A Subclass of Starlike Functions Defined with a Differential Operator

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Abstract: This paper deals with a new subclass of starlike functions $A_{n,\lambda}(p, \mu, \beta, \gamma, \delta)$. Coefficients inequality, distortion theorems and closure theorems have been obtained for this class. Further radii of starlikeness and convexity are also obtained for this class.

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1. Introduction

Let $A_p$ denote the class of functions of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} (p \geq 1),$$ (1)

which are analytic and univalent in the unit disk $\Delta = \{z : |z| < 1\}$. Al-Oboudi [1] had introduced a differential operator $D^n_\lambda$ for a function $f(z) \in A_p$ as

$$D^0_\lambda f(z) = f(z),$$

$$D^1_\lambda f(z) = D_\lambda f(z) = (1 - \lambda)f(z) + \lambda f'(z),$$

$$\ldots \ldots \ldots$$

$$D^n_\lambda f(z) = D_\lambda (D^{n-1}_\lambda f(z)),$$

for $n \in N = \{1, 2, 3, \ldots\}$ and $\lambda \geq 0$. It is easy to see that

$$D^n_\lambda f(z) = z + \sum_{k=1}^{\infty} [1 + (p + k - 1)\lambda^n] a_{p+k} z^{p+k}, \ (p \geq 1).$$ (3)

With the help of differential operator $D^n_\lambda$, we define a subclass $A_{n,\lambda}(p, \mu, \beta, \gamma, \delta)$ for $f(z) \in A_p$ such that

$$\left| \frac{D^{n+1}_\lambda f(z)}{D^n_\lambda f(z)} - \delta \right| < \beta, \ (z \in \Delta),$$ (4)

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where \(0 \leq \mu \leq 1\), \(0 < \beta \leq 1\), \(0 \leq \gamma \leq 1\), \(0 \leq \delta \leq 1\), \(\lambda \geq 0\) and \(n \geq 0\). In particular, the class \(A_{n,1}(1, \mu, \beta, \gamma, 1)\), \(A_{0,1}(1, \mu, \beta, \gamma, 1)\), \(A_{0,1}(1, 1, \beta, 0, 1)\) and \(A_{0,1}(1, 0, 1, 0, 1)\) are studied by Aouf and et. \[2\] Lee and et. \[4\] Padmanabhan \[5\] and Singh \[6\] respectively. Let \(T_p\) denote the subclass of \(A_p\) whose elements can be expressed in the form

\[
f(z) = z - \sum_{k=1}^{\infty} a_{p+k} z^{p+k} (a_{p+k} \geq 0, p \geq 1).
\]

We denote \(A_{n,\lambda}(p, \mu, \beta, \gamma, \delta) \cap T_p = A_{n,\lambda}^*(p, \mu, \beta, \gamma, \delta)\). Taking \(p = 1\), \(\delta = 1\), the class \(A_{n,\lambda}^*(p, \mu, \beta, \gamma, \delta)\) reduces to \(S_{n,\lambda}^*(\mu, \beta, \gamma)\). Which was defined and studied by Hossen \[3\]. The object of this paper is to derive several properties of the class \(A_{n,\lambda}^*(p, \mu, \beta, \gamma, \delta)\) such as coefficients inequality, distortion theorem and closure theorems.

### 2. Coefficient Inequalities

**Theorem 2.1.** A function \(f(z)\) of the form (5) is in the class \(A_{n,\lambda}^*(p, \mu, \beta, \gamma, \delta)\) if and only if

\[
\sum_{k=1}^{\infty} |\{1 + (p + k - 1)\lambda\}(1 + \mu) + \mu(1 - \gamma) - \delta| \{1 + (p + k - 1)\lambda\}^n a_{p+k} \leq \{\beta(1 - \gamma + \mu) + 1 - \delta\}.
\]

The result is sharp. The extremal function function is given by

\[
f(z) = z - \frac{\{\beta(1 - \gamma + \mu) + 1 - \delta\}}{\{1 + (p + k - 1)\lambda\}(1 + \mu) + \mu(1 - \gamma) - \delta\{1 + (p + k - 1)\lambda\}^n a_{p+k}}.
\]

**Proof.** First let \(f(z) \in A_{n,\lambda}^*(p, \mu, \beta, \gamma, \delta)\), then from (4) we have

\[
\left| \frac{D^{n+1} f(z)}{D^1 f(z)} - \delta \right| = \left| \frac{\{z - \sum_{k=1}^{\infty} [1 + (p + k - 1)\lambda]^{n+1} a_{p+k} z^{p+k}\} - \delta \{z - \sum_{k=1}^{\infty} [1 + (p + k - 1)\lambda]^{n} a_{p+k} z^{p+k}\}}{\mu \{z - \sum_{k=1}^{\infty} [1 + (p + k - 1)\lambda]^{n+1} a_{p+k} z^{p+k}\} + (1 - \gamma)\{z - \sum_{k=1}^{\infty} [1 + (p + k - 1)\lambda]^{n} a_{p+k} z^{p+k}\}} \right| < \beta,
\]

or

\[
\left| \frac{(1 - \delta) z - \sum_{k=1}^{\infty} [1 + (p + k - 1)\lambda]^{n+1} a_{p+k} z^{p+k}}{(1 - \gamma + \mu) z - \sum_{k=1}^{\infty} [\mu (1 + (p + k - 1)\lambda) + 1 - \gamma] [1 + (p + k - 1)\lambda]^{n} a_{p+k} z^{p+k}} \right| < \beta.
\]

Since \(\text{Re}(z) \leq |z|\) for all \(z\), we find from (8) that

\[
\text{Re} \left\{ \sum_{k=1}^{\infty} [1 + (p + k - 1)\lambda - \delta][1 + (p + k - 1)\lambda]^{n} a_{p+k} z^{p+k} - (1 - \delta)z \left( \frac{\lambda - \mu}{1 - \gamma + \mu} + \mu(1 + (p + k - 1)\lambda) + 1 - \gamma\{1 + (p + k - 1)\lambda]^{n} a_{p+k} z^{p+k}\right) \right\} < \beta.
\]

Choosing values of \(z\) on the real axis so that \(D^{n+1} f(z)/D^1 f(z)\) is real and letting \(z \to 1^+\) through real values, we have

\[
\sum_{k=1}^{\infty} [1 + (p + k - 1)\lambda - \delta][1 + (p + k - 1)\lambda]^{n} a_{p+k} - (1 - \delta) < \beta(1 - \gamma + \mu) - \sum_{k=1}^{\infty} \beta [\mu (1 + (p + k - 1)\lambda) + 1 - \gamma][1 + (p + k - 1)\lambda]^{n} a_{p+k},
\]

or

\[
\sum_{k=1}^{\infty} \{|1 + (p + k - 1)\lambda| (1 + \mu) + \mu(1 - \gamma) - \delta| \{1 + (p + k - 1)\lambda\}^{n} a_{p+k} \leq \{\beta(1 - \gamma + \mu) + 1 - \delta\}.
\]
Conversely let inequality (6) holds. Then

\[
|D_{\lambda} f(z) - \delta D_{\lambda} f(z)| - \beta [D_{\lambda} f(z) + (1 - \gamma)D_{\lambda} f(z)] = |(1 - \delta)z - \sum_{k=1}^{\infty} [1 + (p + k - 1)\lambda - \delta][1 + (p + k - 1)\lambda]a_{p+k}z^{p+k} - \beta (1 - \gamma + \mu)z - \sum_{k=1}^{\infty} [1 + (p + k - 1)\lambda]a_{p+k} - (1 - \delta) \leq \sum_{k=1}^{\infty} [1 + (p + k - 1)\lambda - \delta][1 + (p + k - 1)\lambda]a_{p+k} - (1 - \delta) - \beta (1 - \gamma + \mu) + \sum_{k=1}^{\infty} \beta [1 + (p + k - 1)\lambda] + 1 - \gamma][1 + (p + k - 1)\lambda]a_{p+k} \leq \sum_{k=1}^{\infty} [(1 + (p + k - 1)\lambda)(1 + \beta \mu + \beta (1 - \gamma) - \delta)(1 + (p + k - 1)\lambda)a_{p+k} - \beta (1 - \gamma + \mu) + 1 - \delta] \leq 0,
\]

by the hypothesis. Hence by the maximum modulus theorem, we have \( f(z) \in A_{n,\lambda}(p, \mu, \beta, \gamma, \delta) \).

\( \square \)

**Corollary 2.2.** Let \( f(z) \in T_p \) be in the class \( A_{n,\lambda}(p, \mu, \beta, \gamma, \delta) \). Then

\[
a_{p+k} \leq \frac{\{\beta (1 - \gamma + \mu) + 1 - \delta\}}{[(1 + (p + k - 1)\lambda)(1 + \beta \mu + \beta (1 - \gamma) - \delta)(1 + (p + k - 1)\lambda)]^n}.
\]

for \( k \geq 1, p \geq 1 \). Equality in (9) holds for the function \( f(z) \) given by (7).

3. **Distortion Theorem**

**Theorem 3.1.** Let \( f(z) \in T_p \) be in the class \( A_{n,\lambda}(p, \mu, \beta, \gamma, \delta) \) with \( 0 \leq \mu \leq 1, 0 < \beta \leq 1, 0 \leq \gamma \leq 1, 0 \leq \delta \leq 1, \lambda \geq 0 \) and \( n \geq 0, p \geq 1 \). Then for \( |z| = r < 1 \),

\[
r - \frac{\{\beta (1 - \gamma + \mu) + 1 - \delta\}}{[(1 + p\lambda)(1 + \beta \mu + \beta (1 - \gamma) - \delta)(1 + p\lambda)]^n} \leq |f(z)| \leq r + \frac{\{\beta (1 - \gamma + \mu) + 1 - \delta\}}{[(1 + p\lambda)(1 + \beta \mu + \beta (1 - \gamma) - \delta)(1 + p\lambda)]^n} r^{p+1}.
\]

The result are sharp.

**Proof.** From inequality (6), it follows that

\[
\sum_{k=1}^{\infty} [(1 + (p + k - 1)\lambda)(1 + \beta \mu + \beta (1 - \gamma) - \delta)(1 + (p + k - 1)\lambda)]^n a_{p+k} \leq \{\beta (1 - \gamma + \mu) + 1 - \delta\}.
\]

This implies that

\[
\sum_{k=1}^{\infty} a_{p+k} \leq \frac{\{\beta (1 - \gamma + \mu) + 1 - \delta\}}{[(1 + p\lambda)(1 + \beta \mu + \beta (1 - \gamma) - \delta)(1 + p\lambda)]^n}.
\]

Consequently, for \( |z| = r < 1 \), we obtain

\[
|f(z)| \leq r + r^{p+1} \sum_{k=1}^{\infty} a_{p+k},
\]

or

\[
|f(z)| \leq r + \frac{\{\beta (1 - \gamma + \mu) + 1 - \delta\}}{[(1 + p\lambda)(1 + \beta \mu + \beta (1 - \gamma) - \delta)(1 + p\lambda)]^n} r^{p+1},
\]

and

\[
|f(z)| \geq r - r^{p+1} \sum_{k=1}^{\infty} a_{p+k}.
\]
or

\[ |f(z)| \geq r - \frac{\{\beta(1 - \gamma + \mu) + 1 - \delta\}}{[(1 + p\lambda)(1 + \beta\mu) + \beta(1 - \gamma) - \delta][1 + p\lambda]} \, p^{p+1}. \tag{13} \]

From (12) and (13) inequality (10) follows. The bounds in (10) are attained for the function \( f(z) \) given by

\[
f(z) = z - \frac{\{\beta(1 - \gamma + \mu) + 1 - \delta\}}{[(1 + p\lambda)(1 + \beta\mu) + \beta(1 - \gamma) - \delta][1 + p\lambda]} \, z^{p+k}. \tag{14} \]

\[ \square \]

4. Closure Theorems

**Theorem 4.1.** The class \( A_{n,\lambda}^*(p, \mu, \beta, \gamma, \delta) \) is closed under convex linear combination.

**Proof.** Let each of the functions \( f_1(z) \) and \( f_2(z) \) given by

\[
f_j(z) = z - \sum_{k=1}^{\infty} a_{p+k,j} z^{p+k} (a_{p+k,j} \geq 0, \, j = 1, 2, \, p \geq 1), \tag{15} \]

be in the class \( A_{n,\lambda}^*(p, \mu, \beta, \gamma, \delta) \). Then it is sufficient to show that the function \( F(z) \) defined by

\[
F(z) = tf_1(z) + (1 - t)f_2(z) \quad (0 \leq t \leq 1), \tag{16} \]

is also in the class \( A_{n,\lambda}^*(p, \mu, \beta, \gamma, \delta) \). Since for \( 0 \leq t \leq 1 \),

\[
F(z) = z - \sum_{k=1}^{\infty} [t a_{p+k,1} + (1-t)a_{p+k,2}] z^{p+k}. \]

Then with the aid of Theorem 2.1, we have

\[
\sum_{k=1}^{\infty} \{1 + (p + k - 1)\lambda\}(1 + \beta\mu) + \beta(1 - \gamma) - \delta\} \{1 + (p + k - 1)\lambda\}^{\alpha} \{t a_{p+k,1} + (1-t)a_{p+k,2}\} \leq \{\beta(1 - \gamma + \mu) + 1 - \delta\}. \]

Which implies that \( F(z) \in A_{n,\lambda}^*(p, \mu, \beta, \gamma, \delta) \). \[ \square \]

**Theorem 4.2.** Let

\[
f_0(z) = z, \quad \text{and} \quad f_k(z) = z - \frac{\{\beta(1 - \gamma + \mu) + 1 - \delta\}}{[(1 + (p + k - 1)\lambda)(1 + \beta\mu) + \beta(1 - \gamma) - \delta][1 + (p + k - 1)\lambda]} \, z^{p+k}. \]

Then \( f(z) \in A_{n,\lambda}^*(p, \mu, \beta, \gamma, \delta) \) if and only if it can be expressed in the form

\[
f(z) = \sum_{k=0}^{\infty} t_k f_k(z), \quad \text{where} \quad t_k \geq 0 \quad (k \geq 1) \quad \text{and} \quad \sum_{k=0}^{\infty} t_k = 1. \tag{17} \]

**Proof.** First let \( f(z) \) can be expressed in the form (17). Then

\[
f(z) = \sum_{k=0}^{\infty} t_k f_k(z)
\]

\[ \begin{align*}
&= z - \sum_{k=1}^{\infty} \frac{\{\beta(1 - \gamma + \mu) + 1 - \delta\}}{[(1 + (p + k - 1)\lambda)(1 + \beta\mu) + \beta(1 - \gamma) - \delta][1 + (p + k - 1)\lambda]} t_k z^{p+k} \\
&\quad + (1 - \sum_{k=0}^{\infty} t_k) z
\end{align*} \]

\[
= z - \frac{\{\beta(1 - \gamma + \mu) + 1 - \delta\}}{[(1 + p\lambda)(1 + \beta\mu) + \beta(1 - \gamma) - \delta][1 + p\lambda]} \, z^{p+k}.
\]
Then, it follows that
\[
\sum_{k=1}^{\infty} \frac{[1 + (p + k - 1)\lambda](1 + \beta\mu + \beta(1 - \gamma) - \delta)[1 + (p + k - 1)\lambda]^n}{[1 + (p + k - 1)\lambda](1 + \beta\mu + \beta(1 - \gamma) - \delta)[1 + (p + k - 1)\lambda]^n} t_k
\]
\[
= \frac{\beta(1 - \gamma + \mu + 1 - \delta)}{\beta(1 - \gamma + \mu) + 1 - \delta} \sum_{k=1}^{\infty} t_k = \frac{\beta(1 - \gamma + \mu) + 1 - \delta}{\beta(1 - \gamma + \mu) + 1 - \delta}(1 - t_0)
\]
\[
\leq \frac{\beta(1 - \gamma + \mu)}{1 - \delta}.
\]
Therefore, by Theorem 2.1, \( f(z) \in A_{n,\lambda}^*(p, \mu, \beta, \gamma, \delta) \).

Conversely, let the function \( f(z) \in T_p \) is in the class \( A_{n,\lambda}^*(p, \mu, \beta, \gamma, \delta) \), then we have
\[
a_{p+k} \leq \frac{\beta(1 - \gamma + \mu) + 1 - \delta}{\left[1 + (p + k - 1)\lambda\right](1 + \beta\mu) + \beta(1 - \gamma) - \delta[1 + (p + k - 1)\lambda]^n}, \quad (k \geq 1, p \geq 1),
\]
Setting
\[
t_k = \frac{[1 + (p + k - 1)\lambda](1 + \beta\mu + \beta(1 - \gamma) - \delta)[1 + (p + k - 1)\lambda]^n}{\beta(1 - \gamma + \mu) + 1 - \delta} a_{p+k}, \quad \text{and}
\]
\[
t_0 = 1 - \sum_{k=1}^{\infty} t_k.
\]
It follows that
\[
f(z) = \sum_{k=0}^{\infty} t_k f_k(z)
\]
This complete the proof.

5. Radius of Starlikeness

**Theorem 5.1.** If \( f(z) \in A_{n,\lambda}^*(p, \mu, \beta, \gamma, \delta) \) then the function \( f(z) \) is starlike in the disk \( 0 < |z| < r = r^*_{n,\lambda}(p, \mu, \beta, \gamma, \delta) \), where
\[
r = \inf \left[ \frac{\left( \left[1 + (p + k - 1)\lambda\right](1 + \beta\mu + \beta(1 - \gamma) - \delta[1 + (p + k - 1)\lambda]^n) \right)^{\frac{1}{p+k-1}}}{\beta(1 - \gamma + \mu) + 1 - \delta} \right],
\]
for \( k \geq 1, p \geq 1 \).

**Proof.** It is enough to show that
\[
\left| \frac{z f'(z)}{f(z)} - 1 \right| < 1 \text{ for } |z| < 1,
\]
or
\[
\left| \frac{z f'(z)}{f(z)} - 1 \right| = \left| \frac{-\sum_{k=1}^{\infty} (p + k - 1)a_{p+k}z^{p+k}}{z - \sum_{k=1}^{\infty} a_{p+k}z^{p+k}} \right| < 1,
\]
or
\[
\sum_{k=1}^{\infty} (p + k - 1)a_{p+k}z^{p+k-1} < 1 - \sum_{k=1}^{\infty} a_{p+k}z^{p+k-1},
\]
or
\[
\sum_{k=1}^{\infty} (p + k)a_{p+k}z^{p+k} < 1.
\]
It is easily to see that (19) holds if
\[
|z|^{p+k-1} < \left[ \frac{\left( \left[1 + (p + k - 1)\lambda\right](1 + \beta\mu + \beta(1 - \gamma) - \delta[1 + (p + k - 1)\lambda]^n) \right)^{\frac{1}{p+k-1}}}{\beta(1 - \gamma + \mu) + 1 - \delta} \right].
\]
This complete the proof.
6. Radius of Convexity

Theorem 6.1. If \( f(z) \in A^*_n,\lambda(p, \mu, \beta, \gamma, \delta) \) then the function \( f(z) \) is convex in the disk \( 0 < |z| < r = r^*_n,\lambda(p, \mu, \beta, \gamma, \delta) \), where

\[
 r = \inf \left[ \frac{\left(1 + (p + k - 1)\lambda\right)(1 + \beta\mu) + \beta(1 - \gamma) - \delta\{1 + (p + k - 1)\lambda\}^n}{\beta(1 - \gamma + \mu) + 1 - \delta}(p + k)^2 \right]^{\frac{1}{p+k-1}},
\]

for \( k \geq 1, p \geq 1 \).

Proof. Upon noting the fact that \( f(z) \) is convex if and only if \( zf'(z) \) is starlike, the Theorem 6.1 follows.

References