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A New Extended Riemann-Liouville Fractional Operator

Research Article

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Abstract: In this paper we will introduce a new and modified Riemann-Liouville fractional operator that results from modifying the extended fractional derivative due to M. Ozarslan [5]. We will study some familiar functions regarding this new operator, Laplace transform and Mellin transform of the potential function are calculated and we will also define a new hypergeometric function in term of extended beta function due to the author [11].

Keywords: Extended beta function, Hypergeometric function, Fractional Calculus, Laplace and Mellin transform.

1. Introduction and Preliminaries

The fractional calculus is considered an the extension of the usual calculus, where the order of integration and derivative is not necessarily integer, an many even be complex. The development of the fractional calculus is as old as the usual calculus. However, it can be considered as a new theory, since there are specific congresses of this in the last two decades. The first mention of extending the meaning of the expression $\frac{d^n}{dx^n}$ since the case of n not integer is in a correspondence between Leibnitz and LHospital and until the nineteenth century it was matter that only some mathematicians like Euler, Laplace, Fourier, Liouville, Riemann and Abel. Many definitions of derivatives of non integer order exist and the best results is the generalization process due to the mathematician Grundwal and Letnikov (see [2, 7]), formulated in the nineteenth century.

Another definition is that of Riemann-Liouville (see [2, 7]), that is chronologically earlier than Grundwal-Letnikov, and is widely used due to the ease of handling through integral transforms and its applicability in solving problems from physics. In the last two decades, fractional calculus has been used successfully in the modeling of phenomena and physical systems studied in many fields of science and engineering.

As it is known in 1997 Chaudhry introduced an extension of the beta function as follows: (see [8, 9])

$$B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{p}{t(1-t)}}\,dt \tag{1}$$

where $p \geq 0$, $R_e(x) > 0$ and $R_e(y) > 0$.

As it is known in 2004 Chaudhry generalized the hypergeometric function in term of the B_p beta function given by: (see [8, 9])

$$F_p(a,b,c,z) = \sum_{n=0}^{\infty} \frac{B_p(b+n,c-b)}{B(b,c-b)}(a)_n \frac{z^n}{n!} \tag{2}$$

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Where $p \geq 0$, $|z| < 1$, $R_e(c) > R_e(b) > 0$ y $(a)_n$ is the Pochhammer symbol and is defined as:

$$(a)_n = \begin{cases} 1 & \text{si } n = 0 \\ a(a+1\ldots(a+n-1) & \text{si } n \in \mathbb{N} \end{cases}$$

For the generalized hypergeometric function, we have the following integral representation.

$$F_p(a,b,c,z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} e^{\frac{-p}{t(1-t)}}\, dt \tag{3}$$

where $p \geq 0$, $R_e(c) > R_e(b) > 0$ y $|arg(1-z)| < \pi$.

Start recalling some definitions of elements that well be used in developing this paper.

Definition 1.1. *Let $f \in L^1_{loc}[a,b]$, $-\infty < a \leq t \leq b < \infty$. Then, the Riemann-Liouville fractional integral of order $v > 0$ is defined as:*

$$I^v f(t) = \frac{1}{\Gamma(v)} \int_a^b (t-\lambda)^{v-1} f(\lambda) d\lambda \tag{4}$$

Definition 1.2. *Let $f \in L^1_{loc}[a,b]$, $-\infty < a \leq t \leq b < \infty$ and $m-1 \leq v < m$, $m \in \mathbb{N}$. Then, the Riemann-Liouville fractional derivative of orden v is defined as:*

$$\begin{aligned} D^v f(t) &= \frac{d^m}{dt^m}\left(\frac{1}{\Gamma(m-v)} \int_a^b (t-\lambda)^{m-v-1} f(\lambda) d\lambda\right) \\ &= \frac{d^m}{dt^m}\left(I^{m-v} f(t)\right) \end{aligned} \tag{5}$$

In 2010 M. Ozarslan and E. Ozergin (see [5]) introduced an extension of the Riemann-Liouville fractional derivative and fractional integral given by the following.

Definition 1.3. *Let $f \in AC[0,b]$, be the space of functions which are absolutely continuous on $[0,b]$, $0 \leq z \leq b$, $p \geq 0$ and $v > 0$. Then, the extended Riemann-Liouville integral fractional of order $v > 0$ is defined as:*

$$I_z^{v,p} f(z) = \frac{1}{\Gamma(v)} \int_0^z (z-t)^{v-1} f(t) e^{\frac{-pz^2}{t(z-t)}}\, dt \tag{6}$$

Definition 1.4. *Let $f \in AC[0,b]$, be the space of functions which are absolutely continuous on $[0,b]$, $0 \leq z \leq b <$, $p \geq 0$, $v > 0$ and $m-1 \leq v < m$, $m \in \mathbb{N}$. Then, the extended Riemann-Liouville fractional derivative of order v is defined as:*

$$\begin{aligned} D_z^{v,p} f(t) &= \frac{d^m}{dt^m}\left(\frac{1}{\Gamma(m-v)} \int_0^z (z-t)^{m-v-1} f(t) e^{\frac{-pz^2}{t(z-t)}dt}\right) \\ &= \frac{d^m}{dt^m}\left(I_z^{m-v,p} f(t)\right) \end{aligned} \tag{7}$$

Remark 1.5. *Note that if $p = 0$, (6) and (7) reduces to the classical Riemann-Liouville fractional integral and fractional derivative of arbitrary order v (4) and (5).*

2. A New Hypergeometric Function

Definition 2.1. *Let $p \geq 0$, $a, b, c \in \mathbb{C}$, that such $R_e(c) > R_e(b) > 0$, $\alpha \in \mathbb{R}^+$ and $|z| < 1$. The new hypergeometric function is defined for following series:*

$$F_p^{\alpha}(a,b,c,z) = \sum_{n=0}^{\infty} \frac{B_p^{\alpha}(b+n, c-b)}{B(b, c-b)} (a)_n \frac{z^n}{n!} \tag{8}$$

where $B_p^{\alpha}(,)$ is the modified and extended beta function due to Pucheta (see [11]) and is defined as:

$$B_p^{\alpha}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} E_{\alpha}(-bt(1-t))dt$$

Lemma 2.2. *Let $p \geq 0$, $a, b, c \in \mathbb{C}$, that such $R_e(c) > R_e(b) > 0$, $\alpha \in \mathbb{R}^+$ and $|arg(1-z)| < \pi$. Then, the new hypergeometric function it has the following integral representation:*

$$F_p^{\alpha}(a,b,c,z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} E_{\alpha}(-pt(1-t))dt \tag{9}$$

Proof. Taking into account that $(1-zt)^{-a} = \sum\limits_{n=0}^{\infty} (a)_n \frac{(tz)^n}{n!}$ and with uniform convergence, we can interchange the orden of integration with the summation:

$$\begin{aligned}
&\frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} E_{\alpha}(-pt(1-t))dt \\
&\frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \sum_{n=0}^{\infty} (a)_n \frac{(tz)^n}{n!} E_{\alpha}(-pt(1-t))dt \\
&\frac{1}{B(b,c-b)} \sum_{n=0}^{\infty} (a)_n \frac{z^n}{n!} \int_0^1 t^{b+n-1}(1-t)^{c-b-1} E_{\alpha}(-pt(1-t))dt \\
&\sum_{n=0}^{\infty} \frac{B_p^{\alpha}(b+n.c-b)}{B(b,c-b)} (a)_n \frac{z^n}{n!} = F_p^{\alpha}(a,b,c,z)
\end{aligned}$$

□

3. A New Extended Riemann-Liouville Fractional Operator

In this section we present the definitions and some properties of the new extended Riemann-Liouville fractional integrals and fractional derivatives of the potential function.

Definition 3.1. *Let $f \in AC[0,b]$, $0 \leq z \leq b$, $p \geq 0$, $\alpha \in \mathbb{R}^+$ and $v > 0$. Then, the new extended Riemann-Liouville fractional integral of order $v > 0$ is defined as:*

$$I_z^{v,p,\alpha} f(z) = \frac{1}{\Gamma(v)} \int_0^z (z-t)^{v-1} f(t) E_{\alpha}\left(\frac{-pt(z-t)}{z^2}\right) dt \tag{10}$$

Definition 3.2. *Let $f \in AC[0,b]$, $0 \leq z \leq b <$, $p \geq 0$, $\alpha \in \mathbb{R}^+$ $v > 0$ and $m-1 \leq v < m$, $m \in \mathbb{N}$. Then, the new extended Riemann-Liouville derivative fractional of order v is defined as:*

$$\begin{aligned}
D_z^{v,p,\alpha} f(t) &= \frac{d^m}{dt^m}\left(\frac{1}{\Gamma(m-v)} \int_0^z (z-t)^{m-v-1} f(t) E_{\alpha}\left(\frac{-pt(z-t)}{z^2}\right)\right) dt \\
&= \frac{d^m}{dt^m}\left(I_z^{m-v,p} f(t)\right)
\end{aligned} \tag{11}$$

Remark 3.3. *Note that if $\alpha = 1$, then, (10) and (11) is reduced to extended Riemann-Liouville fractional derivative and fractional integral (6) y (7).*

Theorem 3.4. *Let $p \geq 0$, $\alpha \in \mathbb{R}^+$, $v > 0$ and $f(z) = z^\lambda$, $\lambda > 0$. Then*

$$I_z^{v,p,\alpha} f(z) = \frac{B_p^\alpha(\lambda+1, v)}{\Gamma(v)} z^{\lambda+v} \tag{12}$$

Proof.

$$I_z^{v,p,\alpha}\left(z^\lambda\right) = \frac{1}{\Gamma(v)} \int_0^z (z-t)^{v-1} z^\lambda E_\alpha\left(\frac{-pt(z-t)}{z^2}\right) dt \tag{13}$$

Making a variable change $u = \frac{t}{z}$, we have:

$dt = zdu$

$t = 0,\ u = 0$

$t = z,\ u = 1$

$(z-t) = z(1-u)$

Thus, replacing in the previous expression (13), we have:

$$\begin{aligned} I_z^{v,p,\alpha}\left(z^\lambda\right) &= \frac{1}{\Gamma(v)} z^{\lambda+v} \int_0^1 u^\lambda (1-u)^{v-1} E_\alpha\left(-pu(1-u)\right) du \\ &= \frac{B_p^\alpha(\lambda+1, v)}{\Gamma(v)} z^{\lambda+v} \end{aligned}$$

□

Theorem 3.5. *Let $p \geq 0$, $\alpha \in \mathbb{R}^+$ $v > 0$ and $m - 1 \leq v < m$, $m \in \mathbb{N}$ and $f(z) = z^\lambda$ $\lambda > 0$. Then*

$$D_z^{v,p,\alpha}(z^\lambda) = \frac{B_p^\alpha(\lambda+1, m-v)}{\Gamma(m-v)} \frac{\Gamma(\lambda+m-v+1)}{\Gamma(\lambda-v+1)} z^{\lambda-v} \tag{14}$$

Proof. From definition (11) and (12), we have:

$$\begin{aligned} D_z^{v,p,\alpha} z^\lambda &= \frac{d^m}{dt^m}\left(I_z^{m-v,p} z^\lambda\right) = \frac{d^m}{dt^m}\left(\frac{B_p^\alpha(\lambda+1, m-v)}{\Gamma(m-v)} z^{\lambda+m-v}\right) \\ &= \frac{B_p^\alpha(\lambda+1, m-v)}{\Gamma(m-v)} \frac{d^m}{dz^m} z^{\lambda+m-v} \\ &= \frac{B_p^\alpha(\lambda+1, m-v)}{\Gamma(m-v)} \frac{\Gamma(\lambda+m-v+1)}{\Gamma(\lambda-v+1)} z^{\lambda-v} \end{aligned}$$

□

Theorem 3.6. *Let $p \geq 0$, $\alpha \in \mathbb{R}^+$, $v > 0$ such that $v - \lambda > 0$, $|z| < 1$ and $f(z) = z^{\lambda-1}(1-z)^{\beta-1}$, $\lambda > 0$ $\beta > 0$. Then*

$$I_z^{v-\lambda,p,\alpha} f(z) = \frac{\Gamma(\lambda) F_p^\alpha(\beta, \lambda, v, z)}{\Gamma(v)} z^{v-1} \tag{15}$$

Proof. From definitions (10), (9) and using $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, we have:

$$I_z^{v-\lambda,p,\alpha}\left(z^{\lambda-1}(1-z)^\beta\right) = \frac{1}{\Gamma(v-\lambda)} \int_0^z t^{\lambda-1}(1-t)^{\beta-1}(z-t)^{v-\lambda-1} \times E_\alpha\left(\frac{-pt(z-t)}{z^2}\right) dt \tag{16}$$

Marking the change of variable $u = \frac{t}{z}$, we have:

$dt = zdu$

$t = 0, \quad u = 0$

$t = z, \quad u = 1$

Thus, replacing in the previous expression (16), we have:

$$\begin{aligned} I_z^{v-\lambda,p,\alpha}\left(z^{\lambda-1}(1-z)^{\beta}\right) &= \frac{1}{\Gamma(v-\lambda)}\int_0^1 t^{\lambda-1}z^{\lambda-1}(1-uz)^{\beta}z^{v-\lambda-1}(1-u)^{v-\lambda-1}\times E_\alpha\left(-pu(1-u)\right)zdu \\ &= \frac{z^{v-1}}{\Gamma(v-\lambda)}\int_0^1 u^{\lambda-1}(1-u)^{v-\lambda-1}(1-uz)^{\beta}\times E_\alpha\left(-pu(1-u)\right)zdu \\ &= \frac{z^{v-1}}{\Gamma(v-\lambda)}B(\lambda,v-\lambda)F_p^{\alpha}(\beta,\lambda,v,z) \\ &= \frac{z^{v-1}\Gamma(\lambda)}{\Gamma(v)}F_p^{\alpha}(\beta,\lambda,v,z) \end{aligned}$$

□

Theorem 3.7. *Let $f(z)$ be an analytic function in the disc $|z| < \rho$, $\rho > 0$, and has the power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then*

$$I_z^{v,p,\alpha}\left(z^{\lambda-1}f(z)\right) = \frac{z^{\lambda+v-1}}{\Gamma(v)}\sum_{n=0}^{\infty} a_n B_p^{\alpha}(\lambda+n,v)z^n \tag{17}$$

Proof. Using the definition (10) and making the change of variables $u = \frac{t}{z}$, we obtain:

$$\begin{aligned} I_z^{v,p,\alpha}\left(z^{\lambda-1}f(z)\right) &= \frac{1}{\Gamma(v)}\int_0^z t^{\lambda-1}\sum_{n=0}^{\infty} a_n t^n (z-t)^{v-1}E_\alpha\left(\frac{-pt(z-t)}{z^2}\right)dt \\ &= \sum_{n=0}^{\infty}\frac{z^{\lambda+v+n-1}}{\Gamma(v)}\int_0^1 u^{\lambda+n-1}(1-u)^{v-1}E_\alpha(-pu(1-u))du \\ &= \sum_{n=0}^{\infty} a_n \frac{z^{\lambda+v+n-1}}{\Gamma(v)}B_p^{\alpha}(\lambda+n,v) \\ &= \frac{z^{\lambda+v-1}}{\Gamma(v)}\sum_{n=0}^{\infty} a_n B_p^{\alpha}(\lambda+n,v) \end{aligned}$$

□

4. Integral Transform

In this section we will evaluate the Laplace and Mellin transform of the new extended Riemann-Liouville fractional integrals and fractional derivatives of the potential function $f(z) = z^w \quad w > 0$.

Definition 4.1. *Let $f : \mathbb{R}^+ \to \mathbb{R}$ an exponential order function and piecewise continuous, then the Laplace transform of f is*

$$\mathcal{L}\left\{f(t)\right\}(s) = \int_0^{\infty} e^{-st}f(t)dt, \quad s \in \mathbb{C} \tag{18}$$

The integral exist for $R_e(s) > 0$.

Definition 4.2. *The Mellin transform of a function $f(t)$ of a real variable $t \in \mathbb{R}^+$ is defined by*

$$M\left\{f(t)\right\}(s) = \int_0^{\infty} t^{s-1}f(t)dt \quad s \in \mathbb{C} \tag{19}$$

4.1. Laplace Transform

Theorem 4.3. *Let* $p \geq 0$, $\alpha \in \mathbb{R}^{+}$, $v > 0$ *and* $f(z) = z^{w}$, $w > 0$. *Then*

$$\mathcal{L}\left\{I_{z}^{v,p,\alpha} z^{w}\right\}(s) = \frac{B_{p}^{\alpha}(w+1, v)}{\Gamma(v)} \frac{\Gamma(w+2)}{s^{w+2}} \tag{20}$$

Proof. From (18), (10) and making the change of variables $u = \frac{t}{z}$, we have

$$\begin{aligned}
\mathcal{L}\left\{I_{z}^{v,o,\alpha} z^{w}\right\}(s) &= \int_{0}^{\infty} e^{-st}\left(\frac{1}{\Gamma(v)} \int_{0}^{z} t^{w}(z-t)^{v-1} E_{\alpha}\left(\frac{-pt(z-t)}{z^{2}}\right) d\lambda\right) dt \\
&= \frac{1}{\Gamma(v)} \int_{0}^{\infty} e^{-st} t^{w+1}\left(\int_{0}^{1} u^{w}(1-u)^{v-1} E_{\alpha}(-pu(1-u)) du\right) dt \\
&= \frac{B_{p}^{\alpha}(w+1, v)}{\Gamma(v)} \int_{0}^{\infty} e^{-st} t^{w+1} dt \\
&= \frac{B_{p}^{\alpha}(w+1, v)}{\Gamma(v)} \frac{\Gamma(w+2)}{s^{w+2}}
\end{aligned}$$

□

Theorem 4.4. *Let* $p \geq 0$, $\alpha \in \mathbb{R}^{+}$ $v > 0$ *and* $m-1 \leq v < m$, $m \in \mathbb{N}$ *and* $f(z) = z^{w}$ $w > 0$. *Then*

$$\mathcal{L}\left\{D_{z}^{v,p,\alpha} z^{w}\right\}(s) = \frac{B_{p}^{\alpha}(w+1, m-v)}{\Gamma(m-v)} \frac{\Gamma(w+2)}{s^{w-m+2}} \tag{21}$$

Proof. Using the definition (11) and (20), it result

$$\begin{aligned}
\mathcal{L}\left\{D_{z}^{v,p,\alpha} z^{w}\right\}(s) &= \mathcal{L}\left\{D^{m}\left(I_{z}^{m-v,p,\alpha} z^{w}\right)\right\}(s) \\
&= s^{m} \mathcal{L}\left\{I_{z}^{m-v,p,\alpha} z^{w}\right\}(s) - \underbrace{\sum_{i=1}^{m} s^{m-i} I_{z}^{i-v,p,\alpha} 0^{w}}_{=0} \\
&= \frac{B_{p}^{\alpha}(w+1, m-v)}{\Gamma(m-v)} \frac{\Gamma(w+2)}{s^{w-m+2}}
\end{aligned}$$

□

Remark 4.5. *Note that if* $p = 0$, $m = 2$, $\alpha = v = 1$ *the expression (20), (21) is reduced to the classic Laplace transform of the integral and derivative of orden 2 of the potential function* z^{w}, $w > 0$.

4.2. Mellin Transform

Theorem 4.6. *Let* $p \geq 0$, $\alpha \in \mathbb{R}^{+}$, $v > 0$ *and* $f(z) = z^{w}$, $w > 0$. *Then*

$$M\left\{I_{z}^{v,p,\alpha} z^{w}\right\}(s) = \frac{\Gamma^{\alpha}(s)}{\Gamma(v)} z^{w+v} B(w-s+3, v-s) \tag{22}$$

Proof.

$$\begin{aligned}
M\left\{I_{z}^{v,p,\alpha} z^{w}\right\}(s) &= \int_{0}^{\infty} p^{s-1}\left(\frac{1}{\Gamma(v)} \int_{0}^{z}(z-t)^{v-1} t^{\lambda} E_{\alpha}\left(\frac{-p}{z^{2}} t(z-t)\right) dt\right) dp \\
&= \frac{z^{v-1}}{\Gamma(v)} \int_{0}^{\infty} p^{s-1} \int_{0}^{z}\left(1-\frac{t}{z}\right)^{v-1} t^{\lambda} E_{\alpha}\left(\frac{-p}{z^{2}} t(z-t)\right) dt dp
\end{aligned}$$

making the change of variables $u = \frac{t}{z}$, we have:

$dt = z du$

$t = 0, \ \ u = 0$

$t = z, \ \ u = 1$

$$\begin{aligned} M\left\{I_z^{v,p,\alpha} z^w\right\}(s) &= \frac{z^{v+\lambda}}{\Gamma(v)} \int_0^\infty p^{s-1} \int_0^1 u^\lambda (1-u)^{v-1} E_\alpha - pu(1-u) du dp \\ &= \frac{z^{v+\lambda}}{\Gamma(v)} \int_0^1 u^\lambda (1-u)^{v-1} \int_0^\infty p^{s-1} E_\alpha - pu(1-u) dp du \end{aligned}$$

Thus, if call $r = pu(1-u)$, we have:

$dr = u(1-u)dp$

$p^{s-1} = \frac{r^{s-1}}{u^{s-1}(1-u)^{s-1}}$

$$\begin{aligned} M\left\{I_z^{v,p,\alpha} z^w\right\}(s) &= \frac{z^{v+\lambda}}{\Gamma(v)} \int_0^1 u^{(\lambda-s+1)+1} (1-u)^{v-s+1} \int_0^\infty r^{s-1} E_\alpha(-r) dr du \\ &= \frac{\Gamma^\alpha(s)}{\Gamma(v)} z^{v+\lambda} B(\lambda - s + 3, v - s) \end{aligned}$$

□

Theorem 4.7. *Let $p \geq 0$, $\alpha \in \mathbb{R}^+$, $v > 0$, $a \in \mathbb{C}$ such that $R_e(a) > 0$ and $|z| < 1$. Then*

$$M\left\{I_z^{v,p,\alpha}(1-z)^{-a}\right\}(s) = \frac{\Gamma^\alpha(s)}{\Gamma(v)} z^v B(-s+3, v-s) F(a, -s+3, v-2s+3, z) \tag{23}$$

Proof. Taking into account that $(1-z)^{-a} = \sum_{n=0}^{\infty} (a)_n \frac{z^n}{n!}$, and as the Mellin transform is a linear operator and (22), we have

$$\begin{aligned} M\left\{I_z^{v,p,\alpha}(1-z)^{-a}\right\}(s) &= \sum_{n=0}^{\infty} \frac{(a)_n}{n!} M\left\{I_z^{v,p,\alpha} z^n\right\}(s) \\ &= \sum_{n=0}^{\infty} \frac{(a)_n}{n!} \frac{\Gamma^\alpha(s)}{\Gamma(v)} z^v B(n-s+3, v-s) z^n \\ &= \frac{\Gamma^\alpha(s)}{\Gamma(v)} z^v \sum_{n=0}^{\infty} (a)_n B(n-s+3, v-s) \frac{z^n}{n!} \\ &= \frac{\Gamma^\alpha(s)}{\Gamma(v)} z^v B(n-s+3, v-s) \times F(a, -s+3, v-2s+3, z) \end{aligned}$$

□

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