International Journal of Mathematics And its Applications
Volume 5, Issue 4–D (2017), 561–564.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Group Extension Through c-left Groupoid

Research Article

Vishal Vincent Henry[1*], Swapnil Srivastava[2] and Ajit Paul[1]

1 Department of Mathematics and Statistics, SHUATS, Allahabad, Uttar Pradesh, India.
2 Department of Mathematics, Ewing Christian College, Allahabad, Uttar Pradesh, India.

Abstract: In this paper the structure of c-left groupoid has been defined and on the basis of that group extension of G of a group H has been solved for which S will be a left transversal to H in G such that the corresponding c-left groupoid is the given one.

Keywords: Groupoid, left transversals, group extension.

1. Introduction and Preliminaries

Let H be a group and S its left transversal with identity "e". The group extension G of H has been obtained by S through c-left groupoid.

Definition 1.1. *A quadruple (S, H, σ, f) where S is a groupoid with binary operation "o" and identity e, H is a group which acts on S from left through a given action θ, σ is a map S to H^H (the set of all maps from H to H) and f is a map from $S \times S$ to H, is called c-left groupoid if it satisfies the following conditions*

(CG1) $[yox = y] \Rightarrow x = e$.

(CG2) For each $x \in S \ \exists \ x' \in S$ *such that* $xox' = e$.

(CG3) $\sigma_e = I_H$, *the identity map on* H, *where* σ_x *denotes the image* $\sigma(x)$ *of* x *under the map* σ *for each* $x \in S$.

(CG4) $f(x, e) = f(e, x) = 1$, *the identity of* H.

(CG5) $\sigma_x(h_1h_2) = \sigma_{h_2\theta x}(h_1)\sigma_x(h_2)$.

(CG6) $xo(yoz) = (xoy)o(f(x, y)\theta z)$.

(CG7) $h\theta(xoy) = (h\theta x)o(\sigma_x(h)\theta y)$.

(CG8) $f(x, yoz)f(y, z) = f(xoy, f(x, y)\theta z)\sigma_z(f(x, y))$.

(CG9) $\sigma_{xoy}(h)f(x, y) = f(h\theta x, \sigma_x(h)\theta y)\sigma_y(\sigma_x(h))$, *where* $x, y, z \in S$ *and* $h_1, h_2, h_3 \in H$.

* *E-mail: vincent.henry98@gmail.com*

Example 1.2. *Let G be a group and S its left transversal to a subgroup H. Let $x, y \in S$ and $h \in H$. Then $x \cdot y = (xoy)f(x,y)$ for some $f(x,y) \in H$ and $(xoy) \in S$. Also $h \cdot x = h\theta x s_x(h)$ for some $\sigma_x(h) \in H$ and $h\theta x \in S$. This gives a map $f : S \times S\theta H$ and a map $\sigma : S\theta H^H$ defined by $f((x,y)) = f(x,y)$ and $s(x)(h) = s_x(h)$. Then (S, H, σ, f) is a c-left groupoid.*

Solution. It is easy to verify CG1, CG2, CG3, CG4. Let $x \in S$ and $h_1, h_2 \in H$. Then

$$\begin{aligned}(h_1h_2)\theta x s_x(h_1h_2) &= (h_1h_2) \cdot x \\ &= h_1(h_2 \cdot x) \\ &= h_1(h_2\theta x)\sigma_x(h_2) \\ &= h_1\theta(h_2\theta x)\sigma_{h_2\theta x}(h_1)\sigma_x(h_2)\end{aligned}$$

This shows that $(h_1h_2)\theta x = h_1\theta(h_2\theta x)$ i.e. θ is a left action of H on S and $\sigma_x\,(h_1h_2) = \sigma_{h_2\theta x}(h_1)\sigma_x(h_2)$ which proves CG5. Next, let $x, y, z \in S$. Then

$$\begin{aligned}(xo(yoz))f(x, yoz)f(y,z) &= x \cdot (yoz)f(y,z) \\ &= x \cdot (y \cdot z) \\ &= (x \cdot y)z \\ &= ((xoy)f(x,y))z \\ &= (xoy)(f(x,y)\theta z)\sigma_z(f(x,y)) \\ &= ((xoy)o(f(x,y)\theta z))f((xoy), f(x,y)\theta z)\sigma_z(f(x,y))\end{aligned}$$

This shows that $xo(yoz) = (xoy)o(f(x,y)\theta z)$ and $f(x, yoz)f(y,z) = f(xoy, f(x,y)\theta z)s_z(f(x,y))$ which proves CG6 and CG8. Finally, let $x, y \in S$ and $h \in H$. Then

$$\begin{aligned}h\theta(xoy)\sigma_{xoy}(h)f(x,y) &= h \cdot (xoy)f(x,y) \\ &= h \cdot (x \cdot y) \\ &= (h \cdot x) \cdot y \\ &= (h\theta x)\sigma_x(h)y \\ &= (h\theta x)\ (\sigma_x(h)\theta y)\sigma_y(\sigma_x(h)) \\ &= ((h\theta x)o(\sigma_x(h)\theta y))f(h\theta x, \sigma_x(h)\theta y)\sigma_y(\sigma_x(h))\end{aligned}$$

This shows that $h\theta(xoy) = (h\theta x)o(\sigma_x(h)\theta y)$ which is CG7 and $\sigma_{xoy}(h)f(x,y) = f(h\theta x, \sigma_x(h)\theta y)\sigma_y(\sigma_x(h))$ which is CG9. Therefore (S, H, σ, f) is a c-left groupoid. □

Lemma 1.3. *Let S be a groupoid with identity e and H be a group which acts on S from left. Then*

(i). $\sigma_x(1) = 1$ *and*

(ii). $h\theta e = e$

where 1 is assumed as identity of H.

Proof. It is given that S be a groupoid with identity e and H be a group which acts on S from left then (i). $\sigma_x(1) = \sigma_x(1 \cdot 1)$

$= \sigma_{1\theta x}(1)\sigma_x(1)$ (using CG5)

$= \sigma_x(1)\sigma_x(1)$ (since $1\theta x = x$)

$\Rightarrow \sigma_x(1) = 1$

(ii). $h\theta e\sigma = h\theta(eoe)$

$= (h\theta e)o(s_e(h)oe)$ (using CG7)

$= (h\theta e)o(h\theta e)$ (using CG3)

$\Rightarrow h\theta e = e$

□

2. Main Result

Theorem 2.1. *Given a c-left groupoid (S, H, σ, f) there is a group extension G to H for which S is a left transversal to H in G such that the corresponding c-left groupoid is (S, H, σ, f).*

Proof. Let SH denote the Cartesian product of S and H. Denote an order pair (x, a) by xa. Define a binary operation $' \cdot '$ in G as follows

$$xa \cdot yb = (xoa\theta y)f(x, a\theta y)\sigma_y(a)b \tag{1}$$

By definition it is closed under the operation of multiplication '·'. Now let us show the associativity of the binary operation

$$\begin{aligned}
(xa \cdot yb) \cdot zc &= [(xoa\theta y)f(x, a\theta y)\sigma_y(a)b] \cdot zc \\
&= [(xoa\theta y)\, o\, (f\,(x, a\theta y)\, \sigma_y\,(a)\, b)\, \theta z]\, f\,(xoa\theta y, f\,(x, a\theta y)\, \sigma_y\,(a)\, b\theta z)\, \sigma_z(f(x, a\theta y)s_y(a)b)c \\
&= [(xoa\theta y)\, o\, (f\,(x, a\theta y)\, \sigma_y\,(a)\, b)\, \theta z]\, f\,(xoa\theta y, f\,(x, a\theta y)\, \sigma_y\,(a)\, b\theta z)\, s_{b\theta z}(f(x, a\theta y)s_y(a))s_z(b)c \\
&= [(xoa\theta y)\, o\, (f\,(x, a\theta y)\, \sigma_y\,(a)\, b)\, \theta z]\, f\,(xoa\theta y, f\,(x, a\theta y)\, \sigma_y\,(a)\, b\theta z)\, \sigma_{\sigma_{y(a)}\theta b\theta z}(f(x, a\theta y))\sigma_{b\theta z}(\sigma_y(a))s_z(b)c \\
&= (xo(a\theta y)of(x, a\theta y)\theta\sigma_y(a)\theta b\theta z f(xoa\theta y, f(x, a\theta y)\theta\sigma_y(a)\theta b\theta z)\sigma_{\sigma_{y(a)}\theta b\theta z}(f(x, a\theta y))\sigma_{b\theta z}(\sigma_y(a))\sigma_z(b)c \\
&= xo\,(a\theta y)\, o\, (\sigma_y\,(a)\, \theta b)\, \theta z f\,(xoa\theta y, f\,(x, a\theta y)\, \theta\sigma_y\,(a)\, \theta b\theta z)\, \sigma_{\sigma_{y(a)}\theta b\theta z}(f(x, a\theta y))\sigma_{b\theta z}(\sigma_y(a))\sigma_z(b)c \\
&= xo\,(a\theta y)\, o\, (\sigma_y\,(a)\, \theta b)\, \theta z f\,(x, (a\theta y)\, o\, (\sigma_y\,(a)\, \theta b)\, \theta z)\, f\,(a\theta y, \sigma_y\,(a)\, \theta\,(b\theta z))\, \sigma_{b\theta z}(\sigma_y(a))\sigma_z(b)c \\
&= (xoa\theta(yob\theta z))f(x, a\theta(yob\theta z))f(a\theta y, (\sigma_y(a)\theta(b\theta z))\sigma_{b\theta z}(\sigma_y(a))\sigma_z(b)c \\
&= (xoa\theta(yob\theta z))f(x, a\theta(yob\theta z))\sigma_{yob\theta z}(a)f(y, b\theta z)\sigma_z(b)c \\
&= xa \cdot [(yob\theta z)f(y, b\theta z)\sigma_z(b)c] \\
&= xa \cdot (yb \cdot zc)
\end{aligned}$$

$e1$ is the right identity of G. For, if $xa \in G$, then

$$\begin{aligned}
xa \cdot e1 &= (xoa\theta e)f(x, a\theta e)\sigma_e(a)1 \\
&= (xoe)f(x, e)a \\
&= x \cdot 1a \\
&= xa
\end{aligned}$$

Next, since

$$\begin{aligned} xa \cdot a^{-1}\theta x'\sigma_{x'}(a^{-1})\big(f(x,x')\big)^{-1} &= xoa\theta(a^{-1}\theta x')f(x, a\theta a^{-1}\theta x')\sigma_{a^{-1}\theta x'}(a)\sigma_{x'}(a^{-1})\big(f(x,x')\big)^{-1} \quad (x' \text{ is right inverse of } x \text{ in } S) \\ &= xo(1\theta x')f(x, 1\theta x')\sigma_{x'}(1)\left(f(x,x')\right)^{-1} \\ &= (xox')f(x,x')1\left(f(x,x')\right)^{-1} \\ &= e1 \end{aligned}$$

This shows that $a^{-1}\theta x'\sigma_{x'}(a^{-1})\left(f(x,x')\right)^{-1}$ is the right inverse of xa. The map $i : H \to G$ defined by $i\,(h) = eh$ is an injective homomorphism. The map $j : S \to G$ defined by $j\,(x) = x1$ is also an injective map. Identifying H and S with their images in G, G becomes a group extension of H for which S is a left transversal to H in G such that the c-left groupoid determined by S is the same as (S, H, σ, f). □

References

[1] Alexandru Nica, *On a Groupoid Construction for Actions of Certain Inverse Semigroups*, International Journal of Mathematics, 5(1994), 349-372.

[2] M.Artin, *Algebra*, Prentice Hall of India, (2000).

[3] R.Lal, *Transversals in Groups*, Journal of Algebra, 181(1996), 70-81.

[4] R.Lal, *Some Problems on Dedekind type Groups*, Journal of Algebra, 181(1996), 223-234.

[5] R.Lal and R.P.Shukla, *Perfectly Stable Subgroups of Finite Groups*, Com. Alg., 24(2)(1996), 643-657.

[6] R.Lal and R.P.Shukla, *Transversals in Non-discrete groups*, Proc. Ind. Acad. Sc., 115(4)(2005), 429-435.

[7] D.J.S.Robinson, *A course in the theory of groups*, Springer-Verlag, New York, (1982).