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On Some Fixed Point Theorems for Generalized Contractive Mappings in an Euclidean Space $\mathbb{R}^n$

Research Article

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Abstract: In this paper a couple of fixed point theorems for contraction and Kannan mappings are proved in an Euclidean space $\mathbb{R}^n$ via calculus method and using the max/mini principle. It is shown that though our approach to Banach and Kannan mappings is different from the constructive one, we are not far away from the usual method. Actually one can take an arbitrary point $x_0 \in \mathbb{R}^n$ and can define a sequence $\{x_n\}$ of iterates of the mapping under consideration. Then it is shown that the sequence converges to a fixed point geometrically.

MSC: 47H10, 54G25.

Keywords: Contraction map, Fixed point theorem, max/mini principle.

1. Introduction

Let $\mathbb{R}$ denote the real line and consider the Euclidean space $\mathbb{R}^n$ of two dimensions. The distance of a point $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ from origin is defined by a function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}^+$ as

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}. \tag{1}$$

The function $\|\cdot\|$ is called a norm in $\mathbb{R}^n$. Here we consider the class of mappings $T : \mathbb{R}^n \to \mathbb{R}^n$. We know that if $i : \mathbb{R}^n \to \mathbb{R}^n$ is an identity mapping, then

$$ix = x \quad \forall\, x \in \mathbb{R}^n.$$

Thus every point x of $\mathbb{R}^n$ is invariant or unaltered under the action of the mapping i on $\mathbb{R}^n$ into itself. Now the question is whether this property remains true if we replace the identity mapping i with some other mapping T on $\mathbb{R}^n$ into itself, that is, whether there is a point $x^* \in \mathbb{R}^n$ with the property that $Tx^* = x^*$. This problem has been discussed in the literature at length under the title fixed point theory. The point x^* in above discussion is called a fixed point of the mapping T and any statement that guarantees the existence of such fixed point for the mapping T is called a fixed point theorem for T. Thus, a fixed point theorem for T is a set of sufficient conditions for the existence of fixed points for the map T on $\mathbb{R}^n$ into $\mathbb{R}^n$.

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The fixed point theorems for contraction mappings in a metric space are proved via constructive method by constructing a sequence of iterations and which is shown to converge to a fixed point of the mapping in question geometrically. However, the fixed point theorems for contractive mappings can also be proved via theoretical method and some fixed point theorems are obtained in Drager and Foote [3] and Dhage [4, 5] by using the following max/mini principle.

Max/Mini principle : Let C be a no-empty, closed and bounded subset of the Euclidean space $\mathbb{R}^n$. If $f : C \to \mathbb{R}$ is a continuous function, then it attains its maximum or minimum on C.

In this paper we prove a fixed point theorem for the self-mappings of $\mathbb{R}^n$ satisfying the general contractive condition by using the max/mini principle.

2. Main Results

The following definition is well-known in the literature.

Definition 2.1. *A mapping $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ is called a* **contraction** *or* **Banach mapping** *if*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \alpha \|x - y\|$$

for all $x, y \in \mathbb{R}^n$ and $0 \leq \alpha < 1$.

The following theorem is proved for the contraction mapping in $\mathbb{R}^n$ for the special case for $n = 1$ under the arguments from calculus using max/min principle.

Theorem 2.2. *Let C be a non-empty, and closed subset of $\mathbb{R}^n$ and let $\mathcal{T} : C \to C$ be a contraction mapping. Then $\mathcal{T}$ has a unique fixed point.*

In this paper we generalize the above fixed point theorem for a wider class of contraction mappings in $\mathbb{R}^n$ via the approach of calculus method. We introduce the following definition.

Definition 2.3. *A mapping $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}^n$ is called a* ***generalized contractive*** *if*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \alpha\|x - y\| + \beta \left[\, \|x - \mathcal{T}x\| + \|y - \mathcal{T}y\| \,\right] + \gamma \left[\, \|x - \mathcal{T}y\| + \|y - \mathcal{T}x\| \,\right] \tag{2}$$

for all $x, y \in \mathbb{R}^n$, where α, β and γ are nonnegative real numbers satisfying the inequality $\alpha + 2\beta + 2\gamma < 1$.

Theorem 2.4. *Let C be a non-empty and closed subset of $\mathbb{R}^n$ and let $\mathcal{T} : C \to C$ be a generalized contractive mapping. If $\mathcal{T}$ is continuous, then $\mathcal{T}$ has a unique fixed point.*

Proof. Define the function $f : C \to \mathbb{R}$ by

$$f(x) = \|x - \mathcal{T}x\|, \ x \in C. \tag{3}$$

We observe that the zero of the function f is a fixed point of $\mathcal{T}$. We show that f is continuous on C. Let $x, y \in C$, then we have

$$\begin{aligned} \|fx - fy\| &= \|x - \mathcal{T}x\| - \|y - \mathcal{T}y\| \\ &\leq \|x - y\| + \|\mathcal{T}x - \mathcal{T}y\| \end{aligned} \tag{4}$$

and so, f is continuous on C.

If C is bounded, then max/min principle implies the existence of a point $u \in C$ such that

$$f(u) = \min\{f(x) \mid x \in C\}. \tag{5}$$

Again, we have

$$f(u) \leq f(\mathcal{T}u) = \|\mathcal{T}u - \mathcal{T}^2 u\|. \tag{6}$$

But, by condition (2), we obtain

$$\begin{aligned} \|\mathcal{T}u - \mathcal{T}^2u\| &\leq \alpha\|u - \mathcal{T}u\| + \beta\left[\, \|u - \mathcal{T}u\| + \|\mathcal{T}u - \mathcal{T}^2u\|\,\right] + \beta\left[\, \|u - \mathcal{T}^2u\| + \|\mathcal{T}u - \mathcal{T}u\|\,\right] \\ &\leq \alpha\|u - \mathcal{T}u\| + \beta\left[\, \|u - \mathcal{T}u\| + \|\mathcal{T}u - \mathcal{T}^2u\|\,\right] + \beta\left[\, \|u - \mathcal{T}u\| + \|\mathcal{T}u - \mathcal{T}^2u\|\,\right] \\ &\leq (\alpha+\beta+\gamma)\|u - \mathcal{T}u\| + (\beta+\gamma)\|\mathcal{T}u - \mathcal{T}^2u\| \\ &\leq \left(\frac{\alpha+\beta+\gamma}{1-(\beta+\gamma)}\right)\|u - \mathcal{T}u\| \\ &\leq \lambda f(u), \end{aligned} \tag{7}$$

where $\lambda = \left[\dfrac{\alpha+\beta+\gamma}{1-(\beta+\gamma)}\right] < 1$. If u is not zero of f, then from (6) and (7) we obtain $f(u) \leq \lambda f(u)$ which is a contradiction since $f(u) \geq 0$ and $\lambda < 1$. Hence u is a zero of f and consequently u is a fixed point of $\mathcal{T}$. If C is not bounded, choose $q \in C$ and set

$$\overline{C} = \{x \in C \mid f(x) \leq f(q)\}. \tag{8}$$

If $x \in \overline{C}$, then

$$\begin{aligned} \|x - q\| &\leq \|x - \mathcal{T}x\| + \|\mathcal{T}x - \mathcal{T}q\| + \|q - \mathcal{T}q\| \\ &\leq \|x - \mathcal{T}x\| + \alpha\|x - q\| + \beta\left[\, \|x - \mathcal{T}x\| + \|q - \mathcal{T}q\|\,\right] + \gamma\left[\, \|x - \mathcal{T}q\| + \|q - \mathcal{T}q\|\,\right] + \|q - \mathcal{T}q\| \\ &\leq \|x - \mathcal{T}x\| + \alpha\|x - q\| + \beta\|x - \mathcal{T}x\| + \beta\|q - \mathcal{T}q\| + \gamma\|x - q\| + \gamma\|q - \mathcal{T}q\| + \gamma\|x - q\| + \gamma\|x - \mathcal{T}x\| + \|q - \mathcal{T}q\| \\ &\leq (\alpha+2\gamma)\|x - q\| + (1+\beta+\gamma)\|x - \mathcal{T}x\| + (1+\beta+\gamma)\|q - \mathcal{T}q\| \\ &\leq 2\left[\frac{1+\beta+\gamma}{1-(\alpha+2\gamma)}\right] f(q), \end{aligned}$$

and so, $\overline{C}$ is closed and bounded. Also we have

$$f(\mathcal{T}x) \leq \lambda f(x) \leq \lambda f(q) \leq f(q)$$

for all $x \in \overline{C}$. This implies that $\mathcal{T}$ maps $\overline{C}$ into itself. Again proceeding as in the previous case, it can be proved that $\mathcal{T}$ has a fixed point. Finally the uniqueness of the fixed point u follows from the definition of the Kannan mapping. This completes the proof. □

Corollary 2.5. *Let C be a non-empty closed subset of $\mathbb{R}^n$ and let $\mathcal{T}: C \to C$ be a continuous mapping. If there exists a positive integer p such that $\mathcal{T}^p$ is a generalized contractive mapping then $\mathcal{T}$ has a unique fixed point.*

Proof. Set $\mathcal{S} = \mathcal{T}^p$. Then $\mathcal{S}: C \to C$ is a continuous mapping. By Theorem 2.4, $\mathcal{S}$ has a unique fixed point $u \in C$, that is, it a point such that $\mathcal{S}(u) = \mathcal{T}^p(u) = u$. Now $\mathcal{T}(u) = \mathcal{T}(\mathcal{T}^p u) = \mathcal{S}(\mathcal{T}u)$, showing that $\mathcal{T}u$ is again a fixed point of $\mathcal{S}$. By the uniqueness of u we get $\mathcal{T}u = u$. The proof is complete. □

Remark 2.6. *We remark that our fixed point theorems formulated in Theorem 2.4 can be extended to the mappings satisfying the more general contractive conditions along the lines of Ciric [6] given in the literature on metric fixed point theory.*

3. Comparison

Though our approach to Kannan mappings is different from the constructive one, we are not far away from the usual method. Now we shall relate our proof of Theorem 2.4 to the usual approach in which one takes an arbitrary point $x_0 \in C$ and defines a sequence $x_0, x_1, \ldots,$ in C in a certain way and which is shown to converge to a fixed point geometrically. This property follows very easily from the arguments given in the proof. For let $x_0 \in C$ and consider a sequence $\{x_n\}$ in C defined by $x_{n+1} = f(x_n), n = 0, 1, \ldots.$ From the definition of f and $\mathcal{T}$ we obtain

$$f(\mathcal{T}x) \leq \lambda f(x), \tag{9}$$

where $0 \leq \lambda = \left(\dfrac{\alpha+\beta+\gamma}{1-(\beta+\gamma)}\right) < 1$. If u is a unique fixed point of $\mathcal{T}$, then

$$\begin{aligned}
\|x-u\| &\leq \|x-\mathcal{T}x\| + \|\mathcal{T}x-\mathcal{T}u\| \\
&\leq \|x-\mathcal{T}x\| + \alpha\|x-u\| + \beta\|x-\mathcal{T}x\| + \gamma\,[\,\|x-u\| + \|u-\mathcal{T}x\|\,] \\
&\leq f(x) + \alpha\|x-u\| + \beta f(x) + 2\gamma\|x-u\| + \gamma f(x) \\
&\leq (1+\beta+\gamma)\, f(x) + (\alpha+2\gamma)\|x-u\|
\end{aligned}$$

and so, we have

$$\|x-u\| \leq \left[\frac{1+\beta+\gamma}{1-(\alpha+2\gamma)}\right] f(x) \tag{10}$$

for all $x \in C$. Combining (9) and (10) for the sequence $\{x_n\}$ yields

$$\|x_n - x\| \leq \left[\frac{1+\beta+\gamma}{1-(\alpha+2\gamma)}\right] f(x_n). \tag{11}$$

This shows that $\{x_n\}$ converges geometrically to a unique fixed point of $\mathcal{T}$. The inequality (11) is also helpful in finding the approximate fixed point of $\mathcal{T}$. If one wans an ϵ-approximate fixed point, then the iteration of the mapping $\mathcal{T}$ should continue until $f(x_n) < \left[\frac{1-(\alpha+2\gamma)}{1+\beta+\gamma}\right]\epsilon$. In case of Corollary 2.5, the iteration should continue until $f(x_{n+p}) < \left[\frac{1-(\alpha+2\gamma)}{1+\beta+\gamma}\right]\epsilon$.

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