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Strong and Delta-Convergence Theorems for Multivalued Mappings in CAT(0) Spaces

Research Article

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Abstract: In this paper we suggest two new three-step iterative schemes for the certain class of multi-valued mappings in CAT(0) spaces and also prove the strong and delta-convergence results for such iterative processes.

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Keywords: Strong convergence, Δ-convergence, Quasi-nonexpansive mappings, Condition (E), CAT(0) spaces.

1. Introduction

Fixed point theory in CAT(0) spaces was first studied by Kirk [6]. He showed that every nonexpansive single-valued mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been developed and a number of papers have appeared. In 2008, Dhompongsa and Panyanak [3] obtained Δ-convergence theorems for the Mann and Ishikawa iterations for nonexpansive single-valued mappings in CAT(0) spaces. On the other hand some authors introduced and studied Mann and Ishikawa iterations for multivalued mappings in Hilbert spaces as well as in Banach spaces. The purpose of this paper is to introduce some iterative processes for quasi-nonexpansive multivalued mappings and also for the multivalued mappings satisfying the condition (E) and prove Δ-convergence and strong convergence theorems for such iterative processes in CAT(0) spaces. The results obtained are analogs of the results of Dhompongsa and Panyanak [3] and Song and Wang [9]. Let (X, d) be a geodesic metric space. We shall denote by $\mathcal{CB}(X)$ the collection of all nonempty closed bounded subsets of X and $\mathcal{K}(X)$ the collection of all nonempty compact subsets of X. Let C be a nonempty convex subset of a CAT(0) space X. A subset C is called proximinal if for each $x \in X$, there exists an element $k \in C$ such that $d(x, k) = d(x, C)$ where $d(x, C) = \inf\{d(x, y) : y \in C\}$. Let $P(X)$ denote the collection of all nonempty proximinal bounded subsets of X. Let $H(.,.)$ be the Hausdorff distance on $\mathcal{CB}(X)$, that is $H(A, B) = \max\{\sup_{a\in A} dist(a, B), \sup_{b\in B} dist(b, A)\}$ for all A, B belonging to $\mathcal{CB}(X)$ where $dist(a, B) = \inf\{d(a, b) : b \in B\}$ is the distance from the point a to the set B. An element $x \in X$ is said to be a fixed point of T if $x \in Tx$. The set of all fixed points of T is denoted by $F(T)$.

In this paper we define two new iterative processes to the CAT(0) space setting in the following manner.

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Let C be a nonempty convex subset of a CAT(0) space X and $T : C \to \mathcal{CB}(X)$ be a given mapping. Then for $x_1 \in C$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences of positive numbers in [0, 1], we have

$$x_{n+1} = (1-\alpha_n)y_n \oplus \alpha_n w_n,$$
$$y_n = (1-\beta_n)z_n \oplus \beta_n u_n,$$
$$z_n = (1-\gamma_n)x_n \oplus \gamma_n T x_n \tag{1}$$

where $w_n \in Ty_n$, $u_n \in Tx_n$ such that $d(u_n, w_n) \leq H(Tx_n, Ty_n) + \mu_n$ and $d(u_{n+1}, w_n) \leq H(Tx_{n+1}, Ty_n) + \mu_n$. We define another iterative process as:

Let $T : C \to P(C)$ be a given mapping and $P_T(x) = \{y \in Tx : \|x - y\| = dist(x, Tx)\}$. For fixed $x_1 \in C$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences of positive numbers in [0, 1], we have

$$x_{n+1} = (1-\alpha_n)y_n \oplus \alpha_n w_n,$$
$$y_n = (1-\beta_n)z_n \oplus \beta_n u_n,$$
$$z_n = (1-\gamma_n)x_n \oplus \gamma_n P_T x_n \tag{2}$$

where $w_n \in P_T y_n$, $u_n \in P_T x_n$ such that $d(u_n, w_n) \leq H(Tx_n, Ty_n) + \mu_n$ and $d(u_{n+1}, w_n) \leq H(Tx_{n+1}, Ty_n) + \mu_n$ and $\mu_n \in (0, \infty)$ such that $\lim_{n\to\infty} \mu_n = 0$. Now we collect some elementary facts which are used in our main results:

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset R$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, 1]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) = \Delta(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom: Let Δ be a geodesic triangle in X and let $\overline{\Delta}$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) inequality if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$, $d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y})$.

Definition 1.1 ([3]). *Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set $r(x, \{x_n\}) = \lim_{n\to\infty} \sup d(x, x_n)$. The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by $r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$ and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$.*

Remark 1.2. *In a CAT(0) space, $A(\{x_n\})$ consists of exactly one point [5].*

Definition 1.3 ([3]). *A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim x_n = x$ and call x the Δ-limit of $\{x_n\}$.*

Notice that given $\{x_n\} \subset X$ such that $\{x_n\}$ is Δ-convergent to x and given $y \in X$ with $x \neq y$, $\lim_{n\to\infty} \sup d(x, x_n) < \lim_{n\to\infty} \sup d(y, x_n)$. Thus every CAT(0) space X satisfies the Opial property.

Definition 1.4. *A multivalued mapping $T : X \to \mathcal{CB}(X)$ is said to be*

1. *Nonexpansive if* $H(Tx, Ty) \leq d(x, y)$ *for all* $x, y \in X$.

2. *Quasi-nonexpansive if* $F(T) \neq \emptyset$ *and* $H(Tx, Tp) \leq d(x, p)$ *for all* $x \in X$ *and for all* $p \in F(T)$.

In 2011, A. Abkar and M. Eslamian [1] stated the condition (E) for multivalued mappings as follows:

Definition 1.5. *A multivalued mapping* $T : X \to \mathcal{CB}(X)$ *is said to satisfy condition* (E_μ) *provided that* $dist(x, Ty) \leq \mu\, dist(x, Tx) + d(x, y)$ *for all* $x, y \in X$. *We say that* T *satisfies condition (E) whenever* T *satisfies* (E_μ) *for some* $\mu \geq 1$.

Lemma 1.6 ([1])**.** *Let* $T : X \to \mathcal{CB}(X)$ *be a multivalued nonexpansive mapping then* T *satisfies the condition* (E_1).

Lemma 1.7 ([8])**.** *Every bounded sequence in a complete CAT(0) space always has a* Δ*-convergent subsequence.*

Lemma 1.8 ([4])**.** *If* C *is a closed convex subset of a complete CAT(0) space and if* $\{x_n\}$ *is a bounded sequence in* C *then the asymptotic center of* $\{x_n\}$ *is in* C.

Lemma 1.9 ([3])**.** *Let* C *be a closed convex subset of a complete CAT(0) space. Let* $\{x_n\}$ *be a bounded sequence in* C *with* $A(\{x_n\}) = \{x\}$ *and* $\{u_n\}$ *is a subsequence of* $\{x_n\}$ *with* $A(\{u_n\}) = \{u\}$ *and the sequence* $\{d(x_n, u)\}$ *converges then* $x = u$.

Lemma 1.10 ([3])**.** *Let* X *be a CAT(0) space. Then* $d((1-t)x \oplus ty, z) = (1-t)d(x, z) + td(y, z)$ *for all* $x, y, z \in X$ *and* $t \in [0, 1]$.

Lemma 1.11 ([3])**.** *Let (X, d) be a CAT(0) space. Then* $d((1-t)x \oplus ty, z)^2 = (1-t)d(x, z)^2 + td(y, z)^2 - t(1-t)d(x, y)^2$ *for all* $t \in [0, 1]$ *and* $x, y, z \in X$.

2. Main Results

In this section we establish the convergence results for the iterative processes for the quasi-nonexpansive multivalued mappings and for the multivalued mappings satisfying condition (E).

Theorem 2.1. *Let* C *be a nonempty closed convex subset of a complete CAT(0) space* X *and* $T : C \to \mathcal{CB}(X)$ *be a quasi-nonexpansive multivalued mapping such that* $F(T) \neq \emptyset$ *and* $Tp = \{p\}$ *for each* $p \in F(T)$. *Let* $\{x_n\}$ *be the iterative process defined by* (1) *and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in [a, b] \subset (0, 1)$. *Assume that* $\lim_{n\to\infty} dist\,(x_n, Tx_n) = 0$ *implies* $\lim_{n\to\infty} \inf dist\,(x_n,\ F(T)) = 0$. *Then* $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof. Let p ∈ F(T). Then using (1) and quasi-nonexpansiveness of T, we have

$$\begin{aligned}
d(y_n, p) &= d((1-\beta_n)z_n \oplus \beta_n u_n, p) \\
&\leq (1-\beta_n)d(z_n, p) + \beta_n d(u_n, p) \\
&= (1-\beta_n)d(z_n, p) + \beta_n dist(u_n, Tp) \\
&\leq (1-\beta_n)d(z_n, p) + \beta_n H(Tx_n, Tp) \\
&\leq (1-\beta_n)d(z_n, p) + \beta_n d(x_n, p) \\
&= (1-\beta_n)d((1-\gamma_n)x_n \oplus \gamma_n Tx_n, p) + \beta_n d(x_n, p) \\
&\leq (1-\beta_n)\{(1-\gamma_n)d(x_n, p) + \gamma_n d(Tx_n, p)\} + \beta_n d(x_n, p) \\
&\leq (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) = d(x_n, p)
\end{aligned}$$

Also we have

$$\begin{aligned} d(x_{n+1},p) &= d((1-\alpha_n)y_n \oplus \alpha_n w_n, p) \\ &\leq (1-\alpha_n)d(y_n,p) + \alpha_n d(w_n,p) \\ &= (1-\alpha_n)d(y_n,p) + \alpha_n \ dist(w_n,Tp) \\ &\leq (1-\alpha_n)d(y_n,p) + \alpha_n H(Ty_n,Tp) \\ &\leq (1-\alpha_n)d(y_n,p) + \alpha_n d(y_n,p) \\ &= d(y_n,p) \leq d(x_n,p) \end{aligned}$$

Thus, the sequence $\{d(x_n,p)\}$ is decreasing and bounded below. It now follows that $\lim_{n\to\infty} d(x_n,p)$ exists for any $p \in F(T)$. Now

$$\begin{aligned} d(x_{n+1},p)^2 &= d((1-\alpha_n)y_n \oplus \alpha_n w_n, p)^2 \\ &\leq (1-\alpha_n)d(y_n,p)^2 + \alpha_n d(w_n,p)^2 - \alpha_n(1-\alpha_n)d(y_n,w_n)^2 \\ &\leq (1-\alpha_n)d(y_n,p)^2 + \alpha_n dist(w_n,Tp)^2 \\ &\leq (1-\alpha_n)d(y_n,p)^2 + \alpha_n H(Ty_n,Tp)^2 \\ &\leq (1-\alpha_n)d(y_n,p)^2 + \alpha_n d(y_n,p)^2 = d(y_n,p)^2 \\ &= d((1-\beta_n)z_n \oplus \beta_n u_n, p)^2 \\ &\leq (1-\beta_n)d(z_n,p)^2 + \beta_n d(u_n,p)^2 - \beta_n(1-\beta_n)d(z_n,u_n)^2 \\ &\leq (1-\beta_n)d(z_n,p)^2 + \beta_n dist(u_n,p)^2 \\ &\leq (1-\beta_n)d(z_n,p)^2 + \beta_n H(Tx_n,Tp)^2 \\ &\leq (1-\beta_n)d(z_n,p)^2 + \beta_n (x_n,p)^2 \\ &= (1-\beta_n)d((1-\gamma_n)x_n \oplus \gamma_n Tx_n, p)^2 + \beta_n d(x_n,p)^2 \\ &\leq (1-\beta_n)\{(1-\gamma_n)d(x_n,p)^2 + \gamma_n d(Tx_n,p)^2 - \gamma_n(1-\gamma_n)d(x_n,Tx_n)^2\} + \beta_n d(x_n,p)^2 \\ &\leq (1-\beta_n)\{(1-\gamma_n)d(x_n,p)^2 + \gamma_n d(x_n,p)^2 - \gamma_n(1-\gamma_n)d(x_n,Tx_n)^2\} + \beta_n d(x_n,p)^2 \\ &\leq (1-\beta_n)d(x_n,p)^2 + \beta_n d(x_n,p)^2 - \gamma_n(1-\gamma_n)(1-\beta_n)d(x_n,Tx_n)^2 \end{aligned}$$

so that

$$\begin{aligned} a^2(1-b)d(x_n,Tx_n)^2 &\leq \gamma_n(1-\gamma_n)(1-\beta_n)d(x_n,Tx_n)^2 \\ &\leq d(x_n,p)^2 - d(x_{n+1},p)^2. \end{aligned}$$

This implies that $\sum_{n=1}^{\infty} a^2(1-b)d(x_n,Tx_n)^2 \leq d(x_1,p)^2 < \infty$ and hence $\lim_{n\to\infty} d(x_n,Tx_n) = 0$. Note that by our assumption $\lim_{n\to\infty} dist(x_n,F(T)) = 0$. Therefore we can choose a sequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence p_k in $F(T)$ such that for all $k \in N$, $d(x_{n_k},p_k) < \frac{1}{2^k}$. Since the sequence $\{d(x_n,p)\}$ is decreasing, we get $d(x_{n_{k+1}},p_k) \leq d(x_{n_k},p_k) < \frac{1}{2^k}$. Hence $d(p_{k+1},p_k) \leq d(x_{n_{k+1}},p_{k+1}) + d(x_{n_{k+1}},p_k) < \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}$. Consequently, we conclude that $\{p_k\}$ is a Cauchy sequence in C and hence converges to $q \in C$. Since $dist(p_k,Tq) \leq H(Tp_k,Tq) \leq d(p_k,q)$ and $p_k \to q$ as $k \to \infty$. It follows that $dist(q,Tq) = 0$ and hence $q \in F(T)$ and $\{x_{n_k}\}$ converges strongly to q. Since $\lim_{n\to\infty} d(x_n,q)$ exists, it follows that $\{x_n\}$ converges strongly to q. □

Theorem 2.2. *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Suppose $T : C \to \mathcal{K}(C)$ satisfies the condition (E). If $\{x_n\}$ is a sequence in C such that $\lim_{n\to\infty} dist\,(x_n, Tx_n) = 0$ and $\Delta\text{-}\lim_n x_n = v$. Then $v \in C$ and $v \in Tv$.*

Proof. Let $\Delta\text{-}\lim_n x_n = v$. We note that by Lemma 1.8, $v \in C$. For each $n \geq 1$, we choose $z_n \in Tv$ such that $d(x_n, z_n) = dist(x_n, Tv)$. By the compactness of Tv, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $\lim_{n\to\infty} z_{n_k} = w \in Tv$. Since T satisfies the condition (E), we have $dist(x_{n_k}, Tv) \leq \mu dist(x_{n_k}, Tx_{n_k}) + d(x_{n_k}, v)$ for some $\mu \geq 1$. Note that

$$\begin{aligned} d(x_{n_k}, w) &\leq d(x_{n_k}, z_{n_k}) + d(z_{n_k}, w) \\ &\leq \mu dist(x_{n_k}, Tx_{n_k}) + d(x_{n_k}, v) + d(z_{n_k}, w). \end{aligned}$$

Thus $\lim_{n\to\infty} \sup d(x_{n_k}, w) \leq \lim_{n\to\infty} \sup d(x_{n_k}, v)$. From the Opial property of CAT(0) space X, we have $v = w \in Tv$. □

Using Theorems 2.1, 2.2, we can prove the following Δ-convergence result for the iterative process (1).

Theorem 2.3. *Let C be a nonempty closed convex subset of a complete CAT(0) space X and $T : C \to \mathcal{K}(C)$ be a quasi-nonexpansive multivalued mapping satisfying condition (E) such that $F(T) \neq \emptyset$ and $Tp = \{p\}$ for each $p \in F(T)$. Let $\{x_n\}$ be the iterative process defined by (1) and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \in [a, b] \subset (0, 1)$. Then $\{x_n\}$ is Δ-convergent to a fixed point of T.*

Proof. From Theorem 2.1, we have $\lim_{n\to\infty} dist(x_n, Tx_n) = 0$. We now let $\omega_w(x_n) = \cup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $\omega_w(x_n) \subset F(T)$. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 1.7 and 1.8, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_n v_n = v \in C$. Since $\lim_{n\to\infty} dist(v_n, Tv_n) = 0$. By Theorem 2.2 we have $v \in F(T)$ and $\lim_{n\to\infty} d(x_n, v)$ exists. Hence $u = v \in F(T)$ by Lemma 1.9. This shows that $\omega_w(x_n) \subset F(T)$. Next we show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in \omega_w(x_n) \subset F(T)$ and $d(x_n, v)$ converges, by Lemma 1.9 we have $x = u$. □

Using Theorem 2.3 along with Lemma 1.6, we obtain the following corollary.

Corollary 2.4. *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $T : C \to \mathcal{K}(C)$ be a multivalued nonexpansive mapping such that $F(T) \neq \emptyset$ and $Tp = \{p\}$ for each $p \in F(T)$. Let $\{x_n\}$ be the iterative process defined by (1) and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \in [a, b] \subset (0, 1)$. Then $\{x_n\}$ is Δ-convergent to a fixed point of T.*

Now we establish the convergence result for the mapping satisfying condition (E).

Theorem 2.5. *Let C be a nonempty compact convex subset of a complete CAT(0) space X. Let $T : C \to \mathcal{CB}(C)$ be a quasi-nonexpansive multivalued mapping satisfying condition (E) such that $F(T) \neq \emptyset$ and $Tp = \{p\}$ for each $p \in F(T)$. Let $\{x_n\}$ be the iterative process defined by (1) and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \in [a, b] \subset (0, 1)$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. From Theorem 2.1 we have $\lim_{n\to\infty} dist\,(x_n, Tx_n) = 0$. Since C is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim x_{n_k} = w$ for some $w \in C$. Since T satisfies the condition (E), for some $\mu \geq 1$, we have

$$\begin{aligned} dist(w, Tw) &\leq d(w, x_{n_k}) + dist(x_{n_k}, Tw) \\ &\leq \mu dist(x_{n_k}, Tx_{n_k}) + 2d(w, x_{n_k}) \to 0 \text{ as } k \to \infty. \end{aligned}$$

This implies that $w \in F(T)$. Since $\{x_{n_k}\}$ converges strongly to a point w and $\lim_{n\to\infty} d\,(x_n, w)$ exists (as the proof of Theorem 2.1 shows), it follows that $\{x_n\}$ converges strongly to w. □

Next we prove the convergence result for the iterative process (2).

Theorem 2.6. *Let C be a nonempty closed convex subset of a complete CAT(0) space X and $T : C \to P(C)$ be a multivalued mapping with $F(T) \neq \emptyset$ such that P_T is nonexpansive. Let $\{x_n\}$ be the iterative process defined by (2) and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \in [a, b] \subset (0, 1)$. Assume that $\lim_{n\to\infty} dist\,(x_n, Tx_n) = 0$ implies $\lim_{n\to\infty} \inf dist\,(x_n,\ F(T)) = 0$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Let $p \in F(T)$. Then $p \in P_T p = \{p\}$. Hence using (2), we have

$$\begin{aligned} d(y_n, p) &= d((1-\beta_n)z_n \oplus \beta_n u_n, p) \\ &\le (1-\beta_n)d(z_n, p) + \beta_n d(u_n, p) = (1-\beta_n)d(z_n, p) + \beta_n dist(u_n, P_T p) \\ &\le (1-\beta_n)d(z_n, p) + \beta_n H(P_T x_n, P_T p) \\ &\le (1-\beta_n)d(z_n, p) + \beta_n d(x_n, p) \\ &\le (1-\beta_n)\{(1-\gamma_n)d(x_n, p) + \gamma_n d(P_T x_n, p)\} + \beta_n d(x_n, p) \\ &\le (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) = d(x_n, p) \end{aligned}$$

Also we have

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_n)y_n \oplus \alpha_n w_n, p) \\ &\le (1-\alpha_n)d(y_n, p) + \alpha_n d(w_n, p) \\ &= (1-\alpha_n)d(y_n, p) + \alpha_n dist(w_n, P_T p) \\ &\le (1-\alpha_n)d(y_n, p) + \alpha_n H(P_T y_n, P_T p) \\ &\le (1-\alpha_n)d(y_n, p) + \alpha_n d(y_n, p) \\ &= d(y_n, p) \\ &\le d(x_n, p) \end{aligned}$$

Thus, the sequence $\{d(x_n, p)\}$ is decreasing and bounded below. It now follows that $\lim_{n\to\infty} d\,(x_n, p)$ exists for any $p \in F(T)$. Now

$$\begin{aligned} d(x_{n+1}, p)^2 &= d((1-\alpha_n)y_n \oplus \alpha_n w_n, p)^2 \\ &\le (1-\alpha_n)d(y_n, p)^2 + \alpha_n d(w_n, p)^2 - \alpha_n(1-\alpha_n)d(y_n, w_n)^2 \\ &\le (1-\alpha_n)d(y_n, p)^2 + \alpha_n dist(w_n, P_T p)^2 \\ &\le (1-\alpha_n)d(y_n, p)^2 + \alpha_n H(P_T y_n, P_T p)^2 \\ &\le (1-\alpha_n)d(y_n, p)^2 + \alpha_n d(y_n, p)^2 = d(y_n, p)^2 \\ &= d((1-\beta_n)z_n + \beta_n u_n, p)^2 \\ &\le (1-\beta_n)d(z_n, p)^2 + \beta_n d(u_n, p)^2 - \beta_n(1-\beta_n)d(z_n, u_n)^2 \\ &\le (1-\beta_n)d(z_n, p)^2 + \beta_n dist(u_n, p)^2 \\ &\le (1-\beta_n)d(z_n, p)^2 + \beta_n H(P_T x_n, P_T p)^2 \\ &\le (1-\beta_n)d(z_n, p)^2 + \beta_n (x_n, p)^2 \\ &\le (1-\beta_n)\{(1-\gamma_n)d(x_n, p)^2 + \gamma_n d(P_T x_n, p)^2 - \gamma_n(1-\gamma_n)d(x_n, P_T x_n)^2\} + \beta_n d(x_n, p)^2 \end{aligned}$$

$$\leq (1-\beta_n)\{(1-\gamma_n)d(x_n,p)^2+\gamma_n d(x_n,p)^2-\gamma_n(1-\gamma_n)d(x_n,P_Tx_n)^2\}+\beta_n d(x_n,p)^2$$
$$\leq (1-\beta_n)d(x_n,p)^2+\beta_n d(x_n,p)^2-\gamma_n(1-\gamma_n)(1-\beta_n)d(x_n,P_Tx_n)^2$$

As in the proof of Theorem 2.1, $\lim_{n\to\infty} dist(x_n,F(T))=0$. Therefore we can choose a sequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence p_k in $F(T)$ such that for all $k\in N$, $d(x_{n_k},p_k)<\frac{1}{2^k}$. Again $\{p_k\}$ is a Cauchy sequence in C and hence converges to $q\in C$. Since $dist(p_k,Tq)\leq dist(p_k,P_Tq)\leq H(P_Tp_k,P_Tq)\leq d(p_k,q)$ and $p_k\to q$ as $k\to\infty$. It follows that $dist(q,Tq)=0$ and hence $q\in F(T)$ and $\{x_{n_k}\}$ converges strongly to q. Since $\lim_{n\to\infty} d(x_n,q)$ exists, it follows that $\{x_n\}$ converges strongly to q. □

Remark 2.7. *The Δ-convergence result for the iterative process* (2) *also holds, the proof is similar as given in Theorems 2.2 and 2.3.*

References

[1] A.Abkar and M.Eslamian, *Common fixed point results in CAT(0) spaces*, Nonlinear Analysis, 74(2011), 1835-1840.

[2] M.Bridson and A.Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer-Verlag, Berlin, Heidelberg, (1999).

[3] S.Dhompongsa and B.Panyanak, *On Δ-convergence theorems in CAT(0) spaces*, Computer and Mathematics with Applications, 56(2008), 2572-2579.

[4] S.Dhompongsa, W.A.Kirk and B.Panyanak, *Nonexpansive set valued mappings in metric and Banach spaces*, Journal of Nonlinear And Convex Analysis, 8(2007), 35-45.

[5] S.Dhompongsa, W.A.Kirk and Sims, *Fixed points of uniformly lipschitzian mappings*, Nonlinear Analysis: TMA, 65(2006), 762-772.

[6] W.A.Kirk, *Geodesic geometry and fixed point theory*, In Seminar of Mathematical Analysis, (Malaga/Seville, 2002/2003), Colecc. Abierta, Univ. Sevilla Secr. Publ., Seville, 64(2003), 195-225.

[7] W.A.Kirk, *Geodesic geometry and fixed point theory II* , In International Conference on Fixed Point Theory and Applications, Yokohama Publ., Yokohama, (2004), 113-142.

[8] W.A.Kirk and B.Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Analysis: Theory, Methods And Applications, 68(2008), 3689-3696.

[9] Y.Song and H.Wang, *Erratum to Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces*, [Computer and Mathematics with Applications, 54(2007), 872-877], Computer and Mathematics with Applications, 55(2008), 2999-3002.