=2}^{n+1} \frac{2}{d_{B4}(v_1) + d_{B4}(e_{1i})} = \frac{2n}{2n+2} = \frac{n}{n+1}$$

$$T_4 = \sum_{i=2}^{n+1} \frac{2}{d_{B4}(v_i) + d_{B4}(e_{1i})} = \frac{2n}{n+3}$$

$$\begin{aligned} H(B_4(K_{1,n})) &= \sum_{i=1}^{4} T_i = \frac{2n}{3n+1} + \frac{n(n-1)}{2(n+1)} + \frac{n}{n+1} + \frac{2n}{n+3} \\ &= n\left[\frac{8(n+1)}{(3n+1)(n+3)} + \frac{1}{2}\right] \end{aligned}$$

□

Remark 3.12. *Geometric Arithmetic index of $B_4(K_{1,n})$ is given by*

$$\begin{aligned} GA(B_4(K_{1,n})) &= \frac{2n\sqrt{2n(n+1)}}{3n+1} + \frac{n(n-1)\sqrt{(n+1)(n+1)}}{2(n+1)} + \frac{n\sqrt{2(2n)}}{n+1} + \frac{2n\sqrt{2(n+1)}}{n+3} \\ &= \frac{2n\sqrt{2n(n+1)}}{3n+1} + \frac{n(n-1)}{2} + \frac{2n\sqrt{n}}{n+1} + \frac{2n\sqrt{2(n+1)}}{n+3} \end{aligned}$$

Theorem 3.13. *Harmonic index of $B_4(K_{m,n})$; $(m, n \geq 2)$ is given by,*

$$H(B_4(K_{m,n})) = \frac{m(m-1)}{2(m+2n-1)} + \frac{n(n-1)}{2(2m+n-1)} + 2mn\left[\frac{3m+3n+2}{(m+2n+1)(2m+n+1)}\right] + \frac{2mn}{3m+3n-2}.$$

Proof. Let [A, B] be the bipartition of $K_{m,n}$ and let $A = \{v_1, v_2, \ldots, v_m\}$ and $B = \{u_1, u_2, \ldots, u_n\}$. Let $e_{ij} = (v_i, u_j)$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$ be the edges of $K_{m,n}$. Then $v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_n, e_{ij} \in V(B_4(K_{m,n}))$ and

$$E(B_4(K_{m,n})) = E(K_{m+n}) \cup \bigcup_{i=1}^{m}\left(\bigcup_{j=1}^{n}(v_i, e_{ij})\right) \bigcup \left(\bigcup_{i=1}^{n}\left(\bigcup_{j=1}^{m}(u_i, e_{ji})\right)\right)$$

and $E(K_{m+n}) = E(K_m) \cup E(K_n) \cup \{(v_i, u_j), i = 1, 2, \ldots, m\ \textit{and}\ j = 1, 2, \ldots, n\}$.

$$d_{B4}(v_i) = p - 1 + d_{Km,n}(v_i) = m + n - 1 + n = m + 2n - 1$$

$$d_{B4}(u_i) = p - 1 + d_{Km,n}(u_i) = m + n - 1 + m = 2m + n - 1$$

$$d_{B4}(e_{ij}) = 2$$

Let F_1 and F_2 be the set of edges in K_m and K_n induced by the vertices $v_1, v_2, \ldots, v_m$ and $u_1, u_2, \ldots, u_n$ respectively. $F_3 = \{(v_i, u_j), i = 1, 2, \ldots, m \;\; and \;\; j = 1, 2, \ldots, n\}$; $F_4 = \left(\bigcup_{i=1}^{m}\left(\bigcup_{j=1}^{n}(v_i, e_{ij})\right)\right)$ and $F_5 = \bigcup_{i=1}^{n}\left(\bigcup_{j=1}^{m}(u_i, e_{ji})\right)$. Therefore $E(K_{m,n}) = \bigcup_{i=1}^{5} F_i$. $H(B_4(K_{m,n}))$ is equal to the sum of $\frac{2}{d_{B_4}(u)+d_{B_4}(v)}$, where the summation is taken over all edges $xy \in E(B_4(K_{m,n}))$. The sum T_i is taken over all edges $xy \in F_i$; $i = 1, 2, 3, 5$.

$$T_1 = \frac{m(m-1)}{2}\frac{2}{d_{B4}(v_i) + d_{B4}(v_j)} = \frac{m(m-1)}{2}\frac{2}{2(m+2n-1)} = \frac{m(m-1)}{2(m+2n-1)}$$
$$T_2 = \frac{n(n-1)}{2}\frac{2}{d_{B4}(u_i) + d_{B4}(u_j)} = \frac{n(n-1)}{2}\frac{2}{2(2m+n-1)} = \frac{n(n-1)}{2(2m+n-1)}$$
$$T_3 = \frac{2mn}{d_{B4}(u_i) + d_{B4}(v_j)} = \frac{2mn}{3m+3n-2}$$
$$T_4 = \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{2}{d_{B4}(v_i) + d_{B4}(e_{ij})} = \frac{2mn}{2m+n-1+2} = \frac{2mn}{m+2n+1}$$
$$T_5 = \sum_{i=1}^{n}\sum_{j=1}^{m}\frac{2}{d_{B4}(u_i) + d_{B4}(e_{ij})} = \frac{2mn}{2m+n+1}$$

Therefore,

$$\begin{aligned} H(B_4(K_{m,n})) &= \sum_{i=1}^{5} T_i \\ &= \frac{m(m-1)}{2(m+2n-1)} + \frac{n(n-1)}{2(2m+n-1)} + \frac{2mn}{3m+3n-2} + \frac{2mn}{m+2n+1} + \frac{2mn}{2m+n+1} \\ &= \frac{m(m-1)}{2(m+2n-1)} + \frac{n(n-1)}{2(2m+n-1)} + 2mn\left[\frac{3m+3n+2}{(m+2n+1)(2m+n+1)}\right] + \frac{2mn}{3m+3n-2} \end{aligned}$$

□

Remark 3.14.

(1). If $m = n$, then $H(B_4(K_{n,n})) = \frac{n(2n-1)}{3n-1} + \frac{4n^2}{3n+1}$.

(2). Geometric-Arithmetic index of $B_4(K_{m,n})$ is given by

$$\begin{aligned} GA(B_4(K_{m,n})) = &\frac{m(m-1)}{2} + \frac{n(n-1)}{2} + \frac{2mn\sqrt{(m+2n-1)(2m+n-1)}}{3m+3n-2} \\ &+ \frac{2mn\sqrt{2(m+2n-1)}}{m+2n+1} + \frac{2mn\sqrt{2(2m+n-1)}}{2m+n+1} \end{aligned}$$

(3). If $m = n$, then Geometric-Arithmetic index of $B_4(K_{n,n})$ is given by $GA(B_4(K_{n,n})) = n(2n-1) + \frac{4n^2\sqrt{2(3n-1)}}{3n+1}$.

Theorem 3.15. *Harmonic index of $B_4(W_n)$ is given by,*

$$H(B_4(W_n)) = (n-1)\left[\frac{5}{3n} + \frac{n-2}{2(n+2)} + \frac{6}{n+4}\right], \;\; n \geq 5$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of W_n with v_1 as the central vertex. and let $e_{1,i} = (v_1, v_i)$, $i = 2, 3, \ldots, n$, $e_{i,i+1} = (v_i, v_{i+1})$, $i = 2, 3, \ldots, n-1$, $e_{n,2} = (v_n, v_2)$ be the edges of W_n. Then $v_1, v_2, \ldots, v_{n1}, e_{1,i}, e_{i,i+1} \in B_4(W_n)$, $d_{B4}(v_1) = 2(n-1)$, $d_{B4}(v_i) = n+2$, $i = 2, 3, \ldots, n$, $d_{B4}(e_{1,i}) = d_{B4}(e_{i,i+1}) = d_{B4}(e_{n,2}) = 2$. Number of edges in $B_4(W_n) = \frac{(n-1)(n+8)}{2}$. Let

$$F_1 = \bigcup_{i=2}^{n}\{(v_1, v_i)\}, \;\; F_2 = \bigcup_{i=2}^{n}\bigcup_{\substack{j=2,\\ i<j}}^{n}\{(v_i, v_j)\}, \;\; F_3 = \bigcup_{i=2}^{n}\{(v_1, e_{1,i})\}, \;\; F_4 = \bigcup_{i=2}^{n}\{(v_i, e_{1,i})\}$$
$$F_5 = \bigcup_{i=3}^{n-1}\{(v_i, e_{i-1,i}), (v_i, e_{i,i+1})\}$$
$$F_6 = \{(v_2, e_{2,3}), (v_2, e_{n,2}), (v_n, e_{n,2}), (v_n, e_{n-1,n})\}$$

$H(B_4(W_n))$ is equal to the sum of $\frac{2}{d_{B_4}(u)+d_{B_4}(v)}$, where the summation is taken over all edges $xy \in E(B_4(W_n))$. The sum T_i is taken over all edges $xy \in F_i$, $i = 1, 2, \ldots, 6$.

$$T_1 = \sum_{i=2}^{n} \frac{2}{d_{B4}(v_1) + d_{B4}(v_i)} = \sum_{i=2}^{n} \frac{2}{2(n-1)+n+2} = \frac{2(n-1)}{3n}$$

$$T_2 = \sum_{i=2}^{n} \sum_{\substack{j=2,\\ i<j}}^{n} \frac{2}{d_{B4}(v_i) + d_{B4}(v_j)} = \sum_{i=2}^{n} \sum_{\substack{j=2,\\ i<j}}^{n} \frac{2}{d_{B4}(v_i) + d_{B4}(v_j)} \frac{(n-1)(n-2)}{2} \frac{2}{n+2+n+2} = \frac{(n-1)(n-2)}{2(n+2)}$$

$$T_3 = \sum_{i=2}^{n} \frac{2}{d_{B4}(v_1) + d_{B4}(e_{1i})} = (n-1)\frac{2}{2(n-1)+2} = \frac{n-1}{n}$$

$$T_4 = \sum_{i=2}^{n} \frac{2}{d_{B4}(v_i) + d_{B4}(e_{1i})} = \frac{2(n-1)}{n+4}$$

$$T_5 = \sum_{i=3}^{n-1} \left[\frac{2}{d_{B4}(v_i) + d_{B4}(e_{i-1,i})} + \frac{2}{d_{B4}(v_i) + d_{B4}(e_{i,i+1})} \right] = 2(n-3)\frac{2}{n+2+2} = \frac{4(n-3)}{n+4}$$

$$T_6 = \frac{2}{d_{B4}(v_2) + d_{B4}(e_{23})} + \frac{2}{d_{B4}(v_2) + d_{B4}(e_{n2})} + \frac{2}{d_{B4}(v_n) + d_{B4}(e_{n-1,n})} + \frac{2}{d_{B4}(v_n) + d_{B4}(e_{n2})} = \frac{8}{n+4}$$

Therefore

$$\begin{aligned} H(B_4(W_n)) &= \sum_{i=1}^{6} T_i = \frac{2(n-1)}{3n} + \frac{(n-1)(n-2)}{2(n+2)} + \frac{n-1}{n} + \frac{2(n-1)}{n+4} + \frac{4(n-3)}{n+4} + \frac{8}{n+4} \\ &= (n-1)\left[\frac{5}{3n} + \frac{n-2}{2(n+2)} + \frac{6}{n+4}\right] \end{aligned}$$

□

Remark 3.16. *Geometric Arithmetic index of $B_4(W_n)$ is given by*

$$GA(B_4(W_n)) = \frac{2(n-1)^{3/2}}{n}\left[1+\sqrt{2(n+2)}\right] + \frac{(n-1)(n-2)}{2} + \frac{6(n-1)\sqrt{2(n+2)}}{n+4}, \; n \geq 5.$$

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