International Journal of Mathematics And its Applications
Volume 5, Issue 4–D (2017), 529–534.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

On ω-compact Spaces

Research Article

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Abstract: In [3] the author, introduced the notion of ω-compactness and investigated its fundamental properties. In this paper, we investigate some more properties of this type of compact space.

MSC: 54C60.

Keywords: Topological spaces, ω-open set, ω-compact spaces.

1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, seperation axioms etc. by utilizing generalized open sets. Compactness and properties closely related to compactness play an important role in the applications of General Topology to Real Analysis and Functional Analysis. In [3] the author, introduced the notion of ω-compactness and investigated its fundamental properties. In this paper, we investigate some more properties of this type of compact space.

2. Preliminaries

Throughout this paper, spaces always means topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A and the interior of A in X, respectively. A subset A of X is said to be semiopen [1] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$. A subset A of a space (X, τ) is called an ω-closed set [2] if $\mathrm{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is semiopen in (X, τ). The complement of an ω-closed set is called an ω-open set [2]. The intersection of all ω-closed sets containing $A \subset X$ is called the ω-closure [3] of A and is denoted by $\omega\,\mathrm{Cl}(E)$. The union of all ω-open sets contained in $A \subset X$ is called the ω-interior [3] of A and is denoted by $\omega\,\mathrm{Int}(E)$. Let (X, τ) be a topological space. The family of all ω-openr (resp. ω-closed) sets of (X, τ) is denoted by $\omega O(X)$ (resp. $\omega C(X)$). The family of all ω-open (resp. ω-closed) sets of (X, τ) containing a point $x \in X$ is denoted by $\omega O(X, x)$ (resp. $\omega C(X, x)$). A subset K of a nonempty set X is said to be ω-compact relative to (X, τ) if every cover of K by ω-open sets of (X, τ) has a finite subcover. We say that (X, τ) is ω-compact if X is ω-compact.

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3. ω-compact Spaces

We will give several characterizations of the ω-compact spaces. The first characterization makes use of the finite intersection condition.

Theorem 3.1. *The following statements are equivalent for any topological space (X, τ):*

(i). X is ω-compact.

(ii). Given any family $\mathcal{F}$ of ω-open sets, if no finite subfamily of $\mathcal{F}$ covers X, then $\mathcal{F}$ does not cover X.

(iii). Given any family $\mathcal{F}$ of ω-closed sets, if $\mathcal{F}$ satisfies the finite intersection condition, then $\cap\{A : A \in \mathcal{F}\} \neq \emptyset$.

(iv). Given any family $\mathcal{F}$ of subsets of X, if $\mathcal{F}$ satisfies the finite intersection condition, then $\cap\{\omega\,\mathrm{Cl}(A) : A \in \mathcal{F}\} \neq \emptyset$.

Proof. $(i) \Leftrightarrow (ii)$ and $(ii) \Leftrightarrow (iii)$ are obivious. $(iii) \Rightarrow (iv)$: If $\mathcal{F} \subset P(X)$ satisfies the finite intersection condition, then $\cap\{\omega\,\mathrm{Cl}(A) : A \in \mathcal{F}\}$ is a family of ω-closed sets, which obviously satisfies the finite intersection condition. $(iv) \Rightarrow (iii)$ Follows from the fact that $A = b\,\mathrm{Cl}(A)$ for every ω-closed set A. □

Theorem 3.2 ([3]). *If $A \subseteq B \subseteq (X, \tau)$ where A is ω-open relative to B and B is open in (X, τ), then A is ω-open in X.*

Theorem 3.3. *Every open subset of a ω-compact space is ω-compact, in particular, ω-compactness is hereditary with respect to open sets.*

Proof. Let A be an open subset of an ω-compact space X. If $\{U_\alpha : \alpha \in \Delta\}$ is an ω-open cover of (A, τ_A), then by Theorem 3.2, each U_α is ω-open in X. Then $\{U_\alpha : \alpha \in \Delta\} \cup \{X\backslash A\}$ is ω-open cover of X. Since X is ω-compact, there exists a finite subset $\Delta_0 \subset \Delta$ such that $\{U_\alpha : \alpha \in \Delta_0\}$ covers A. □

Definition 3.4. *A point $x \in X$ is said to be ω-cluster point of a net $\{x_\alpha\}_{\alpha\in\Delta}$ if $\{x_\alpha\}_{\alpha\in\Delta}$ is frequently in every ω-open set containing x. We denote by ω-cp$\{x_\alpha\}_{\alpha\in\Delta}$ the set of all ω-cluster points of a net $\{x_\alpha\}_{\alpha\in\Delta}$.*

Theorem 3.5. *The set of all ω-cluster points of an arbitrary net in X is ω-closed.*

Proof. Let $\{x_\alpha\}_{\alpha\in\Delta}$ be a net in X. Set $A = \omega - cp\{x_\alpha\}_{\alpha\in\Delta}$. Let $x \in X\backslash A$. Then there exists a ω-open set U_x containing x and $\alpha_x \in \Delta$ such that $X_\beta \notin U_x$ whenever $\beta \in \Delta$, $\beta \geq \alpha_x$. It turns out that $U_x \subset X\backslash A$, hence $x \in \omega\,\mathrm{Int}(X\backslash A) = X\backslash\omega\,\mathrm{Cl}(A)$. This shows that $\omega\,\mathrm{Cl}(A) \subset A$; hence A is ω-closed. □

Theorem 3.6. *A topological space X is ω-compact if and only if each net $\{x_\alpha\}_{\alpha\in\Delta}$ in X, has atleast one ω-cluster point.*

Proof. Let X be a ω-compact space. Assume that there exist some net $\{x_\alpha\}_{\alpha\in\Delta}$ in X such that ω-cp$\{x_\alpha\}_{\alpha\in\Delta}$ is empty. Then for every $x \in X$, there exist $U(x) \in \omega O(X, x)$ and $\alpha(x) \in \Delta$, such that $x_\beta \notin U(x)$ whenever $\beta \geq \alpha(x)$, $\beta \in \Delta$. Then the family $\{U(x) : x \in X\}$ is a cover of X by ω-open sets and has a finite subcover, say, $\{U_k : k = 1, 2, ...n\}$ where $U_k = U(x_k)$ for k = 1,2,....n, $\{x_k : k = 1, 2, ...n\}$. Let us take $\alpha \in \Delta$ such that $\alpha \geq \alpha(x_k)$ for all $k \in \{1, 2, ...n\}$. For every $\beta \in \Delta$ such that $\beta \geq \alpha$ we have, $x_\beta \notin U_k$, k = 1, 2,....n, hence $x_\beta \notin X$, which is a contradiction. Conversely, if X is not ω-compact, there exists $\{U_i : i \in I\}$ a cover of X by ω-open sets, which has no finite subcover. Let $P(I)$ be the family of all finite subsets of I. Clearly, $(P(I), \subseteq)$ is a directed set. For each $J \in \mathcal{I}$. we may choose $x_j \in X\backslash \cup \{U_i : i \in J\}$. Let us consider the net $\{x_j\}_{j\in P(I)}$. By hypothesis, the set ω-$cp\{x_j\}_{j\in P(I)}$ is nonempty. Let $x \in \omega$-$cp\{x_j\}_{j\in P(I)}$ and let $i_0 \in I$ such that $x \in U_{i0}$. By the definition of ω-cluster point, for each $J \in P(I)$ there exist $J^* \in P(I)$ such that $J \subset J^*$ and $x^*_j \in U_{i0}$. For $J = \{i_0\}$, there exists $J^* \in P(I)$ such that $i_0 \in J^*$ and $x^*_j \in U_{i0}$. But $x^*_j \in X\backslash \cup \{U_i : i \in J^*\} \subset X\backslash U_{i_0}$. The contradiction we obtained shows that X is ω-compact. □

In the following, we will give a characterization of ω-compact spaces by means of filterbases.

Let us recall that a nonempty family $\mathcal{F}$ of subsets of X is said to be a filterbase on X if $\emptyset \notin \mathcal{F}$ and each intersection of two members of $\mathcal{F}$ contains a third member of $\mathcal{F}$. Notice that each chain in the family of all filterbase on X (ordered by inclusion) has an upper bound, for example, the union of all members of the chain. Then, by Zorn's Lemma, the family of all filterbases on X has atleast one maximal element. Similarly, the family of all filterbases on X containing a given filterbase $\mathcal{F}$ has atleast one maximal element.

Definition 3.7. *A filterbase $\mathcal{F}$ on a topological space X is said to be:*

(i). ω-converge to a point $x \in X$ if for each ω-open set U containing x, there exists $B \in \mathcal{F}$ such that $B \subset U$.

(ii). ω-accumulate at $x \in X$ if $U \cap B \neq \emptyset$ for every ω-open set U containing x and every $B \in \mathcal{F}$.

Remark 3.1. *A filterbase $\mathcal{F}$ ω-accumulate at x if and only if $x \in \cap\{\omega \operatorname{Cl}(B) : B \in \mathcal{F}\}$. Clearly, if a filterbase $\mathcal{F}$ ω-converges to $x \in X$, then $\mathcal{F}$ ω-accumulates at x.*

Lemma 3.8. *If a maximal filterbase $\mathcal{F}$ ω-accumulate at $x \in X$, then $\mathcal{F}$ ω-converges to x.*

Proof. Let $\mathcal{F}$ be a maximal filterbase which ω-accumulate at $x \in X$. If $\mathcal{F}$ does not ω-converges to x, then there exists a ω-open set U_0 containing x such that $U_0 \cap B \neq \emptyset$ and $(X \backslash U_0) \cap B \neq \emptyset$ for every $B \in \mathcal{F}$. Then $\mathcal{F} \cup \{U_0 \cap B : B \in \mathcal{F}\}$ is a filterbase which strictly contains $\mathcal{F}$, which is a contradiction. □

Theorem 3.9. *For a topological space X, the following statements are equivalent:*

(i). X is ω-compact;

(ii). Every maximal filterbase ω-converges to some point of X;

(iii). Every filterbase ω-accumulates at some point of X.

Proof. $(i) \Rightarrow (ii)$: Let $\mathcal{F}_0$ be a maximal filterbase on X. Suppose that $\mathcal{F}_0$ does not ω-converge to any point of X. Then, by Lemma 3.8, $\mathcal{F}_0$ does not ω-accumulate at any point of X. For each $x \in X$, there exists a ω-open set U_x containing x and $B_x \in \mathcal{F}_0$ such that $U_x \cap B_x = \emptyset$. The family $\{U_x : x \in X\}$ is a cover of X by ω-open sets. By (i), there exists a finite subset $\{x_1, x_2, ...x_n\}$ of X such that $X = \cup\{U_{x_k} : k = 1, 2, ...n\}$. Since $\mathcal{F}_0$ is a filterbase, there exists $B_0 \in \mathcal{F}_0$ such that $B_0 \subset \cap\{B_{x_k} : k = 1, 2, ...n\} = X \backslash \cup \{U_{x_k} : k = 1, 2, ...n\}$, hence $B_0 = \emptyset$. This is a contradiction. $(ii) \Rightarrow (iii)$: Let $\mathcal{F}$ be a filterbase on X. There exists a maximal filterbase $\mathcal{F}_0$ such that $\mathcal{F} \subset \mathcal{F}_0$. By (ii), $\mathcal{F}_0$ ω-converges to some point $x_0 \in X$. Let $B \in \mathcal{F}$. For every $U \in \omega O(X, x_0)$, there exists $B_U \in \mathcal{F}_0$ such that $B_U \subset U$, hence $U \cap B \neq \emptyset$, since it contains the member $B_U \cap B$ of $\mathcal{F}_0$. This shows that $\mathcal{F}$ ω-accumulates at x_0. $(iii) \Rightarrow (i)$: Let $\{V_i : i \in I\} = \emptyset$ be any family of ω-closed sets such that $\cap\{V_i : i \in I\} = \emptyset$. We shall prove that there exists a finite subset I_0 of I such that $\cap\{V_i : i \in I\}$. By Theorem 3.1, this implies (i). Let $P(I)$ be the family of finite subsets of I. Assume that $\cap\{V_i : i \in J\} = \emptyset$ for every $J \in P(I)$ (say $*$). Then the family $\mathcal{F} = \{\cap\{V_i : i \in J\} : J \in P(I)\}$ is a filterbase on X. By (iii), $\mathcal{F}$ ω-accumulates to some point $x_0 \in X$. Since $\{X \backslash V_i : i \in I\}$ is a cover of X, there exists $i_0 \in I$ such that $x_0 \in X \backslash V_{i_0}$. Then $X \backslash V_{i0}$ is a ω-open set containing x_0, $V_{i_0} \in \mathcal{F}$ and $(X \backslash V_{i_0}) \cap V_{i_0} = \emptyset$. This is a contradiction with the fact that $\mathcal{F}$ ω-accumulates at x_0 shows that (*) is false. □

Definition 3.10. *A point x in a topological space X is said to be a ω-complete accumulation point of a subset S of X if $n(S \cap A) = n(S)$ for each $A \in \omega O(X, x)$, where $n(S)$ denotes the cardinality of S.*

Definition 3.11. *In a topological space (X, τ), a point x is said to be a ω-adherent point of a filterbase $\mathcal{F}$ on X if it lies in the ω-closure of all sets of $\mathcal{F}$.*

Theorem 3.12. *A topological space (X, τ) is ω-compact if and only if each infinite subset of X has a ω-complete accumulation point.*

Proof. Let the topological space (X, τ) be ω-compact and A an infinite subset of X. Let K be the set of all points x in X which are not ω-complete accumulation points of S. Now it is obvious that for each point x in K, we are able to find $U(x) \in \omega O(X, x)$ such that $n(A \cap U(x)) \neq n(S)$. If K is the whole space X, then $\mathcal{F} = \{U(x)x \in X\}$ is a ω-cover of X. By hypothesis, X is ω-compact. So, there exists a finite subcover $\mathcal{G} = \{U(x_i) : i = 1, 2, ...n\}$, such that $A \subset \cup\{U(x_i) \cap A : i = 1, 2, ...n\}$. Then $n(S) = max\{n(U(x_i) \cap A) : i = 1, 2, ...n\}$ which does not agree with what we assumed. This implies that A has a ω-complete accumulation point. Now assume that X is not ω-compact and that every infinite subset A of X has a ω-complete accumulation point in X. It follows that there exists a ω-cover $\mathcal{S}$ with no finite subcover. Set $\alpha = min\{n(\Psi)$: $\Psi \subset \mathcal{S}$, where Ψ is a ω-cover of X$\}$. Fix $\Psi = \mathcal{S}$ for which $n(\Psi) = \alpha$ and $\cup\{U : U \in \Psi\} = X$. Then, by hypothesis $\alpha \geq n(N)$, where N denotes the set of all natural numbers. By well-ordering of Ψ by some minimal well-ordering "$\sim$", suppose that U is any member of Ψ. By minimal well-ordering "$\sim$" we have $n(\{V : V \in \Psi, V \sim U\}) < n(\{V : V \in \Psi\})$. Since Ψ cannot have any subcover with cardinality less than α, then for each $U \in \Psi$ we have $X \neq \cup\{V; V \in \Psi, V \sim U\})$. For each $U \in \Psi$, choose a point $x(U) \in X \backslash \cup \{V \cup \{x(V)\}; V \in \Psi, V \sim U\})$. We are always able to do this if not one can choose a cover of smaller cardinality from Ψ. If $H = \{x(U) : U \in \Psi\}$, then to finish the proof we will show that H has no ω-complete accumulation point in X. Suppose $z \in X$. Since Ψ is a ω-cover of X, z is a point of some set, say W in Ψ. By the fact that $U \sim W$, we have $x(U) \in W$. It follows that $T = \{U : U \in \Psi$ and $x(U) \in W\} \subset \{V; V \in \Psi, V \sim W\}$. But $n(T) < \alpha$. Therefore, $n(H \cap W) < \alpha$. But $n(H) = \alpha \geq n(N)$. Since for two distinct points U and W in Ψ, we have $x(U) \neq x(W)$. This means that H has no ω-complete accumulation point in X, which contradicts our assumption. Therefore X is ω-compact. □

Theorem 3.13. *For a topological space (X, τ), the following statements are equivalent:*

(i). X is ω-compact;

(ii). Every net in X with a well-ordered directed set as its domain ω-accumulates to some point of X.

Proof. $(i) \Rightarrow (ii)$: Suppose that X is ω-compact and $A = \{x_\alpha : \alpha \in \Delta\}$ a net with a well-ordered directed set Δ as domain. Assume that A has no ω-adherent point in X. Then for each $x \in X$, there exists $V(x) \in \omega O(X, x)$ and an $\alpha(x) \in \Delta$ such that $V(x) \cap \{x_\alpha : \alpha \geq \alpha(x)\} = \emptyset$. This implies that $\{x_\alpha : \alpha \geq \alpha(x)\}$ is a subset of $X \backslash V(x)$. Then the collection $\mathcal{F} = \{V(x) : x \in X\}$ is a ω-open cover of X. Since X is ω-compact, $\mathcal{F}$ has a finite subfamily $\{V_{xi} : i = 1, 2, ...n\}$ such that $X = \overset{n}{\underset{i=1}{\cup}}\{V(x_i) : i = 1, 2, ...n\}$. Suppose that the corresponding elements of Δ are $\{\alpha(x_i)\}$, where $i = 1, 2, ...n$. Since Δ is well-ordered and $\{\alpha(x_i) : i = 1, 2, ...n\}$ is finite, the largest element of $\{\alpha(x_i)\}$ exists. Suppose it is $\{\alpha(x_i)\}$. Then for $\beta \geq \{\alpha(x_i)\}$, we have $\{x_\delta : \delta \geq \beta\} \subset \overset{n}{\underset{i=1}{\cap}}\{X \backslash V(x_i)\} = X \backslash \overset{n}{\underset{i=1}{\cup}} V(x_i) = \emptyset$, which is impossible. This shows that A has atleast one ω-adherent point in X. $(ii) \Rightarrow (i)$: Now it is enough to prove that each infinite subset has a ω-complete accumulation point by utilizing Theorem 3.12. Suppose that S is an infinite subset of X. According to Zorn's Lemma, the infinite set S can be well-ordered. This means that we can assume S to be a net with a domain which is a well-ordered index set. It follows that S has a ω-adherent point z. Therefore, z is a ω-complete accumulation point of S. This shows that X is ω-compact. □

Theorem 3.14. *A topological space X is ω-compact if and only if each family of ω-closed subsets of X with the finite intersection property has a nonempty intersection.*

Theorem 3.15. *A topological space X is ω-compact if and only if each filterbase in X has atleast one ω-adherent point.*

Proof. Suppose that X is ω-compact and $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ a filterbase in it. Since all finite intersections of $\mathcal{F}_\alpha$'s are nonempty, it follows that all finite intersection of $\omega\,\mathrm{Cl}(F_\alpha)$'s are also nonempty. Now it follows from Theorem 3.13 that $\bigcap_{\alpha\in\Delta} \omega\,\mathrm{Cl}(F_\alpha) \neq \emptyset$. This implies that $\mathcal{F}$ has atleast one ω-adherent point. Now suppose $\mathcal{F}$ is a family of ω-closed sets. Let each finite intersection be nonempty. The sets F_α with their finite intersection establish a filterbase $\mathcal{F}$. Therefore, $\mathcal{F}$ ω-accumulates to some point $z \in X$. It follows that $z \in \bigcap_{\alpha\in\Delta} F_\alpha$. Now we have by Theorem 3.13 X is ω-compact. □

Theorem 3.16. *A topological space X is ω-compact if and only if each filterbase on X with atleast one ω-adherent point is ω-convergent.*

Proof. Suppose that X is ω-compact, $x \in X$ and $\mathcal{F}$ is a filterbase on X. The ω-adherence of $\mathcal{F}$ is a subset of $\{x\}$. Then the ω-adherence of $\mathcal{F}$ is equal to $\{x\}$ by Theorem 3.13. Assume that there exists $V \in \omega O(X, x)$ such that for all $F \in \mathcal{F}$, $F \cap (X\backslash V) \neq \emptyset$. Then $\Psi = \{F\backslash V : F \in \mathcal{F}\}$ is a filterbase on X. It follows that the ω-adherence of Ψ is nonempty. However, $\bigcap_{F\in\mathcal{F}} \omega\,\mathrm{Cl}(F\backslash V) = (\bigcap_{F\in\mathcal{F}} \omega\,\mathrm{Cl}(F)) \cap (X\backslash V) = \{x\} \cap (X\backslash V) = \emptyset$, a contradiction. Hence for each $V \in \omega O(X, x)$, there exists an $F \in \mathcal{F}$ with $F \subset V$. This shows that $\mathcal{F}$ ω-converges to x. To prove the converse, it suffices to show that each filterbase in X has atleast one ω-accumulation point. Assume that $\mathcal{F}$ is a filterbase on X with no ω-adherent point. By hypothesis, $\mathcal{F}$ ω-converges to some point $z \in X$. Suppose F_α is an arbitrary element of $\mathcal{F}$. Then for each $V \in \omega O(X, x)$, there exists $F_\beta \in \mathcal{F}$ such that $F_\beta \subset V$. Since $\mathcal{F}$ is a filterbase, there exists a ω such that $F_\omega \subset F_\alpha \cap F_\beta \subset F_\alpha \cap V$, where $F_\alpha \neq \emptyset$. This means that $F\alpha \cap V \neq \emptyset$ for every $V \in \omega O(X, x)$ and corresponding for each α, z is a point of $\omega\,\mathrm{Cl}(F_\alpha)$. It follows that $z \in \bigcap_\alpha \omega\,\mathrm{Cl}(F_\alpha)$. Therefore, z is a ω-adherent point of $\mathcal{F}$, a contradiction. This shows that X is ω-compact. □

Definition 3.17. *Let X be a topological space. A point $x \in X$ is said to be ω-θ-cluster point of a net $\{x_\alpha\}_{\alpha\in\Delta}$ if $\{x_\alpha\}_{\alpha\in\Delta}$ is frequently in the ω-closure of every ω-open set containing x.*

Theorem 3.18. *A topological space X is ω-compact if and only if each net $\{x_\alpha\}_{\alpha\in\Delta}$ in X, has atleast one ω-cluster point.*

Proof. Similar to the proof of Theorem 3.6 □

4. Firmly ω-continuous Functions

Definition 4.1. *A function $f : X \to Y$, where X and Y are topological spaces, is said to have property ξ if for every ω-open cover Δ of Y there exists a finite cover (the members of which need not be necessarily ω-open) $\{A_1, A_2, ...A_n\}$ of X such that for each $i \in \{1, 2, ..., n\}$, there exists a set $U_i \in \Delta$ such that $f(A_i) \subset U_i$.*

Theorem 4.2. *A topological space X is ω-compact if and only if for every topological space Y and every ω-irresolute function $f : X \to Y$, f has the property ξ.*

Proof. Suppose that the topological space X is ω-compact and the function $f : X \to Y$ is ω-irresolute. Let Θ be an ω-open cover of Y. Thr set $f(X)$ is ω-compact relative to Y. This means that there exists a finite subfamily $\{U_1, U_2, ..., U_n\}$ of Θ which cover $f(X)$. Then the sets $A_1 = f^{-1}(U_1)$, $A_2 = f^{-1}(U_2)$,....$A_n = f^{-1}(U_n)$ form a cover of X such that $f(A_i) \subset U_i$ for each $i \in \{1, 2, ..., n\}$. Conversely, assume that X is a topological space such that for every topological space Y and every ω-irresolute function $f : X \to Y$, f has the property ξ. It follows that the identity function $id : X \to X$ has also the property ξ. Therefore, for every ω-open cover Θ, there exists a finite cover $A_1, A_2, ..., A_n$ of X such that for each $i \in \{1, 2, ..., n\}$, there exists a set $U_i \in \Theta$ such that $A_i = id(A_i) \subset U_i$. Then $\{U_1, U_2, ..., U_n\}$ is a subcover og θ. Since Θ was an arbitrary ω-open cover of X, the space X is ω-compact. □

Definition 4.3. *A function $f : X \to Y$ is said to be firmly ω-continuous if for every ω-open cover Δ of Y there exists a finite ω-open cover Θ of X such that for every $U \in \Theta$ there exists $G \in \Delta$ such that $f(U) \subset G$.*

Remark 4.4. *It should be noticed that if the topological space X is ω-compact and Y is an arbitrary topological space, then every ω-irresolute function $f : X \to Y$ is firmly ω-continuous.*

Lemma 4.5. *Let X, Y, Z and W be topological spaces. Let $g : X \to Y$ and $h : Z \to W$ be ω-irresolute functions and let $f : Y \to Z$ be firmly ω-continuous. Then the functions $f \circ g : X \to Z$ and $h \circ f : Y \to W$ are firmly ω-continuous.*

Lemma 4.6. *If $f : X \to Y$ is a ω-irresolute function which has the property ξ, then f is firmly ω-continuous.*

Theorem 4.7. *For a topological space (X, τ), the following properties are equivalent:*

(1). X is ω-compact.

(2). The the identity function $id : X \to X$ is firmly ω-continuous.

(3). Every ω-irresolute function from X to X is firmly ω-continuous.

(4). Every ω-irresolute function from X to a topological space Y is firmly ω-continuous.

(5). Every ω-irresolute function from X to a topological space Y has the property ξ.

(6). For each topological space Y and each ω-irresolute function $f : Y \to X$, f is firmly ω-continuous.

Proof. (1)$\Rightarrow$(2): Suppose that X is ω-compact. The identity function $id : X \to X$ is ω-irresolute and by Remark 4.4, $id : X \to X$ is firmly ω-continuous. (2)$\Rightarrow$(3): Let $f : X \to X$ be any ω-irresolute function. By (2), the identity function $id : X \to X$ is firmly ω-continuous and hence by Lemma 4.5 $f = id \circ f : X \to X$ is firmly ω-continuous. (3)$\Rightarrow$(4): Let $f : X \to X$ be any ω-irresolute function. The identity function $id : X \to X$ is ω-irresolute and by (3) id is firmly ω-continuous. It follows from 4.5 that $f = f \circ id : X \to Y$ is firmly ω-continuous. (4)$\Rightarrow$(5): This is obvious. (5)$\Rightarrow$(1): This follows immediately from Theorem 4.2. (6)$\Rightarrow$(2): Let $id : X \to X$ be the identity function. Then id is ω-irresolute and by (6) id is firmly ω-continuous. (1)$\Rightarrow$(6): Let Θ be an ω-open cover of X. Since X is ω-compact, then there exists a finite subcover $\{U_1, U_2, ..., U_n\}$ of Θ. Assume that $A_i = f^{-1}(U_i)$ for $i \in I$, where $I = \{1, 2, ..n\}$. It follows that $f(A_i) \subset U_i$ for $i \in I$. This shows that f is firmly ω-continuous. $\square$

References

[1] N.Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, 70(1963), 36-41.

[2] P.Sundaram and M.Sheik John, *Weakly closed and weakly continuous maps in topological spaces*, Proc. 82^{nd} session of the Indian Science Congress, Calcutta, (1995), (P-49).

[3] M.Sheik John, *A study on generalizations of closed sets and continuous maps in topological spaces*, Ph. D. Thesis, Bharathiyar University, Coimbatore, India, (2002).