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A New Construction of Apollonius Circle and a New Proof of Secant-Tangent Theorem

Research Article

Surajit Dutta[1]*

1 Gouri Apartment, Block B, 2nd Floor, Flat #301, Dankuni, Hooghly, West-Bengal, India.

Abstract: Here in this paper we give an easy and elegant construction of Apollonius circle by using a simple property of isosceles triangles. We also give a simple proof of the Secant-Tangent theorem by using the same property of isosceles triangles.

MSC: 51M04.

Keywords: Apollonius Circle, Secant-Tangent Theorem.

1. Introduction

Isosceles triangles have the following property: Consider triangle ABC where $AB = BC$, has a line segment drawn from A to a point D on the ray BC, and let E be the intersection point of the ray BC and the reflection of AD around AC, then BC is the geometric mean of BD and BE i.e. $BC^2 = BD.BE$, see [1]. Let us name this property as *"Property: P"*. Let A and B be two different points on the line L. Consider a point P for which $\frac{PA}{PB} = K \neq 1$, then the locus of the point P would be a circle and this circle is known as Apollonius Circle, see [2]. Here in this paper we will give a new construction of Apollonius circle by using *"Property: P"*, other constructions could be found here [4, 5]. If a point is taken outside a circle and from that point a secant and a tangent are drawn, then the product of the secant and its external segment is equal to the square of the tangent, this is known as Secant-Tangent theorem, see [3]. Here in this paper we give an alternative proof of this theorem by using *"Property: P"*.

2. Applications

2.1. Construction of Apollonius Circle

Let A and B be two different points on the line L. Consider a point P for which $\frac{PA}{PB} = K \neq 1$, see Figure 1. We have to construct the circle for which every point X on the circle satisfies the above equation i.e. $\frac{XA}{XB} = K \neq 1$. Here we will consider the following three steps to construct the circle:

Step 1: Draw the internal angle bisector of $\angle APB$, let C be the intersection point of L and the internal angle bisector, (see Figure 1).

* *E-mail: surajitdutta53@gmail.com*

Step 2: Select the acute angle between $\angle ACP$ and $\angle BCP$. This selection can easily be done by drawing a perpendicular line on L at point C. At point P draw an angle $\angle RPC$ equals the selected acute angle such that $\angle RPC$ and the drawn acute angle lie in the same side of PC. Here we considered $\angle ACP$ is acute angle.

Step 3: Let point $O = PR \cap L$. Draw a circle ε by considering OC as the radius. ε is the Apollonius Circle.

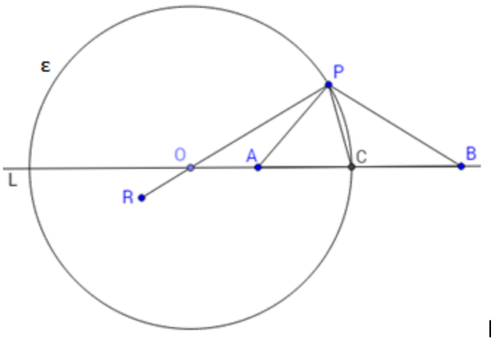

Figure 1.

Proof. Since $\frac{PA}{PB} = K \neq 1$, then it's clear that PC is not the perpendicular bisector of AB, So none of the $\angle ACP$ and $\angle PCB$ is right angle. So either of $\angle ACP$ or $\angle PCB$ is acute angle. Here we considered $\angle ACP$ as acute angle. Since $\angle ACP$ is acute angle and $\angle RPC = \angle ACP$, so $\angle RPC + \angle ACP < 180°$, So According to Euclid's fifth postulate PR and CA will meet at a point O. Now $\angle OPC = \angle OCP > \angle CPB = \angle APC$. The first equality holds by drawing, the second inequality holds by Exterior Angle Theorem, the third equality holds since PC is the internal bisector of $\angle APB$. So point A lies between point O and C. Now OPC is an isosceles triangle where $OP = OC$ and $\angle APC = \angle CPB$, so by applying *"Property: P"* we get $OC^2 = OA.OB$, i.e. A and B are inverse points with respect to the circle ε.
Since A and B are inverse points with respect to the circle ε, so ε is the Apollonius Circle for A, B and some positive number l, see [6]. Since P is on the circle ε, so $\frac{PA}{PB} = K = l$. It's true for all the points which are on the circle ε. □

2.2. Proof of the Secant-tangent Theorem

Consider ε be a circle with center O. Consider B is an external point and BA is the tangent at point A. BC is the secant which intersects the circle at point D and C. Now we have to prove that $AB^2 = BD.BC$ (see Figure 2).

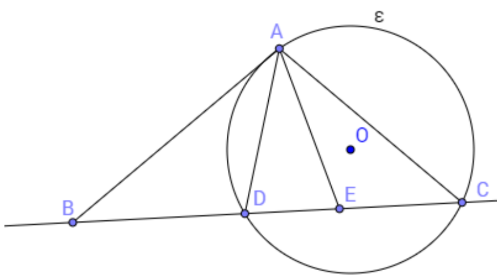

Figure 2.

Connect A with D and C. Now $\angle BDA > \angle ACD = \angle BAD$, the first inequality holds by Exterior Angle Theorem, the second equality holds by Tangent-chord theorem where AB is tangent at point A of the circle. So in triangle ABD, $AB > BD$ since $\angle BDA > \angle BAD$.

In triangle ABC it's clear that $\angle BAC > \angle BAD = \angle ACB$, the second equality holds by Tangent-chord theorem where AB is tangent at point A of the circle. So in triangle ABC, $BC > AB$, since $\angle BAC > \angle ACB$. From the above two results we see that $BC > AB > BD$. So we can take a point E on DC such that $AB = BE$. Connect A and E. Now,

$$\angle EAC = \angle AED - \angle ACE = \angle BAE - \angle ACE = \angle BAD + \angle DAE - \angle ACE = \angle DAE,$$

the first equality holds by Exterior Angle Theorem, the second equality holds since triangle ABC is isosceles where $AB = BE$, the third since $\angle BAE = \angle BAD + \angle DAE$ and the fourth since $\angle BAD = \angle ACE$ as AB is tangent at point A of the circle. Now triangle ABE is isosceles triangle where $AB = BE$ and $\angle DAE = \angle EAC$, so by applying the *"Property: P"* we get, $BE^2 = BD.BC$ i.e. $AB^2 = BD.BC$ since $AB = BE$.

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