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Unit-Regular Semigroups and Rings

Research Article

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Abstract: Multiplicative semigroups of rings form an important class of semigroups and one theme in the study of semigroups is how the structure of this semigroup affects the structure of the ring. An important tool in analyzing the structure of a semigroup are the Green's relations. In this paper, we study some properties of these relations on the multiplicative semigroup of a unit regular ring with unity. This also gives easier proofs of some known results on ring theory.

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1. Introduction

A semigroup is a set with an associative binary operation. In particular every ring is a semigroup, considering its multiplication alone. The concept of regularity for elements of a ring was introduced by von Neumann (see [8]): an element x of a ring R is said to be *regular*, if there exists an element x' in R such that $xx'x = x$. The ring itself is said to be regular, if all its elements are regular. An element u of a ring R with multiplicative identity 1 is said to be a *unit*, if it has a multiplicative inverse in R. An element x of R is said to be *unit-regular*, if there exists a unit u in R such that $xux = x$. If all elements of R are unit-regular, then R is called a unit-regular ring.

2. Unit-regularity

The concept of regularity can be extended to any semigroup and the concept of unit regularity can be extended to any monoid, that is, a semigroup with identity. We start with simple alternative characterizations of unit-regularity, stated as a remark for unit-regular rings in [3]:

Proposition 2.1. *The following are equivalent for an element x in a monoid S:*

(i) x is unit-regular.

(ii) There exists an idempotent e and a unit u in S such that $x = eu$.

(iii) There exists an idempotent f and a unit v in S such that $x = vf$.

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Proof. Assume (i), so that there exists a unit v in S such that $xvx = x$. It is easily seen that $e = xv$ is an idempotent with $x = ev^{-1}$. Thus we have (ii) with $u = v^{-1}$. Next assume (ii), and let $f = u^{-1}eu$. Then e is easily seen to be an idempotent and $x = eu = uf$, which gives (iii). Finally, assume (iii) and let $u = v^{-1}$. Then $f = ux$, so that $xux = xf = vf^2 = vf = x$ which gives (i). □

Ideals play an important role in the study of semigroups and rings. Note that by a *left ideal* in a semigroup S, we mean a subset I of S such that $xI \subseteq I$ for each x in S. By the *principal left ideal* generated by an element a in S, we mean the smallest ideal containing a and it is easily seen to be equal to $\{xa : x \in S\} \cup \{a\}$. Right ideals and principal right ideals are defined analogously and we denote the principal right ideal generated by a in S, which is equal to $\{ax : x \in S\} \cup \{a\}$, as aS^1. In the theory of semigroups, the study of ideals is facilitated by certain equivalence relations, collectively called Green's relations, based on principal ideals. The relations on $\mathscr{L}$ and $\mathscr{R}$ on elements of a semigroups are defined by

$$a \mathscr{L} b \text{ iff } S^1a = S^1b \quad \text{and} \quad a \mathscr{R} b \text{ iff } aS^1 = bS^1$$

The smallest equivalence relation containing $\mathscr{L}$ and $\mathscr{R}$ is denoted by $\mathscr{D}$. It can be shown that the relations $\mathscr{L}$ and $\mathscr{R}$ commute and $\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$. Thus for elements a and b of the semigroup, $a \mathscr{D} b$ iff there exist elements p and q such that $a \mathscr{L} p \mathscr{R} b$ and $a \mathscr{R} q \mathscr{L} b$. We represent these relations as an array $\begin{bmatrix} a & q \\ p & b \end{bmatrix}$ and call it a *D-square*. In particular, if e and f are idempotents in the semigroup, then there exists a D-square of the form $\begin{bmatrix} e & a \\ a' & f \end{bmatrix}$ such that $aa' = e$, $a'a = f$ ([6], Proposition II.3.6). Such a D-square will be called an *I-square*. The cited proposition also shows that if a and a' are elements of the a semigroup such that $aa'a = a$ and $a'aa' = a'$, then $e = aa'$ and $f = a'a$ are idempotents and $\begin{bmatrix} e & a \\ a' & f \end{bmatrix}$ is an I-square.

It is a characteristic property of regular semigroups that every principal left ideal is generated by an idempotent, and so is every principal right ideal (see [1, 2]). It is easily seen that for idempotents e and f in a semigroup, we have

$$e \mathscr{L} f \text{ iff } ef = e,\ fe = f \quad \text{and} \quad e \mathscr{R} f \text{ iff } ef = f,\ fe = e$$

Now in any semigroup S with identity 1, every unit is easily seen to be both $\mathscr{L}$-related and $\mathscr{R}$-related to 1, Also, the $\mathscr{L}$ or $\mathscr{R}$ class of 1 contains no other idempotent. For if $e \mathscr{L} 1$ for some idempotent e in a semigroup S, the $xe = 1$ for some x in S, so that $e = 1e = (xe)e = xe = 1$ and similarly for the $\mathscr{R}$ relation. In a unit-regular semigroup, we have a stronger result:

Proposition 2.2. *In a unit-regular semigroup, the $\mathscr{D}$-class of 1 contains no other idempotent.*

Proof. Let S be a unit-regular semigroup and let e be an idempotent in S with $e \mathscr{D} 1$. Then by definition of the $\mathscr{D}$-relation, there exists x in S such that $e \mathscr{L} x \mathscr{R} 1$. Since x is unit-regular, there exists an idempotent f and a unit u in S such that $x = fu$. Then $f = xu^{-1}$ also, so that $f \mathscr{R} x \mathscr{R} 1$ and so $f = 1$ by the discussion preceding the proposition. Hence $x = u$ and so $x \mathscr{L} 1$. Since $e \mathscr{L} x$ also, we have $e \mathscr{L} 1$ and so $e = 1$. □

The condition that the $\mathscr{D}$-class of 1 contains no other idempotent can be given another equivalent form:

Proposition 2.3. *The following are equivalent in any semigroup S with identity 1:*

(i) The $\mathscr{D}$-class of 1 contains no other idempotent in S.

(ii) The $\mathscr{D}$-class of 1 is the group of units in S.

Proof. Let D_1 be the $\mathscr{D}$-class of 1 in S and let G be the group of units in S. Then $G \subseteq D_1$, for if $u \in G$, then $u = u1$ and $1 = u^{-1}u$, so that $u \mathscr{L} 1$ and so $u \mathscr{D} 1$.

Suppose that (i) holds and let $x \in D_1$. Since 1 is regular, every element of D_1 and hence in particular x is regular (Prelim), so that it has a generalized inverse x' in S. Also, $x'x$ and xx' are idempotents $\mathscr{D}$-related to x (Prelim) and so are in D_1, since $x \mathscr{D} 1$. Hence $x'x = xx' = 1$, by (i), so that $x \in G$. Thus $D_1 \subseteq G$ also and hence $D_1 = G$, which gives (ii).

Conversely if $D_1 = G$ and e is an idempotent in D_1, then $e \in G$ so that e has an inverse f in G so that $e = e1 = e(ef) = e^2 f = ef = 1$. Thus we have (i). □

This result and the previous one give the following (cf.[9], Proposition 2.2):

Proposition 2.4. *In any unit-regular semigroup S, the $\mathscr{D}$-class containing 1 is the group of units of S.*

Both (i) and (ii) in Proposition 2.3 give necessary conditions of unit-regularity, but neither is sufficient for unit-regularity. To see another characterization of semigroups satisfying these conditions, let x, y be elements of such a semigroup S with $xy = 1$. Then yx is easily seen to be an idempotent with $xyx = x$, so that $yx \mathscr{L} x \mathscr{R} xy = 1$ and hence $yx = 1$, by (i). So for x, y in a semigroup S as above, $xy = 1$ implies $yx = 1$.

This property holds in any finite semigroup with identity. To see this, let S be a finite semigroup and let a, b be in S with $ab = 1$. Then $x \mapsto xa$ is a bijection of S onto Sa, for if $xa = ya$ in S, then $x = x(ab) = (xa)b = (ya)b = y(ab) = y$ Hence Sa has the same number of elements as S, and since $Sa \subseteq S$ and S is finite, it follows that $Sa = S$. So there exists c in S with $ca = 1$. Again, $c = c1 = c(ab) = (ca)b = b$ so that $ba = ca = 1$. Thus the condition $xy = 1$ implies $yx = 1$ is a weaker form of finiteness. We define a monoid S to be *Dedekind-finite*, if $xy = 1$ implies $yx = 1$ for x, y in S. A ring is called Dedekind-finite, if its multiplicative semigroup is Dedekind-finite.

Thus the above argument shows that a semigroup satisfying conditions (i) or (ii) above is Dedekind-finite. We can show that the converse is also true. Let S be a Dedekind-finite semigroup and let e be an idempotent in the $\mathscr{D}$-class of 1. Then there exists an I-square $\begin{bmatrix} e & x' \\ x & 1 \end{bmatrix}$. That is, $x'x = e$ and $xx' = 1$. Since S is Dedekind-finite, it follows that $x'x = 1$, so that $e = 1$. Thus we can extend Proposition 2.3 as follows:

Proposition 2.5. *The following are equivalent in any semigroup S with identity* 1

(i) S is Dedekind-finite.

(ii) The $\mathscr{D}$-class of 1 contains no other idempotent.

(iii) The $\mathscr{D}$-class of 1 is the group of units in S □.

And Proposition 2.2 gives the following:

Corollary 2.6. *A unit-regular semigroup is Dedekind-finite.*

A generalization of (ii) above can be given in terms of the partial order ω on the idempotents of a semigroup defined in [7]: for idempotents e and f, we define $e \,\omega\, f$ iff $ef = fe = e$. If a semigroup has identity 1, then $e \,\omega\, 1$. Thus (ii) says that the the $\mathscr{D}$-class D_1 does not contain distinct comparable idempotents under ω. More generally, a regular semigroup is said to be *completely semisimple*, if no $\mathscr{D}$-class contains distinct idempotents comparable under the partial order ω. (Though this is not the original definition of completely semisimple semigroups, it can be proved to be equivalent to this condition. See [4]). A ring is said to be completely semisimple, if its multiplicative semigroup is completely semisimple.

Now in a completely semisimple semigroup S with identity 1, if e an idempotent with $e \mathscr{D} 1$, then $e = 1$, since $e \,\omega\, 1$. Thus we have the following:

Corollary 2.7. *Every completely semisimple monoid is Dedekind-finite.*

We can show that the converse is true for a regular ring with multiplicative identity. In fact we can prove a slightly more general result, as in Theorem 1 of [5]. This can be put in terms of the $\mathscr{D}$-relation. For this, we first prove a lemma. Note that two idempotents e and f in a ring are said to be *orthogonal*, written $e \perp f$, if $ef = fe = 0$.

Proposition 2.8. *If* $\begin{bmatrix} e & a \\ a' & f \end{bmatrix}$ *and* $\begin{bmatrix} g & b \\ b' & h \end{bmatrix}$ *be I-squares with $e \perp g$ and $f \perp h$, then* $\begin{bmatrix} e+g & a+b \\ a'+b' & f+h \end{bmatrix}$ *is also an I-square.*

Proof. First note that since $e \perp g$ and $f \perp h$, the elements $e+g$ and $f+h$ are idempotents. Also, since $a \mathscr{L} f$, we have $a = af$ and since $b' \mathscr{R} h$, we have $b' = b'bb' = hb'$, so that $ab' = a(fh)b' = 0$. Similarly, $ba' = 0$. Hence $(a+b)(a'+b') = e+g$, since $aa' = e$ and $bb' = g$, by definition of I-squares. By similar arguments, we get $(a'+b')(a+b) = f+h$. Again, $a = aa'a = ea$ and $b = bb'b = gb$, so that

$$(a+b)(a'+b')(a+b) = (e+g)(a+b) = a + eb + ga + b = a + egb + gea + b = a + b$$

Similarly, we can show that $(a'+b')(a+b)(a'+b') = a'+b'$. Thus $\begin{bmatrix} e+g & a+b \\ a'+b' & f+h \end{bmatrix}$ is an I-square. □

We also some notions from [7]: for idempotents e and f of a semigroup, we write $e\ \omega^l\ f$ if $S^1e \subseteq S^1f$ and $e\ \omega^r\ f$ iff $eS^1 \subseteq fS^1$; it can be easily seen that

$$e\,\omega^l\,f \text{ iff } ef = e \quad \text{and} \quad e\,\omega^r\,f \text{ iff } ef = f$$

Now we can prove the result mentioned above:

Proposition 2.9. *Let R be a ring with multiplicative identity. Then the following are equivalent:*

(i) R is Dedekind-finite.

(ii) For idempotents e and f of R, if $e \mathscr{D} f$ and $e\,\omega^l\,f$, then $e \mathscr{L} f$.

(iii) For idempotents e and f of R, if $e \mathscr{D} f$ and $e\,\omega^r\,f$, then $e \mathscr{R} f$.

Proof. We show that both (ii) and (iii) are equivalent to (i). First suppose that R is Dedekind-finite and let e and f be idempotents in R with $e \mathscr{D} f$ and $e\,\omega^l\,f$. Then fe can be easily seen to be an idempotent with $e \mathscr{L} fe\ \omega\ f$ (see [7]). Now since $fe \mathscr{L} e$ and $e \mathscr{D} f$, we have $fe \mathscr{D} f$, so that here exists an I-square $\begin{bmatrix} fe & x \\ x' & f \end{bmatrix}$. Also, since $fe\ \omega\ f$, it follows that $fe \perp (1-f)$. Hence we have the I-square $\begin{bmatrix} fe+(1-f) & x+(1-f) \\ x'+(1-f) & 1 \end{bmatrix}$, by Corollary 2.8. In particular, this means the idempotent $fe+(1-f)$ is in the $\mathscr{D}$-class of 1 and so $fe+(1-f) = 1$, by Proposition 2.5. This equation gives $fe = f$ and hence $f\,\omega^l\,e$. Together with $e\,\omega^l\,f$, this gives $e \mathscr{L} f$. Thus we have (ii).
Conversely assume (ii) and let e be an idempotent in R with $e \mathscr{D} 1$. Since $e1 = e$, we have $e\,\omega^l\,1$ and so by (ii), we have $e \mathscr{L} 1$. This gives $1e = 1$, so that $e = 1$. Thus the $\mathscr{D}$-class of 1 contains no other idempotent and so R is Dedekind-finite, by Proposition 2.5.
By the left-right dual of the above arguments, we can show that (i) is equivalent to (iii) also. □

This leads to the following:

Corollary 2.10. *Every Dedekind-finite regular ring is completely semisimple.*

Proof. Let R be a Dedekind-finite ring and let e, f be idempotents in R with $e \mathscr{D} f$ and $e \,\omega\, f$. Then $e \,\omega^l\, f$, so that $e \mathscr{L} f$ and $e \,\omega^r\, f$, so that $e \mathscr{R} f$, by Proposition 2.9. Now since $e \mathscr{L} f$ we have $ef = e$ and since $e \mathscr{R} f$, we have $ef = f$. Thus $e = f$ and it follows that R is completely semisimple. □

Thus we can strengthen Corollary 2.7 for rings:

Proposition 2.11. *A regular ring with multiplicative identity is Dedekind-finite if and only if it is completely semisimple.*

Together with Corollary 2.6, this gives the following:

Corollary 2.12. *Every unit-regular ring is completely semisimple.*

Now in the case of a regular ring R with multiplicative identity 1, for each idempotent e we have the idempotent $1 - e$ also. It is easily seen that the map $e \mapsto (1 - e)$ is transforms ω^l to $(\omega^r)^{-1}$ and vice versa:

Proposition 2.13. *Let e and f be idempotents in R. If $e \,\omega^l\, f$, then $(1 - f) \,\omega^r\, (1 - e)$ and if $e \,\omega^r\, f$, the $(1 - f) \,\omega^l\, (1 - e)$. In particular, if $e \mathscr{L} f$, then $(1 - e) \mathscr{R} (1 - f)$ and if $e \mathscr{R} f$, then $(1 - e) \mathscr{L} (1 - f)$.* □

However, not much could be said about the effect of the map $e \mapsto (1 - e)$ on the $\mathscr{D}$-relation on arbitrary regular rings. We next see that in a unit-regular ring, $e \mathscr{D} f$ implies $(1 - e) \mathscr{D} (1 - f)$ for idempotents e and f and this property in fact characterizes unit regularity. To prove this, we make use of the following lemma. Note that for every element x of a semigroup with identity and for every unit u in S, we have $ux \mathscr{L} x$, since ux is a left multiple of x and $x = u^{-1}(ux)$ is a left multiple of ux; and similarly $xu \mathscr{R} x$.

Lemma 2.14. *If e and f are idempotents in a unit-regular semigroup S with $e \mathscr{D} f$, then there exist idempotents g, h in S with $g \mathscr{L} e$ and $h \mathscr{R} f$ and a unit u in S such that $h = u^{-1}gu$.*

Proof. Let $e\mathscr{D}f$ in S, so that there exists x in S with $e \mathscr{L} x \mathscr{R} f$, and since S is unit-regular, there exists a unit u in S with $xux = x$. Let $g = ux$ and $h = xu$. Then g and h are idempotents in S with $g \mathscr{L} x \mathscr{L} e$ and $h \mathscr{R} x \mathscr{R} f$. Also $u^{-1}gu = xu = h$. □

Using this, we can prove the claimed equivalence for unit-regular rings:

Proposition 2.15. *For a regular ring R with multiplicative identity 1, the following are equivalent:*

(i) R is unit-regular.

(ii) For idempotents e, f in R, if $e \mathscr{D} f$, then $(1 - e) \mathscr{D} (1 - f)$.

Proof. First suppose that R is unit-regular and let e and f be idempotents in R with $e \mathscr{D} f$. Then by the above lemma, there exist idempotents g and h in R with $g \mathscr{L} e$ and $h \mathscr{R} f$ and a unit u in R such that $h = u^{-1}gu$. Then $(1-e) \mathscr{R} (1-g)$ and $(1-f) \mathscr{L} (1-h)$, by Proposition 2.13 and also $1-h = 1-u^{-1}gu = u^{-1}(1-g)u$. Hence $(1-e) \mathscr{D} (1-g)$ and $(1-h) \mathscr{D} (1-f)$, since $\mathscr{R}$ and $\mathscr{L}$ are contained in $\mathscr{D}$; also $1 - g \mathscr{L} u^{-1}(1 - g) \mathscr{R} u^{-1}(1 - g)u = 1 - h$, so that $(1 - g) \mathscr{D} (1 - h)$. Since the $\mathscr{D}$-relation is transitive, it follows that $(1 - e) \mathscr{D} (1 - f)$.

Conversely, assume (ii) and let $x \in R$. Since R is regular, x has a generalized inverse x', and writing $e = xx'$ and $f = x'x$, we have the I-square $\begin{bmatrix} e & x \\ x' & f \end{bmatrix}$. In particular, $e \mathscr{D} f$, so that $(1 - e) \mathscr{D} (1 - f)$ by (ii), and so there exists an I-square

$\begin{bmatrix} 1-e & y \\ y' & 1-f \end{bmatrix}$. Since $e \perp (1-e)$ and $f \perp (1-f)$, we get the I-square $\begin{bmatrix} 1 & x+y \\ x'+y' & 1 \end{bmatrix}$, by Proposition 2.8. Hence $(x+y)(x'+y') = 1 = (x'+y')(x+y)$, so that $u = x+y$ is a unit in R with $u^{-1} = x'+y'$. Now

$$xu^{-1}x = x(x'+y')x = xx'x + xy'x = x + xy'x$$

Also, $x = xf$ since $x \mathscr{L} f$ and $y' = (1-f)y'$ since $y' \mathscr{R} (1-f)$, so that $xy' = (xf)((1-f)y') = x(f(1-f))y' = 0$. Thus $xu^{-1}x = x$. It follows that R is unit-regular. □

Using this, we can strengthen Lemma 2.14 as follows:

Corollary 2.16. *If e and f are idempotents in a unit-regular ring R with $e \mathscr{D} f$, then there exists a unit u in R such that $f = u^{-1}eu$.*

Proof. Let e and f be idempotents in R with $e \mathscr{D} f$, so that there exists an I-square $\begin{bmatrix} e & x \\ x' & f \end{bmatrix}$. By the proposition above, we have $(1-e) \mathscr{D} (1-f)$ also, and so there exists an I-square $\begin{bmatrix} 1-e & y \\ y' & 1-f \end{bmatrix}$ also. As in the proof of the proposition, these two I-squares can be added to yield the I-square $\begin{bmatrix} 1 & x+y \\ x'+y' & 1 \end{bmatrix}$, so that $u = x+y$ is a unit in R with $u^{-1} = x'+y'$. Now we have $u^{-1}e = (x'+y')e = x'e + y'e = x' + y'(1-e)e = x'$ since $x' \mathscr{L} e$ and $y' \mathscr{L} (1-e)$; and $eu = e(x+y) = ex+ey = x+e(1-e)y = x$ since $x \mathscr{R} e$ and $y \mathscr{R} (1-e)$. Hence $u^{-1}eu = (u^{-1}e)(eu) = x'x = f$. □

The following definition is from [9] and we extend it to ring in the usual manner:

Definition 2.17. *A unit-regular semigroup S is said to be* strongly unit-regular *if for any pair e, f of $\mathscr{D}$-related idempotents in S, there exists a unit u in S such that $f = u^{-1}eu$. A ring is called strongly unit-regular, if its multiplicative semigroup is strongly unit-regular.*

Using this terminology, the above corollary can be written as follows:

Proposition 2.18. *Every unit-regular ring is strongly unit-regular.*

References

[1] A.H.Clifford and G.B.Preston, *The algebraic theory of semigroups, Part-1*, Amer.Math.Soc., Providence, (1961).

[2] A.H.Clifford and G.B.Preston, *The algebraic theory of semigroups, Part-1*, Amer.Math.Soc., Providence, (1967).

[3] G.Ehrlich, *Unit-Regular Rings*, Portugaliae Mathematica, 27(1968), 209-212.

[4] T.E.Hall, *On Regular Semigroups*, J.Algebra, 24(1973), 1-24.

[5] R.E.Hartwig and Jiang Luh, *On finite Regular Rings*, Pacific Journal of Mathematics, 69(1)(1977), 73-95.

[6] J.M.Howie, *An Introduction to Semigroup Theory*, Academic Press, London, (1976).

[7] K.S.S.Nambooripad, *Structure of Regular Semigroups I*, Mem. Amer. Math. Soc., 224(1979).

[8] J.von Neumann, *Continuous Geometry*, Princeton University Press, (1998).

[9] R.Veeramony, *Unit regular Semigroups*, Proceeding of the International Symposium on Regular Semigroups and Applications, University of Kerala, (1986), 228-240.