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CR-Iterative Process for Asymptotically Quasi-Nonexpansive Mappings

Research Article

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Abstract: In this paper strong convergence theorems for CR-iterative process for asymptotically quasi-nonexpansive mappings are established in the framework of CAT(0) spaces.

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1. Introduction

Let X be a CAT(0) space and let C be a nonempty subset of X and $T : C \to C$ be a mapping. A point $x \in C$ is called a fixed point of T if $Tx = x$. Denote $F(T)$ by the set of fixed points of T, i.e., $F(T) = \{x \in C : Tx = x\}$. The concept of quasi-nonexpansiveness was introduced by Diaz and Metcalf [4] in 1967, the concept of asymptotically nonexpansiveness was introduced by Goebel and Kirk [6] in 1972.

Definition 1.1. *Let C be nonempty subset of a CAT(0) space X and $T : C \to C$ be a mapping. Then T is said to be*

(1). Nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$.

(2). Asymptotically nonexpansive if there exists a sequence $\{u_n\} \in [0, \infty)$ with the property $\lim_{n\to\infty} u_n = 0$ and such that $d(T^n x, T^n y) \leq (1 + u_n)d(x, y)$ for all $x, y \in C$.

(3). Quasi-nonexpansive if $d(Tx, p) \leq d(x, p)$ for all $x \in C$, $p \in F(T)$.

(4). Asymptotically quasi-nonexpansive if there exists a sequence $\{u_n\} \in [0, \infty)$ with the property $\lim_{n\to\infty} u_n = 0$ and such that $d(T^n x, p) \leq (1 + u_n)d(x, p)$ for all $x \in C$, $p \in F(T)$.

Remark 1.2. *From the definition, it is clear that the class of quasi-nonexpansive mappings and asymptotically nonexpansive mappings includes nonexpansive mappings, whereas the class of asymptotically quasi-nonexpansive mappings is larger than that of quasi-nonexpansive mappings and asymptotically nonexpansive mappings. Or we can say that an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive mapping but the converse does not hold. So, the following*

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example which showed that the mapping T is asymptotically nonexpansive mapping but it is not nonexpansive also showed that this mapping is asymptotically quasi-nonexpansive mapping but not nonexpansive mapping. For example [6], Let $X = R$, $C = [0, 1]$ and $\frac{1}{2} < k < 1$. For each $x \in C$, define

$$Tx = \begin{cases} kx, & \textit{if } 0 \leq x \leq \frac{1}{2} \\ -\frac{k(x-k)}{2k-1}, & \textit{if } \frac{1}{2} \leq x \leq k \\ 0, & \textit{if } k \leq x \leq 1 \end{cases}$$

Then $T : C \to C$ is asymptotically nonexpansive mapping but it is not nonexpansive.

In 2012, R. Chugh, V. Kumar and S. Kumar [3] introduced CR-iterative process and showed that this iteration converges much faster than all the existing iterative processes in the literature. The CR-iterative process is defined by a sequence $\{x_n\}$:

$$\begin{aligned} x_{n+1} &= \alpha_n T y_n + (1 - \alpha_n) y_n, \\ y_n &= \beta_n T z_n + (1 - \beta_n) T x_n, \\ z_n &= \gamma_n T x_n + (1 - \gamma_n) x_n, \end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences of positive numbers in [0, 1] with $\sum_{n=0}^{\infty} \alpha_n = \infty$. We now modify this notion of iterative process in CAT(0) spaces for asymptotically quasi-nonexpansive mappings. Let C be a non-empty closed convex subset of a complete CAT(0) space X and $T : C \to C$ be an asymptotically quasi-nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated iteratively by

$$\begin{aligned} x_{n+1} &= \alpha_n T^n y_n \oplus (1 - \alpha_n) y_n, \\ y_n &= \beta_n T^n z_n \oplus (1 - \beta_n) T^n x_n, \\ z_n &= \gamma_n T^n x_n \oplus (1 - \gamma_n) x_n, \end{aligned} \tag{1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences of positive numbers in [0, 1]. In this paper, we study the CR-iterative process for asymptotically quasi-nonexpansive mappings in CAT(0) space and generalize some results of Sahin and Basarir [10] which studied the SP-iterative process in a CAT(0) space for nonexpansive mappings. Let us recall some definitions and known results in the existing literature on this concept.

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset R$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) = \Delta(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom: Let Δ be a geodesic triangle in X and let $\bar{\Delta}$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) inequality if for all $x, y \in \Delta$ and all comparison points $\bar{x}$, $\bar{y} \in \bar{\Delta}$, $d(x,y) \leq d_{\mathbb{E}^2}(\bar{x},\bar{y})$. If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d(x,y_0)^2 \leq \frac{1}{2}d(x,y_1)^2 + \frac{1}{2}d(x,y_2)^2 - \frac{1}{4}d(y_1,y_2)^2 \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [2]. In fact, a geodesic space is a CAT(0) space if and only if it satisfy (CN) inequality. In 2008, Dhompongsa and Panyanak [5] gave the following lemma which will be used frequently in the proof of our main results.

Lemma 1.3. *Let X be a CAT(0) space. Then*

$$d((1-t)x \oplus ty, z) = (1-t)d(x,z) + td(y,z) \tag{2}$$

for all $x, y, z \in X$ and $t \in [0,1]$.

In 2002, Zhou et. al. [12] obtained the following lemma.

Lemma 1.4. *Let $\{a_n\}$ and $\{u_n\}$ be two sequences of positive real numbers satisfying $a_{n+1} \leq (1+u_n)a_n$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} u_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

2. Main Results

In this section we prove the strong convergence theorems of the CR-iterative process in the CAT(0) spaces.

Theorem 2.1. *Let C be a nonempty closed convex subset of a complete CAT(0) space X and $T : C \to C$ be asymptotically quasi-nonexpansive mapping with $F(T) \neq \emptyset$ and $\{u_n\}$ be a nonnegative real sequence with $\sum_{n=1}^{\infty} u_n < \infty$. Suppose that $\{x_n\}$ is defined by the iteration process (1). If $\lim_{n\to\infty} \inf d(x_n, F(T)) = 0$ or $\lim_{n\to\infty} \sup d(x_n, F(T)) = 0$, where $d(x, F(T)) = \inf_{p\in F(T)} d(x,p)$, then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Let $p \in F(T)$. Since T is an asymptotically quasi-nonexpansive mapping, there exists a sequence $\{u_n\} \in [0, \infty)$ with the property $\lim_{n\to\infty} u_n = 0$ and such that $d(T^n x, p) \leq (1+u_n)d(x,p)$ for all $x \in C$ and $p \in F(T)$. By combining this inequality and Lemma 1.3, we get

$$\begin{aligned}
d(x_{n+1},p) &= d(\alpha_n T^n y_n \oplus (1-\alpha_n)y_n, p) \\
&\leq \alpha_n d(T^n y_n, p) + (1-\alpha_n)d(y_n,p) \\
&\leq \alpha_n(1+u_n)d(y_n,p) + (1-\alpha_n)d(y_n,p) \\
&= (1+\alpha_n u_n)d(y_n,p) \\
&= (1+\alpha_n u_n)d(\beta_n T^n z_n \oplus (1-\beta_n)T^n x_n, p) \\
&\leq (1+\alpha_n u_n)[\beta_n d(T^n z_n,p) + (1-\beta_n)d(T^n x_n,p)] \\
&\leq (1+\alpha_n u_n)[\beta_n(1+u_n)d(z_n,p) + (1-\beta_n)(1+u_n)d(x_n,p)] \\
&= (1+\alpha_n u_n)[\beta_n(1+u_n)d(\gamma_n T^n x_n \oplus (1-\gamma_n)x_n, p) + (1-\beta_n)(1+u_n)d(x_n,p)] \\
&\leq (1+\alpha_n u_n)[\beta_n(1+u_n)\{\gamma_n d(T^n x_n,p) + (1-\gamma_n)d(x_n,p)\} + (1-\beta_n)(1+u_n)d(x_n,p)] \\
&\leq (1+\alpha_n u_n)[\beta_n(1+u_n)\{\gamma_n(1+u_n)d(x_n,p) + (1-\gamma_n)d(x_n,p)\} + (1-\beta_n)(1+u_n)d(x_n,p)]
\end{aligned}$$

$$
\begin{aligned}
&= (1+\alpha_n u_n)[\beta_n(1+u_n)(1+\gamma_n u_n)d(x_n,p) + (1-\beta_n)(1+u_n)d(x_n,p)] && (3)\\
&= (1+\alpha_n u_n)(1+u_n)(1+\beta_n\gamma_n u_n)d(x_n,p)\\
&\le (1+u_n)^3 d(x_n,p). && (4)
\end{aligned}
$$

When $x \ge 0$ and $(1+x) \le e^x$, $(1+x)^3 \le e^{3x}$. Thus

$$
\begin{aligned}
d(x_{n+m},p) &\le (1+u_{n+m-1})^3 d(x_{n+m-1},p)\\
&\le e^{3u_{n+m-1}} d(x_{n+m-1},p)\\
&\le e^{3\sum\limits_{k=n}^{n+m-1} u_k} d(x_n,p).
\end{aligned}
$$

Let $e^{3\sum\limits_{k=n}^{n+m-1} u_k} = M$. Thus, there exists a constant $M > 0$ such that $d(x_{n+m},p) \le Md(x_n,p)$ for all $n,m \in \mathbb{N}$ and $p \in F(T)$. By (4), $d(x_{n+1},p) \le (1+u_n)^3 d(x_n,p)$. This gives

$$d(x_{n+1},F(T)) \le (1+u_n)^3 d(x_n,F(T)) = (1+u_n^3+3u_n+3u_n^2)d(x_n,F(T)).$$

Since $\sum\limits_{n=1}^{\infty} u_n < \infty$, we have $\sum\limits_{n=1}^{\infty}(u_n^3+3u_n+3u_n^2) < \infty$. Lemma 1.4 and $\lim\limits_{n\to\infty}\inf d(x_n,F(T)) = 0$ or $\lim\limits_{n\to\infty}\sup d(x_n,F(T)) = 0$ gives that

$$\lim_{n\to\infty} d(x_n,F(T)) = 0. \tag{5}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in C. Since $\lim\limits_{n\to\infty} d(x_n,F(T)) = 0$, for each $e > 0$, there exists $n_1 \in \mathbb{N}$ such that $d(x_n,F(T)) < \frac{e}{M+1}$ for all $n > n_1$. Thus there exists $p_1 \in F(T)$ such that $d(x_n,p_1) < \frac{e}{M+1}$ for all $n > n_1$ and we obtain that

$$
\begin{aligned}
d(x_{n+m},x_n) &\le d(x_{n+m},p_1) + d(p_1,x_n)\\
&\le Md(x_n,p_1) + d(p_1,x_n)\\
&= (M+1)d(x_n,p_1)\\
&< \frac{e}{M+1}(M+1) = e \ \text{ for all } m,n > n_1.
\end{aligned}
$$

Therefore $\{x_n\}$ is a Cauchy sequence in C. Since the set C is complete, the sequence $\{x_n\}$ must converge to a fixed point in C. Let $\lim\limits_{n\to\infty} x_n = p \in C$. Now we show that p is a fixed point. By $\lim\limits_{n\to\infty} x_n = p$ for all $e_1 > 0$, there exists $n_2 \in \mathbb{N}$ such that

$$d(x_n,p) < \frac{e_1}{2(2+u_1)} \tag{6}$$

for all $n > n_2$. From (5), for each $e_1 > 0$, there exists $n_3 \in \mathbb{N}$ such that $d(x_n,F(T)) < \frac{e_1}{2(4+3u_1)}$ for all $n > n_3$. In particular, $\inf\{d(x_{n_3},p) : p \in F(T)\} < \frac{e_1}{2(4+3u_1)}$. Thus there must exists $p^* \in F(T)$ such that

$$d(x_{n_3},p^*) < \frac{e_1}{2(4+3u_1)} \quad \text{for all } n > n_3. \tag{7}$$

From (6) and (7),

$$d(Tp,p) \le d(Tp,p^*) + d(p^*,Tx_{n_3}) + d(Tx_{n_3},p^*) + d(p^*,x_{n_3}) + d(x_{n_3},p)$$

$$
\begin{aligned}
&\leq d(Tp,p^*) + 2d(Tx_{n_3},p^*) + d(p^*,x_{n_3}) + d(x_{n_3},p)\\
&\leq (1+u_1)d(p,p^*) + 2(1+u_1)d(x_{n_3},p^*) + d(p^*,x_{n_3}) + d(x_{n_3},p)\\
&\leq (1+u_1)\{d(p,x_{n_3}) + d(x_{n_3},p^*)\} + 2(1+u_1)d(x_{n_3},p^*) + d(p^*,x_{n_3}) + d(x_{n_3},p)\\
&= (2+u_1)d(x_{n_3},p) + (4+3u_1)d(x_{n_3},p^*)\\
&< (2+u_1)\frac{e_1}{2(2+u_1)} + (4+3u_1)\frac{e_1}{2(4+3u_1)} = e_1.
\end{aligned}
$$

Since e_1 is arbitrary, so $d(Tp,p) = 0$, that is, $Tp = p$. Therefore, $p \in F(T)$. This completes the proof. □

Remark 2.2. *The above Theorem is also satisfied for the cases when $T : C \to C$ is an asymptotically nonexpansive or quasi-nonexpansive mapping. Since the class of asymptotically quasi-nonexpansive mappings includes quasi-nonexpansive mappings and asymptotically nonexpansive mappings.*

Corollary 2.3. *Under the hypothesis of Theorem 2.1, T satisfies the following conditions:*

(1). $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

(2). If the sequence $\{z_n\}$ in C satisfies $\lim_{n\to\infty} d(z_n, Tz_n) = 0$, then

$$\lim_{n\to\infty} \inf d(z_n, F(T)) = 0 \ \text{ or } \ \lim_{n\to\infty} \sup d(z_n, F(T)) = 0.$$

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. It follows from the hypothesis that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. From given condition (2), $\lim_{n\to\infty} \inf d(x_n, F(T)) = 0$ or $\lim_{n\to\infty} \sup d(x_n, F(T)) = 0$. Therefore the sequence $\{x_n\}$ must converge to a fixed point of T by Theorem 2.1. □

Corollary 2.4. *Under the hypothesis of Theorem 2.1, T satisfies the following conditions:*

(1). $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

(2). There exists a function $f : [0,\infty) \to [0,\infty)$ which is right continuous at 0, $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $d(x,Tx) \geq f(d(x,F(T)))$ for all $x \in C$, where $d(x,F(T)) = \inf_{p\in F(T)} d(x,p)$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. It follows from the hypothesis that

$$\lim_{n\to\infty} f(d(x_n, F(T))) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

That is, $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since $f : [0,\infty) \to [0,\infty)$ is right continuous at 0, $f(0) = 0$, therefore we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Thus $\lim_{n\to\infty} \inf d(x_n, F(T)) = \lim_{n\to\infty} \sup d(x_n, F(T)) = 0$. By Theorem 2.1, the sequence $\{x_n\}$ converges strongly to p, a fixed point of T. This completes the proof. □

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