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Exceptionality of Edge Graceful Labeling of Tricycle Graphs

Research Article

S. Venkatesan[1]* and P. Sekar[2]

1 Senior Lecturer in Mathematics, Sri Ramachandra University, Chennai, Tamilnadu, India.

2 Former Principal, C. Kandasamy Naidu College, Chennai, Tamilnadu, India.

Abstract: Graceful labeling is assignment of distinct labels from $f\{0, |E|\}$ to vertices of graph, where edges are labeled by difference of absolute values of adjacent vertices and every label from $f\{1, 2, \ldots, |E|\}$ is used exactly once as an edge label. The edge - gracefulness is reversal of gracefulness. It involves labeling edges, rather than vertices, with an induced numbering consisting of sums of edge- labels, rather than differences of vertex labels. In this paper, we are discussing about exceptionality of edge graceful labeling of tricycle graphs by the verification by Lo's theorem.

Keywords: Graph labeling, graceful labeling, edge - graceful labeling, tricycle graph.

1. Introduction and Preliminaries

Alex Rosa, a design theorist, introduced β - valuations (i.e. graceful labeling) as well as many other labeling as tools for decomposing the complete graph into isomorphic sub graphs. First, these β - valuations were used as a means of attacking Ringel's conjecture that K_{2n+l} can be decomposed into $(2n + 1)$ many sub graphs that are all isomorphic to a given tree with n edges was popularized in 1967 by Alex Rosa [1]. Rosa called a function f a β - valuation of a graph G with m edges if f is an injection from the vertices of a graph G to the set $\{0, 1, \ldots, q\}$ such that, when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edge labels are distinct. Golomb [3] called this particular method of labeling as graceful, and this is the popular term used today. We call a graph graceful if such an f exists. A vertex labeling (or valuation) of a graph $G = (V, E)$ is an assignment f of labels to the vertices of $V(G)$ that induces for each edge $uv \in E(G)$ a label depending on the vertex labels $f(u)$ and $f(v)$. Let G be a graph with q edges and let $f : V(G) \to \{0, 1, \ldots, q = E(G)\}$ be an injection. A vertex labeling f is called a graceful labeling if for each edge uv the absolute value $f(u) - f(v) = w(u, v)$ is assigned and the resulting edge labels are mutually distinct. The value $w(u, v)$ is called the edge-weight of the edge uv. A graph possessing a graceful labeling is called a graceful graph. The graceful labeling is one of the majority well-liked graph labeling. Numerous variations of graceful labeling have been introduced in latest years by researchers. A comprehensive narration of graph labeling problems and associated results is presented by Gallian [2].

Definition 1.1 (Graceful labeling)**.** *A graph $G = (V, E)$ with n vertices and m edges is a 1-1 mapping f of the vertex set $V(G)$ into the set $\{0, 1, 2, q\}$. If we define, for any edge $e = (u, v) \in E(G)$, the value $\Omega(e) = |f(u) - f(v)|$, then Ω is a 1-1 mapping of the set $E(G)$ onto the set $\{1, 2, q\}$. A graph is called graceful if it has a graceful labeling.*

* *E-mail: numericalguru@gmail.com*

Edge - graceful graphs were defined by Sheng-Ping in 1985 is originated from graceful graphs. In fact, edge - gracefulness is reversal of gracefulness. It involves labeling edges, rather than vertices, with an induced numbering consisting of sums of edge- labels, rather than differences of vertex labels.

Definition 1.2 (Edge-Graceful Graph). *A (p, q) - graph $G = (V, E)$ is called edge-graceful if there exist a bijection $f : E - \{1, 2, \ldots q\}$ and the induced mapping $f^+ : V - Z_p = \{0, 1, \ldots, p-1\}$ defined by $f^+ \equiv \sum_{uv \in E} f(uv)\ (mod\ \ p)$ is a bijection . This concept was introduced by Lo [5] in 1985.Investigation has shown that star graphs, odd unicyclic graphs, regular complete k - partite graphs $(kn_1, n_2, \ldots n_k)$ and complete graphs are some of the classes of graphs that are edge - graceful.*

Definition 1.3 (k-partite Graph). *A graph G is k-partite, $k > 1$, if it is feasible to partition the set of vertices that every element of the set of edges $E(G)$ joins a vertex of V_i to a vertex of V_j where $i \neq j$. A graph is regular when $n_1 = n_2 = \cdots = n_k$. A complete graph, K_p, has p vertices and an edge between any pair of vertices.*

Theorem 1.4 ([5]). *If a graph G of p nodes and q edges is edges-graceful, then $p \left| \left(q^2 + q - \frac{p(p-1)}{2} \right) \right.$.*

Proof. Since G is edge- graceful, we have an edge-graceful labeling. Thus $2(1+2+\cdots+q) = 0+1+2+\cdots+(p-1)(\text{mod } p) \Rightarrow q^2 + q = \frac{p(p-1)}{2}(\text{mod } p)$. Therefore $p \left| \left(q^2 + q - \frac{p(p-1)}{2} \right) \right.$. □

Theorem 1.5 ([5]). *k_p is edge- graceful for p odd.*

Theorem 1.6. *The regular complete k - partite graph $kn_1, n_2 \ldots n_k$ is edge - graceful if and only if n is odd, and k is either odd or a multiple of 4.*

Proof. By Lo's theorem, if a graph is edge- graceful, then $p|(q^2 + q - p(p-1)/2)$. Assume that $kn_1, n_2 \ldots n_k$ is edge - graceful. For $kn_1, n_2 \ldots n_k$ there are k partite sets of vertices, each with n vertices, so $p = n_k$. Every vertex has degree of $n(k-1)$ and there are nk vertices the number of edges is obtained by multiplying the degree for each vertex by the number of vertices and dividing by 2.Since each edge will be counted twice, once on each vertex, so $q = n^2k(k-1)/2$. So, $nk| \left(\frac{n^2k(k-1)}{2}\right)^2 + \frac{n^2k(k-1)}{2} - \frac{nk(k-1)}{2}$ must be true. When the right side is simplified and divided by nk, the relation becomes $\frac{n^3k(k-1)^2}{4} - \frac{(n-1)}{2}$ and it must be $\in Z$.

Case 1. n is even

$\rightarrow n^3k(k-1)^2$ Even & divisible by 4 it implies that $\frac{(n-1)}{2} \in Z$ is shows that $(n-1)$ is even obviously n is odd $\Leftrightarrow$ the original assumption.

Case 2. n is odd

$\rightarrow \frac{(n-1)}{2} \in Z$ implies that $\frac{n^3k(k-1)^2}{4} \in Z$ and n^3 odd. Let k be even $\rightarrow (k-1)^2$ odd imply $\frac{n^3k(k-1)^2}{4}$ is divisible by 4 if k is divisible by 4. Let k be odd $\rightarrow (k-1)^2$ is even and divisible by 4, consequently, n must be odd, and k must be odd or a multiple of 4. To determine whether the all the tricycle graphs are edge- graceful or not, we are using Lo's theorem that states that if a graph is edge - graceful then $p \left| \left(q^2 + q - \frac{p(p-1)}{2} \right) \right.$ □

Definition 1.7 (Tricycle graph). *A tricycle graph is consists of 3 cycles that intersect at a vertex, several edges, an edge, or are joined by a path.*

2. Main Theorem

Theorem 2.1. *All Tricycles graphs are edge-graceful*

Proof. Let us assume that T be any tricycle. T has q edges and $q-2$ vertices. According to Lo's theorem, if a graph with p vertices and q edges is edge-graceful, then $p \left| \left(q^2 + q - \frac{p(p-1)}{2} \right) \right.$.

We assumed that T is edge-graceful, so $(q-2)\Big|\Big(q^2+q-\frac{[(q-2)((q-2)-1)]}{2}\Big) \Rightarrow (q-2)\Big|\Big(\frac{2q^2+2q-(q^2-3q-2q+6)}{2}\Big) \Rightarrow (q-2)\Big|\Big(\frac{2q^2+2q-q^2+3q+2q-6}{2}\Big) \Rightarrow (q-2)\Big|\Big(\frac{q^2+7q-6}{2}\Big) \Rightarrow (q-2)\Big|\Big(\frac{q^2-2q+9q-19+12}{2}\Big) \Rightarrow (q-2)\Big|\Big(\frac{q(q-2)+9(q-2)+12}{2}\Big)$ Dividing by $q-2$ the relation is as follows $\Rightarrow \frac{q}{2}+\frac{9}{2}+\frac{6}{(q-2)}$. From the relation we have the following choices

(1). $\frac{q}{2}$ will be a integer or it will be a rational number

(2). $\frac{9}{2}$ is a rational number

(3). $\frac{6}{(q-2)}$ will be a integer or it will be rational number

Case 1: For $q=4$ the relation $\frac{q}{2}+\frac{9}{2}+\frac{6}{(q-2)}=2+\frac{9}{2}+3=\frac{19}{2}$ which is a rational number.

Case 2: For $q=6$ the relation $\frac{q}{2}+\frac{9}{2}+\frac{6}{(q-2)}=\frac{6}{2}+\frac{9}{2}+\frac{6}{4}=9$ which is a integer.

Case 3: For $q=8$ the relation $\frac{q}{2}+\frac{9}{2}+\frac{6}{(q-2)}=4+\frac{9}{2}+\frac{6}{6}=\frac{19}{2}$ which is a rational number.

Case 4: For $q=10$ the relation $\frac{q}{2}+\frac{9}{2}+\frac{6}{(q-2)}=4+\frac{9}{2}+\frac{6}{8}=\frac{41}{4}$ which is a rational number.

Case 5: For $q=12$ the relation $\frac{q}{2}+\frac{9}{2}+\frac{6}{(q-2)}=6+\frac{9}{2}+\frac{6}{10}=\frac{111}{4}$ which is a rational number.

Case 6: For $q=14$ the relation $\frac{q}{2}+\frac{9}{2}+\frac{6}{(q-2)}=7+\frac{9}{2}+\frac{6}{12}=12$ which is a integer.

For verification purpose by Microsoft Excel -2013 the calculation of integer and rational numbers by taking $q=4$ to 102 have been made for the above relation

Table 1.

q	$q/2$	$9/2$	$6/(q-2)$	$q/2+9/2+6/(q-2)$	q	$q/2$	$9/2$	$6/(q-2)$	$q/2+9/2+6/(q-2)$
4	2	4.5	3	9.5	54	27	4.5	0.115385	31.61538
6	3	4.5	1.5	9.0000	56	28	4.5	0.111111	32.61111
8	4	4.5	1	9.5	58	29	4.5	0.107143	33.60714
10	5	4.5	0.75	10.25	60	30	4.5	0.103448	34.60345
12	6	4.5	0.6	11.1	62	31	4.5	0.1	35.6
14	7	4.5	0.5	12.0000	64	32	4.5	0.096774	36.59677
16	8	4.5	0.428571	12.92857	66	33	4.5	0.09375	37.59375
18	9	4.5	0.375	13.875	68	34	4.5	0.090909	38.59091
20	10	4.5	0.333333	14.83333	70	35	4.5	0.088235	39.58824
20	10	4.5	0.333333	14.83333	70	35	4.5	0.088235	39.58824
22	11	4.5	0.3	15.8	72	36	4.5	0.085714	40.58571
24	12	4.5	0.272727	16.77273	74	37	4.5	0.083333	41.58333
26	13	4.5	0.25	17.75	76	38	4.5	0.081081	42.58108
28	14	4.5	0.230769	18.73077	78	39	4.5	0.078947	43.57895
30	15	4.5	0.214286	19.71429	80	40	4.5	0.076923	44.57692
32	16	4.5	0.2	20.7	82	41	4.5	0.075	45.575
34	17	4.5	0.1875	21.6875	84	42	4.5	0.073171	46.57317
36	18	4.5	0.176471	22.67647	86	43	4.5	0.071429	47.57143
38	19	4.5	0.166667	23.66667	88	44	4.5	0.069767	48.56977
40	20	4.5	0.157895	24.65789	90	45	4.5	0.068182	49.56818
42	21	4.5	0.15	25.65	92	46	4.5	0.066667	50.56667
44	22	4.5	0.142857	26.64286	94	47	4.5	0.065217	51.56522
46	23	4.5	0.136364	27.63636	96	48	4.5	0.06383	52.56383
48	24	4.5	0.130435	28.63043	98	49	4.5	0.0625	53.5625
50	25	4.5	0.125	29.625	100	50	4.5	0.061224	54.56122
52	26	4.5	0.12	30.62	102	51	4.5	0.06	55.5613

The computation table reveals that excluding $q=6$ and $q=14$ all the values are not an integer till $q=100$ i.e., all are rational numbers. So it's a negation to our assumption that all tricycles are edge graceful. Therefore by By Lo's theorem a tricycle graph G with p vertices and $q-2$ edges is edge-graceful, then $p\Big|\Big(q^2+q-\frac{p(p-1)}{2}\Big)$ is valid only in the cases $q=6$

and $q = 14$. So it is quiet adequate to articulate that not all the Tricycles are edge-graceful except $q = 6$ and $q = 14$. Hence the theorem. □

3. Conclusion

Graceful labeling and edge-gracefulness have been studied for over three decades, and these topics persist to be a charming one in the globe of graph theory and discrete mathematics. Here is the paper discussed Edge Gracefulness of Tricycle graph by Lo's theorem.

References

[1] A.Rosa, *On certain valuations of the vertices of a graph*, Theory of graphs (Internat. Symp, Rome, July 1966), Gordan and Breach, N.Y & Paris (1967), 349-355.

[2] J.A.Gallian, *Dynamic survey of graph labeling*, The Electronic Journal Combinatorics, (2008), #DS6 .

[3] G.S.Bloom and S.W.Golomb, *Numbered complete graphs, unusual rulers, and assorted applications*, Theory and Applications of Graphs.

[4] S.Cabaniss, R.Low and J.Mitchem, *On edge - graceful regular graphs and trees*, Ars combinatorial, 34 (1992), 129-142.

[5] S.Lo, *On edge graceful labeling of graphs*, Congressus numeratium, 50(1985), 231-241.