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Common Fixed Point of Contractive Modulus on Complete Metric Space

Research Article

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Abstract: In this paper, we have proved the existence of unique common fixed point of contractive maps on complete metric space through a weakly compatible maps and contractive modulus.

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Keywords: Complete Metric Space, Common Fixed Point, Contractive Modulus, Weakly Compatible.

1. Introduction and Preliminaries

In 1922, S. Banach proved a fixed point theorem for contraction mapping in complete metric space. The study of fixed points of mappings satisfying certain contraction conditions has been at the center of rigorous research activity. In many years, authors generalized the Banach contraction principle in various spaces such as quasi-metric spaces, fuzzy metric spaces, 2-metric spaces, cone metric spaces, cone Banach space, partial metric spaces and generalized metric spaces and so on. In 1984 [5], Khan M.S., Swaleh M., Sessa S are discuss the concepts of fixed point theorems by altering distances between the points of a complete metric space. In 2012 [3], Hemant Kumar Nashine, Hassen Aydi are extended the concept of Common fixed point theorems for four mappings through generalized altering distances in ordered metric spaces. In 2017 [9], R.Krishnakumar and D.Dhamodharan are establish the concepts of common fixed point of four mapping with contractive modulus on cone Banach space. In this paper, we proved the existence of unique common fixed point of four mapping in contractive modulus and weakly compatible maps on complete metric space.

Definition 1.1. *Let X be a nonempty set, a distance function $d : X \times X \to [0, \infty)$ is called a metric on X if it satisfies the following conditions with*

(1). $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$, $\forall\, x, y \in X$,

(2). $d(x, y) = d(y, x)$, $\forall\, x, y \in X$,

(3). $d(x, y) \leq d(x, z) + d(z, y)$, $\forall\, x, y, z \in X$.

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Then (X, d) is called a metric space.

Example 1.2. *Let $X = R$ and $d : X \times X \to [0, \infty)$ such that $d(x, y) = |x - y|$. Then (X, d) is a metric space (Euclidean metric space).*

Definition 1.3. *Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x in X whenever for every $\epsilon > 0$ there is a natural number $n \in N$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$. It is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.*

Definition 1.4. *Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence whenever for every $\epsilon > 0$ there is a natural number $n_0 \in N$, such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq n_0$.*

Definition 1.5. *Let (X, d) be a metric space, if every Cauchy sequence is convergent in X, then X is called a complete metric space.*

Definition 1.6. *Let f and g be two self maps defined on a set X maps f and g are said to be commuting of $fgx = gfx$ for all $x \in X$.*

Definition 1.7. *Let f and g be two self maps defined on a set X maps f and g are said to be weakly compatible if they commute at coincidence points that is if $fx = gx$ for all $x \in X$ then $fgx = gfx = x$.*

Definition 1.8. *Let f and g be two self maps on set X. If $fx = gx$, for some $x \in X$ then x is called coincidence point of f and g.*

Lemma 1.9. *Let f and g be weakly compatible self mapping of a set X. If f and g have a unique point of coincidence, that is $w = fx = gx$ then w is the unique common fixed point of f and g.*

2. Main Result

Theorem 2.1. *Let (X, d) be a complete metric space. Suppose that the mappings P, Q, S and T are four self maps of (X, d) such that $T(X) \subseteq P(X)$ and $S(X) \subseteq Q(X)$ and satisfying*

$$d(Ty, Sx) \leq ad(Px, Qy) + b\{d(Px, Sx) + d(Qy, Ty)\} + c\{d(Px, Ty) + d(Qy, Sx)\} \tag{1}$$

for all $x, y \in X$, where $a, b, c \geq 0$ and $a + 2b + 2c < 1$. suppose that the pairs $\{P, S\}$ and $\{Q, T\}$ are weakly compatible, then P, Q, S and T have a unique common fixed point.

Proof. Choose x_0 is an arbitrary initial point of X and define the sequence $\{y_n\}$ in X such that

$$y_{2n} = Sx_{2n} = Qx_{2n+1}$$
$$y_{2n+1} = Tx_{2n+1} = Px_{2n+2}$$

By (1) implies that

$$\begin{aligned} d(y_{2n+1}, y_{2n}) &= d(Tx_{2n+1}, Sx_{2n}) \\ &\leq ad(Px_{2n}, Qx_{2n+1}) + b\{d(Px_{2n}, Sx_{2n}) + d(Qx_{2n+1}, Tx_{2n+1})\} + c\{d(Px_{2n}, Tx_{2n+1}) + d(Qx_{2n+1}, Sx_{2n})\} \\ &\leq ad(y_{2n-1}, y_{2n}) + b\{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})\} + c\{d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})\} \\ &\leq ad(y_{2n-1}, y_{2n}) + b\{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})\} + cd(y_{2n-1}, y_{2n+1}) \end{aligned}$$

$$d(y_{2n+1}, y_{2n}) \leq (a+b+c)d(y_{2n-1}, y_{2n}) + (b+c)d(y_{2n}, y_{2n+1})$$
$$d(y_{2n+1}, y_{2n}) \leq \frac{a+b+c}{1-(b+c)} d(y_{2n}, y_{2n-1})$$
$$d(y_{2n+1}, y_{2n}) \leq hd(y_{2n}, y_{2n-1})$$

where $h = \frac{a+b+c}{1-(b+c)} < 1$ for all $n \in N$

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &\leq hd(y_{2n-1}, y_{2n}) \\ &\leq h^2 d(y_{2n-2}, y_{2n-1}) \\ &\vdots \\ &\leq h^{2n-1} d(y_0, y_1) \end{aligned}$$

for all $m > n$

$$\begin{aligned} d(y_n, y_m) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m) \\ &\leq (h^n + h^{n+1} + \cdots h^{m-1}) d(y_0, y_1) \\ &\leq h^n (1 + h + h^2 + \cdots + h^{m-1-n}) d(y_0, y_1) \\ &\leq \frac{h^n}{1-h} d(y_0, y_1) \end{aligned}$$

Hence $\{y_n\}$ is a Cauchy sequence. There exists a point l in (X, d) such that $\lim_{n\to\infty} \{y_n\} = l$, $\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} Q_{2n+1} = l$ and $\lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Px_{2n+2} = l$ that is,

$$\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} Q_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Px_{2n+2} = x^*$$

Since $T(X) \subseteq P(X)$, there exists a point z in X Such that $x^* = Pz$ then by (1)

$$\begin{aligned} d(Sz, x^*) &\leq d(Sz, Tx_{2n-1}) + d(Tx_{2n-1}, x^*) \\ &\leq ad(Pz, Qx_{2n-1}) + b\{d(Pz, Sz) + d(Qx_{2n-1}, Tx_{2n-1})\} + c\{d(Pz, Tx_{2n-1}) + d(Qx_{2n-1}, Sz)\} + d(Tx_{2n-1}, x^*) \end{aligned}$$

Taking the limit as $n \to \infty$

$$\begin{aligned} d(Sz, x^*) &\leq ad(x^*, x^*) + b\{d(x^*, x^*) + d(x^*, Sz)\} + c\{d(x^*, x^*) + d(x^*, Sz)\} + d(x^*, x^*) \\ &\leq 0 + b\{d(x^*, Sz) + 0\} + c\{0 + d(x^*, Sz)\} + 0 \\ &\leq (b+c) d(x^*, Sz) \end{aligned}$$

Which is a contradiction since $a + 2b + 2c < 1$. Therefore $Sz = Pz = x^*$. Since $S(X) \subseteq Q(X)$ there exists a point $w \in X$ such that $x^* = Qw$. By (1)

$$\begin{aligned} d(Sz, x^*) &\leq d(Sz, Tw) \\ &\leq ad(Pz, Qw) + b\{d(Pz, Sz) + d(Qw, Tw)\} + c\{d(Pz, Tw) + d(Qw, Sw)\} \\ &\leq ad(x^*, x^*) + b\{d(x^*, x^*) + d(x^*, Tw)\} + c\{d(x^*, Tw) + d(x^*, x^*)\} \\ &\leq 0 + b\{0 + d(x^*, Tw)\} + c\{d(x^*, Tw) + 0\} \\ d(x^*, Tw) &\leq (b+c) d(x^*, Tw) \end{aligned}$$

which is a contradiction since $a+2b+2c<1$. Therefore $Tw=Qw=x^*$. Thus $Sz=Pz=Tw=Qw=x^*$. Since P and S are weakly compatible maps. Then $SP(z)=PS(z)$; $Sx^*=Px^*$.

To prove that x^* is a fixed point of S. Suppose $Sx^*\neq x^*$ then by (1)

$$\begin{aligned}
d(Sx^*,x^*) &\leq d(Sx^*,Tx^*)\\
&\leq ad(Px^*,Qw)+b\{d(Px^*,Sx^*)+d(Qw,Tw)\}+c\{d(Px^*,Tw)+d(Qw,Sx^*)\}\\
&\leq ad(Sx^*,x^*)+b\{d(Sx^*,Sx^*)+d(x^*,x^*)\}+c\{d(Sx^*,x^*)+d(x^*,Sx^*)\}\\
&\leq ad(Sx^*,x^*)+b\{0+0\}+2cd(Sx^*,x^*)\\
d(Sx^*,x^*) &\leq (a+2c)d(Sx^*,x^*)
\end{aligned}$$

Which is a contradiction, Since $a+2b+2c<1$.

$$Sx^*=x^*$$

Hence $Sx^*=Px^*=x^*$. Similarly, Q and T are weakly compatible maps then $TQw=QTw$, that is $Tx^*=Qx^*$

To prove that x^* is a fixed point of T. Suppose $Tx^*\neq x^*$ by (1)

$$\begin{aligned}
d(Tx^*,x^*) &\leq d(Sx^*,Tx^*)\\
&\leq ad(Px^*,Qx^*)+b\{d(Px^*,Sx^*)+d(Qx^*,Tx^*)\}+c\{d(Px^*,Tx^*)+d(Qx^*,Sx^*)\}\\
&\leq ad(x^*,Tx^*)+b\{d(x^*,x^*)+d(Tx^*,Tx^*)\}+c\{d(x^*,Tx^*)+d(Tx^*,x^*)\}\\
&\leq ad(Tx^*,x^*)+b\{0+0\}+2cd(Tx^*,x^*)\\
d(Tx^*,x^*) &\leq (a+2c)d(Tx^*,x^*)
\end{aligned}$$

which is a contradiction since $a+2b+2c<1$.

$$Tx^*=x^*.$$

Hence $Tx^*=Qx^*=x^*$. Thus $Sx^*=Px^*=Tx^*=Qx^*=x^*$. That is, x^* is a common fixed point of P,Q,S and T.

To prove that the uniqueness of x^*. Suppose that x^* and y^*, $x^*\neq y^*$ are common fixed points of P,Q,S and T respectively, by (1) we have,

$$\begin{aligned}
d(x^*,y^*) &\leq d(Sx^*,Ty^*)\\
&\leq ad(Px^*,Qy^*)+b\{d(Px^*,Sx^*)+d(Qy^*,Ty^*)\}+c\{d(Px^*,Ty^*)+d(Qy^*,Sx^*)\}\\
&\leq ad(x^*,y^*)+b\{d(x^*,x^*)+d(y^*,y^*)\}+c\{d(x^*,y^*)+d(y^*,x^*)\}\\
&\leq ad(x^*,y^*)+b\{0+0\}+c\{d(x^*,y^*)+d(y^*,x^*)\}\\
&\leq (a+2c)d(x^*,y^*)
\end{aligned}$$

which is a contradiction. Since $a+2b+2c<1$. Therefore $x^*=y^*$. Hence x^* is the unique common fixed point of P,Q,S and T respectively. □

Corollary 2.2. *Let (X,d) be a Complete metric space. Suppose that the mappings P,S and T are three self maps of (X,d) such that $T(X)\subseteq P(X)$ and $S(X)\subseteq P(X)$ and satisfying $d(Sx,Ty)\leq ad(Px,Py)+b\{d(Px,Sy)+d(Px,Ty)\}+c\{d(Px,Ty)+d(Py,Sx)\}$ for all $x,y\in X$, where $a,b,c\geq 0$ and $a+2b+2c<1$. Suppose that the pairs $\{P,S\}$ and $\{P,T\}$ are weakly compatible, then P,S and T have a unique common fixed point.*

Proof. The proof of the corollary immediate by taking $P = Q$ in the above Theorem 2.1. □

Definition 2.3 ([9]). *A function $\Phi : [0, \infty) \to [0, \infty)$ is said to be contractive modulus if Φ is continuous function and $\Phi(t) < t$ for $t > 0$.*

Theorem 2.4. *Let (X, d) be a Complete metric space. Suppose that the mappings P, Q, S and T are four self maps of (X, d) such that $T(X) \subseteq P(X)$ and $S(X) \subseteq Q(X)$ satisfying*

$$d(Sx, Ty) \leq \Phi(\lambda(x, y)), \tag{2}$$

where Φ is an upper semi continuous contractive modulus and

$$\lambda(x, y) = \max\left\{d(Px, Qy), d(Px, Sx), d(Qy, Ty), \frac{1}{2}\{d(Px, Ty) + d(Qy, Sx)\}\right\}.$$

The pair $\{S, P\}$ and $\{T, Q\}$ are weakly compatible. Then P, Q, S and T have a unique common fixed point.

Proof. Let us take x_0 is an arbitrary point of X and define a sequence $\{y_{2n}\}$ in X such that

$$y_{2n} = Sx_{2n} = Qx_{2n+1}$$
$$y_{2n+1} = Tx_{2n+1} = Px_{2n+2}$$

By (2) implies that

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\
&\leq \Phi(\lambda(x_{2n}, x_{2n+1})) \\
&\leq \lambda(x_{2n}, x_{2n+1}) \\
&= \max\left\{d(Px_{2n}, Qx_{2n+1}), d(Px_{2n}, Sx_{2n}), d(Qx_{2n+1}, Tx_{2n+1}), \frac{1}{2}\{d(Px_{2n}, Tx_{2n+1}) + d(Qx_{2n+1}, Sx_{2n})\}\right\} \\
&= \max\left\{d(Tx_{2n-1}, Sx_{2n}), d(Tx_{2n-1}, Sx_{2n}), d(Sx_{2n}, Tx_{2n+1}), \frac{1}{2}\{d(Tx_{2n-1}, Tx_{2n+1}) + d(Sx_{2n}, Sx_{2n})\}\right\} \\
&= \max\left\{d(Tx_{2n-1}, Sx_{2n}), d(Tx_{2n-1}, Sx_{2n}), d(Sx_{2n}, Tx_{2n+1}), \frac{1}{2}d(Tx_{2n-1}, Tx_{2n+1})\right\} \\
&= \max\left\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), \frac{1}{2}d(y_{2n-1}, y_{2n+1})\right\} \\
&\leq \max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1})\}
\end{aligned}$$

Since Φ is an contractive modulus, $\lambda(x_{2n} - x_{2n+1}) = d(y_{2n}, y_{2n+1})$ is not possible. Thus,

$$d(y_{2n}, y_{2n+1}) \leq \Phi(d(y_{2n-1}, y_{2n})) \tag{3}$$

Since Φ is an upper semi continuous, contractive modulus. Equation (3) implies that the sequence $\{d(y_{2n+1}, y_{2n})\}$ is monotonic decreasing and continuous. There exists a real number, say $r \geq 0$ such that $\lim_{n\to\infty} d(y_{2n+1}, y_{2n}) = r$, as $n \to \infty$ equation (3) $\Rightarrow r \leq \Phi(r)$ which is only possible if $r = 0$ because Φ is a contractive modulus. Thus $\lim_{n\to\infty} d(y_{2n+1}, y_{2n}) = 0$.

Claim: $\{y_{2n}\}$ is a Cauchy sequence. Suppose $\{y_{2n}\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ and sub sequence $\{n_i\}$ and $\{m_i\}$ such that $m_i < n_i < m_{i+1}$

$$d(y_{m_i}, y_{n_i}) \geq \epsilon \quad \text{and} \quad d(y_{m_i}, y_{n_{i-1}}) \leq \epsilon \tag{4}$$

$\epsilon \leq d(y_{m_i}, y_{n_i}) \leq d(y_{m_i}, y_{n_{i-1}}) + d(y_{n_{i-1}}, y_{n_i})$ therefore $\lim_{i\to\infty} d(y_{m_i}, y_{n_i}) = \epsilon$. Now $\epsilon \leq d(y_{m_{i-1}}, y_{n_{i-1}}) \leq d(y_{m_{i-1}}, y_{m_i}) + d(y_{m_i}, y_{n_{i-1}})$ by taking limit $i \to \infty$ we get, $\lim_{i\to\infty} d(y_{m_{i-1}}, y_{n_{i-1}}) = \epsilon$, from (3) and (4)

$$\epsilon \leq d(y_{m_i}, y_{n_i}) = d(Sx_{m_i}, Tx_{n_i}) \leq \Phi(\lambda(x_{m_i}, x_{n_i}))$$

where implies

$$\epsilon \leq \Phi(\lambda(x_{m_i}, x_{n_i})) \tag{5}$$

$$\begin{aligned}\lambda(x_{m_i}, x_{n_i}) &= \max\left\{d(Px_{m_i}, Qx_{n_i}), d(Px_{m_i}, Sx_{m_i}), d(Qx_{n_i}, Tx_{n_i}), \frac{1}{2}(d(Px_{m_i}, Tx_{n_i}) + d(Qx_{n_i}, Sx_{m_i}))\right\}\\ &= \max\left\{d(Tx_{m_{i-1}}, Sx_{n_{i-1}}), d(Tx_{m_{i-1}}, Sx_{m_i}), d(Sx_{n_{i-1}}, Tx_{n_i}), \frac{1}{2}(d(Tx_{m_{i-1}}, Tx_{n_i}) + d(Sx_{n_{i-1}}, Sx_{m_i}))\right\}\\ &= \max\left\{d(y_{m_{i-1}}, y_{n_{i-1}}), d(y_{m_{i-1}}, y_{m_i}), d(y_{n_{i-1}}, y_{n_i}), \frac{1}{2}(d(y_{m_{i-1}}, y_{n_i}) + d(y_{n_{i-1}}, y_{m_i}))\right\}\end{aligned}$$

Taking limit as $i \to \infty$, we get

$$\begin{aligned}&\lim_{i\to\infty} \lambda(x_{m_i}, x_{n_i}) = \max\{\epsilon, 0, 0, \frac{1}{2}(\epsilon, \epsilon)\}\\ &\lim_{i\to\infty} \lambda(x_{m_i}, x_{n_i}) = \epsilon\end{aligned}$$

Therefore from (5) we have, $\epsilon \leq \Phi(\epsilon)$. This is a contradiction because $\epsilon > 0$ and Φ is contractive modulus. Therefore $\{y_{2n}\}$ is Cauchy sequence in X. There exits a point z in X such that $\lim_{n\to\infty} y_{2n} = z$. Thus,

$$\begin{aligned}&\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Qx_{2n+1} = z \quad \text{and}\\ &\lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Px_{2n+2} = z\\ (i.e)\quad &\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Qx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Px_{2n+2} = z\end{aligned}$$

$T(X) \subseteq P(X)$, there exists a point $u \in X$ such that $z = Pu$

$$\begin{aligned}d(Su, z) &\leq d(Su, Tx_{2n+1}) + d(Tx_{2n+1}, z)\\ &\leq \Phi(\lambda(u, x_{2n+1})) + d(Tx_{2n+1}, z), \quad \text{where}\\ \lambda(u, x_{2n+1}) &= \max\left\{d(Pu, Qx_{2n+1}), d(Pu, Su), d(Qx_{2n+1}, Tx_{2n+1}), \frac{1}{2}(d(Pu, Tx_{2n+1}) + d(Qx_{2n+1}, Su))\right\}\\ &= \max\left\{d(z, Sx_{2n}), d(z, Su), d(Sx_{2n}, Tx_{2n+1}), \frac{1}{2}(d(z, Tx_{2n+1}) + d(Sx_{2n}, Su))\right\}.\end{aligned}$$

Now taking the limit as $n \to \infty$ we have,

$$\begin{aligned}\lambda(u, x_{2n+1}) &= \max\{d(z, Su), d(z, Su), d(Su, Tu), \frac{1}{2}(d(z, Tu) + d(z, Su))\}\\ &= \max\{d(z, Su), d(z, Su), d(Su, z), \frac{1}{2}(d(z, z) + d(z, Su))\}\\ &= d(z, Su)\end{aligned}$$

Thus

$$d(Su, z) \leq \Phi(d(Su, z)) + d(z, z) = \Phi(d(Su, z))$$

If $Su \neq z$ then $d(Su, z) > 0$ and hence as Φ is contracive modulus $\Phi(d(Su, z)) < d(Su, z)$. Which is a contradiction, $Su = z$ so, $Pu = Su = z$. So u is a coincidence point if P and S. The pair of maps S and P are weakly compatible $SPu = PSu$ that is $Sz = Pz$. $S(X) \subseteq Q(X)$, there exists a point $v \in X$ such that $z = Qv$. Then we have

$$\begin{aligned} d(z, Tv) &= d(Su, Tv) \\ &\leq \Phi(\lambda(u, v)) \leq \lambda(u, v) \\ &= \max\left\{d(Pu, Qv), d(Pu, Su), d(Qv, Tv), \frac{1}{2}(d(Pu, Tv) + d(Qv, Su))\right\} \\ &= \max\left\{d(z, z), d(z, z), d(z, Tv), \frac{1}{2}(d(z, Tv) + d(z, z))\right\} \\ &= d(z, Tv) \end{aligned}$$

Thus $d(z, Tv) \leq \Phi(d(z, Tv))$. If $Tv \in z$ then $d(z, Tv) \geq 0$ and hence as Φ is contractive modulus $\Phi(d(z, Tv)) < d(z, Tv)$. Therefore $d(z, Tv) < d(z, Tv)$ which is a contradiction. Therefore $Tv = Qv = z$. So, v is a coincidence point of Q and T. Since the pair of maps Q and T are weakly compatible, $QTv = TQv$ (i.e) $Qz = Tz$. Now show that z is a fixed point of S. We have

$$\begin{aligned} d(Sz, z) &= d(Sz, Tv) \\ &\leq \Phi(\lambda(z, v)) \leq \lambda(z, v) \\ &= \max\{d(Pz, Qv), d(Pz, Sz), d(Qv, Tv), \frac{1}{2}(d(Pz, Tv) + d(Qv, Sz))\} \\ &= \max\{d(Sz, z), d(Sz, Sz), d(z, z), \frac{1}{2}(d(Sz, z) + d(z, Sz))\} \\ &= d(Sz, z) \end{aligned}$$

Thus $d(Sz, z) \leq \Phi(d(Sz, z))$. If $Sz \neq z$ then $d(Sz, z) > 0$ and hence as Φ is contractive modulus $\Phi(d(Sz, z)) < d(Sz, z)$ which is a contradiction. There exits $Sz = z$. Hence $Sz = Pz = z$. Show that z is a fixed point of T. We have

$$\begin{aligned} d(z, Tz) &= d(Sz, Tz) \\ &\leq \Phi(\lambda(z, z)) \leq \lambda(z, z) \\ &= \max\left\{d(Pz, Qz), d(Pz, Sz), d(Qz, Tz), \frac{1}{2}(d(Pz, Tz) + d(Qz, Sz))\right\} \\ &= \max\left\{d(z, Tz), d(z, z), d(Tz, Tz), \frac{1}{2}(d(z, Tz) + d(Tz, z))\right\} \\ &= d(z, Tz) \end{aligned}$$

Thus $d(z, Tz) \leq \Phi(d(z, Tz))$. If $z \neq Tz$ then $d(z, Tz) > 0$ and hence as Φ is contractive modulus $\Phi(d(z, Tz)) < d(z, Tz)$ which is a contradiction. Hence $z = Tz$. Therefore $Tz = Qz = z$. Therefore $Sz = Pz = Tz = Qz = z$. That is z is common fixed point of P, Q, S and T.

Uniqueness: Suppose, z and w $\quad (z \neq w)$ are common fixed point of P, Q, S and T, we have

$$\begin{aligned} d(z, w) &= d(Sz, Tw) \\ &\leq \Phi(\lambda(z, w)) \leq \lambda(z, w) \\ &= \max\left\{d(Pz, Qw), d(Pz, Sz), d(Qw, Tw), \frac{1}{2}(d(Pz, Tw) + d(Qw, Sz))\right\} \\ &= \max\left\{d(z, w), d(z, z), d(w, w), \frac{1}{2}(d(z, w) + d(w, z))\right\} \\ &= d(z, w) \end{aligned}$$

Thus, $d(z,w) \leq \Phi(d(z,w))$. Since $z \neq w$, then $d(z,wd) > 0$ and hence as Φ is contractive modulus $\Phi(d(z,w)) < d(z,w)$. Therefore $d(z,w) < d(z,w)$ which is a contradiction, therefore $z = w$. Thus z is the unique common fixed point of P, Q, S and T. □

Corollary 2.5. *Let (X,d) be a complete metric space. Suppose that the mappings P, S and T are three self maps of (X,d) such that $T(X) \subseteq P(X)$ and $S(X) \subseteq P(X)$ satisfying*

$$d(Sx,Ty) \leq \Phi(\lambda(x,y)), \tag{6}$$

where Φ is an upper semi continuous contractive modulus and

$$\lambda(x,y) = \max\left\{d(Px,Py), d(Px,Sx), d(Py,Ty), \frac{1}{2}\{d(Px,Ty) + d(Py,Sx)\}\right\}.$$

The pair $\{S,P\}$ and $\{T,P\}$ are weakly compatible. Then P, S and T have a unique common fixed point.

Proof. The proof of the corollary immediate by taking $P = Q$ in the above Theorem 2.4. □

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