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Study and Analysis of Almost Para Sasakian Type-Riemannian Manifold

Research Article

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Abstract: In this paper, author studied and analyses Almost Para Sasakian Type-Riemannian manifold. The first section of this paper is introductory in nature, which deals with basic definition and literature review with previous known defined results. Second section deals with Almost Para Sasakian Type-Riemannian manifold (APST-Riemannian manifold) and its various applications and the third section is devoted for Para-K-contact type Riemannian manifold (PKCT-Riemannian manifold). As the outcomes of this work further detail theorems are suggested for future work.

Keywords: APST-Riemannian manifold, PQST manifold, PKCT-Riemannian manifold, tensor analysis, differential geometry.

1. Introduction

The notation of almost para-contact manifold and para-contact metric manifold was introduced by I. Sato [1]. He and K. Matsumoto [2] defined and studied special cases of Almost Para Contact-Riemannian manifold (APC- Riemannian manifold) known as para sasakian manifold and special para sasakian manifold. Later on many others contributed to the study of several classes of almost para-contact manifold endowed with a Reimannian metric T. Adati et. al. [3] and G. Chuman [4].

Definition 1.1 ([5, 6])**.** *Let an n dimensional Riemannian manifold M_n, on which there are defined a tensor field F of type (1, 1), a vector field T, a 1-form A and metric tensor g satisfying for arbitrary vector field X, Y, Z, … satisfying,*

(a). $F^2X = X - A(X)T$,

(b). $F(T) = 0$,

(c). $A(FX) = 0$,

(d). $A(T) = 1$,

(e). $g(Fx, Fy) = -g(x, y) + A(x)A(y)$.

then structure (F, T, A, g) is called almost para contact metric structure and manifold M_n will be called Almost para contact metric Riemannian manifold. Let us put

(f). $'F(x, y) = g(\overline{X}, y)$ *where* $\overline{X} = Fx$, *it can be verified that* $'F$ *is skew symmetric, i.e.* $'F(x, y) + {'F}(y, x) = 0$, *one can check from Definition 1.1 (e), that*

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(g). $g(T, X) = A(X)$.

Definition 1.2 ([5]). *An almost para contact metric manifold on which the fundamental 2-form ‘F satisfies*

$$2`F = dA \tag{1}$$

is called an almost para sasakian type manifold (APST-Riemannian manifold) or contact Riemannian manifold [7].

2. Some Properties of APST-Riemannian Manifold

From (1)

$$\begin{aligned} 2`F &= dA, \\ \text{or } \; 2`F(X,Y) &= dA(X,Y), \\ &= XA(Y) - YA(X) - A([X,Y]), \\ &= (D_x A)(Y) - (D_y A)(X), \end{aligned}$$

where ‘D’ is Riemannian connexion. Thus

Theorem 2.1. *On APST-Riemannian manifold,*

$$`F(X,Y) = \frac{1}{2}[(D_x A)(Y) - (D_y A)(X)]. \tag{2}$$

From (1)

$$(d`F) = d^2 A = 0 \tag{3}$$

$$(d`F)(X,Y,Z) = X`F(Y,Z) - Y`F(X,Z) + Z`F(X,Y) \tag{4}$$

$$\begin{aligned} -`F([X,Y],Z) + `F([X,Z],Y) - `F([Y,Z],X) &= D_x`F(Y,Z) + `F(D_xY,Z) + `F(Y,D_xZ) - (D_y`F)(X,Z) \\ &\quad - `F(D_yX,Z) - `F(X,D_yZ) + (D'_zF)(X,Y) + `F(D_zX,Y) + `F(X,D_zY) \\ &\quad - `F((D_xY - D_yX),Z) + `F((D_xZ - D_zX),Y) - `F((D_yZ - D_zY),X) \\ &= (D_x`F)(Y,Z) + (D_y`F)(Z,X) + (D_z`F)(X,Y). \end{aligned} \tag{5}$$

Theorem 2.2. *On APST-Riemannian manifold,*

$$(d`F) = 0 \text{ (i.e ‘F is closed ) } \Leftrightarrow (D_x`F)(Y,Z) + (D_y`F)(Z,X) + (D_z`F)(X,Y) = 0. \tag{6}$$

Definition 2.3 ([8, 9]). *An almost para contact metric manifold on which ‘F is closed is called Para Quasi-Sasakian type Manifold (PQST) manifold.*

Definition 2.4 ([10, 11]). *An APST – Riemannian manifold, on which*

$$(D_x A)(Y) + (D_y A)(X) = 0 \tag{7}$$

holds, is called para-K-contact type Riemannian manifold (PKCT)-Riemannian manifold.

3. Theorems on PKCT-Riemannian Manifold

From (2) and (7)

$$2`F(X,Y)+0=[(D_xA)(Y)-(D_yA)(X)]+[(D_xA)(Y)+(D_yA)(X)]$$
$$2`F(X,Y)=2(D_xA)(Y)$$
$$`F(X,Y)=(D_xA)(Y)=-(D_yA)(X).$$

Theorem 3.1. *On PKCT-Riemannian manifold,*

$$`F(X,Y)=(D_xA)(Y)=-(D_yA)(X). \tag{8}$$

From (8)

$$`F(X,Y)=(D_xA)(Y)$$
$$(D_z`F)(X,Y)+`F(D_zX,Y)+`F(X,D_zY)=(D_zD_xA)(Y)+(D_xA)(D_zY)$$
$$(D_z`F)(X,Y)+(D_{D_zx}A)(Y)+(D_xA)(D_zY)=(D_zD_xA)(Y)+(D_xA)(D_zY)$$
$$(D_z`F)(X,Y)=(D_zD_xA)(Y)-(D_{D_zx}A)(Y) \tag{9}$$
$$(D_x`F)(Y,Z)=(D_xD_yA)(Z)-(D_{D_xy}A)(Z) \tag{10}$$
$$(D_y`F)(Z,X)=(D_yD_zA)(X)-(D_{D_yz}A)(X) \tag{11}$$

Replace Z by X

$$(D_y`F)(X,Z)=(D_yD_xA)(Z)-(D_{D_yx}A)(Z) \tag{12}$$

Subtracting (12) *from* (10), *it comes*

$$(D_x`F)(Y,Z)-(D_y`F)(X,Z)=(D_xD_yA)(Z)-(D_yD_xA)(Z)-(D_{D_xy}A-D_{D_yx}A)(Z) \tag{13}$$
$$(D_x`F)(Y,Z)+(D_y`F)(Z,X)=(D_xD_yA)(Z)-(D_yD_xA)(Z)-(D_{[X,Y]}A)(Z) \tag{14}$$

Using (6), *it comes*

$$-(D_z`F)(X,Y)=-A(K(X,Y,Z)),$$
$$(D_z`F)(X,Y)=A(K(X,Y,Z)).$$

Theorem 3.2. *On PKCT-Riemannian manifold,*

$$(D_z`F)(X,Y)=A(K(X,Y,Z)). \tag{15}$$

From Definition 1.1 (d)

$$A(T)=1$$
$$(D_xA)T+A(D_xT)=0$$

Using (8) it comes

$$`F(X,T) + A(D_xT) = 0$$

Using Definition 1.1 (f) it comes

$$g(\overline{X},T) + A(D_xT) = 0$$
$$A(\overline{X}) + A(D_xT) = 0$$
$$A(\overline{X} + D_xT) = 0$$
$$D_xT = -\overline{X}.$$

Theorem 3.3. *On PKCT-Riemannian manifold,*

$$D_xT = -\overline{X}. \tag{16}$$

Alternative definition of PKCT-Riemannian manifold is given by

Definition 3.4. *An almost para contact metric type Riemannian manifold, on which $D_xT = -\overline{X}$ is called PKCT-Riemannian manifold.*

From Definition 1.1 (f) and (e)

$$`F(Y,T) = g(\overline{Y}, T,) = A(\overline{Y}) = 0$$
$$`F(Y,T) = 0 \tag{17}$$
$$(D_x`F)(Y,T) + `F(D_xY,T) + `F(Y,D_xT) = 0$$
$$(D_x`F)(Y,T) + 0 - `F(D_xT,Y) = 0$$

Using (16), it comes

$$(D_x`F)(Y,T) = `F(-\overline{X},Y)$$
$$(D_x`F)(Y,T) = -`F(\overline{X},Y)$$
$$(D_x`F)(Y,T) = `F(Y,\overline{X}) \tag{18}$$

Using Definition 1.1 (f)

$$(D_x`F)(Y,T) = g(\overline{Y},\overline{X})$$
$$(D_x`F)(Y,T) = g(\overline{X},\overline{Y}).$$

Theorem 3.5. *On PKCT-Riemannian manifold,*

$$D_x`F(Y,T) = g(\overline{X},\overline{Y}). \tag{19}$$

From Definition 1.1 (f)

$$`F(X,Y) = g(\overline{X},Y)$$

$$
\begin{aligned}
{}`F(\overline{X},\overline{Y}) &= g(\overline{\overline{X}},\overline{Y})\\
{}`F(\overline{X},\overline{Y}) &= g(X - A(X)T,\overline{Y})\\
{}`F(\overline{X},\overline{Y}) &= g(X,\overline{Y}) - A(X)g(T,\overline{Y})\\
{}`F(\overline{X},\overline{Y}) &= g(X,\overline{Y}) - A(X)A(\overline{Y})\\
{}`F(\overline{X},\overline{Y}) &= g(X,\overline{Y})\\
{}`F(\overline{X},\overline{Y}) &= {}`F(Y,X)\\
{}`F(\overline{X},\overline{Y}) &= -{}`F(X,Y) \qquad (20)
\end{aligned}
$$

$$
\begin{aligned}
&(D_z{}`F)(\overline{X},\overline{Y}) + {}`F((D_zF)(X) + F(D_zX),\overline{Y})\\
&\quad + {}`F(\overline{X},(D_zF)(Y) + F(D_zY)) = -(D_z'F)(X,Y) - {}`F(D_zX,Y) - {}`F(X,D_zY) \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
&(D_z{}`F)(\overline{X},\overline{Y}) + {}`F((D_zF)(X),\overline{Y}) + {}`F(F(D_zX),\overline{Y}) + {}`F(\overline{X},(D_zF)(Y))\\
&\quad + {}`F(\overline{X},F(D_zY)) = -(D_z{}`F)(X,Y) - {}`F(D_zX,Y) - {}`F(X,D_zY) \qquad (22)
\end{aligned}
$$

Using (20), it comes

$$(D_z{}`F)(\overline{X},\overline{Y}) + {}`F((D_zF)(X),\overline{Y}) + {}`F(\overline{X},(D_zF)(Y)) = -(D_z{}`F)(X,Y) \qquad (23)$$

Using Definition 1.1 (f) it comes

$$(D_z{}`F)(\overline{X},\overline{Y}) + g((\overline{D_zF)\,(X}),\overline{Y}) - g((\overline{D_zF)(Y}))\overline{X}) = -(D_z{}`F)(X,Y) \qquad (24)$$

Using Definition 1.1 (e), it comes

$$(D_z{}`F)(\overline{X},\overline{Y}) - g((D_zF)(X),Y) + A((D_zF)(X))A(Y) + g((D_zF)(Y),X) - A((D_zF)(Y)).A(X) = -(D_z{}`F)(X,Y) \qquad (25)$$

$$(D_z{}`F)(\overline{X},\overline{Y}) + (D_z{}`F)(Y,X) + A((D_zF)(X))A(Y) - A((D_zF)(Y))A(X) = 0 \qquad (26)$$

$$(D_z{}`F)(\overline{X},\overline{Y}) + (D_z{}`F)(Y,X) + g(\overline{Z,X})A(Y) - g(\overline{Z,Y})A(X) = 0 \qquad (27)$$

Using Definition 1.1 (e) it comes

$$(D_z{}`F)(\overline{X},\overline{Y}) - (D_z{}`F)(X,Y) - g(Z,X)A(Y) + g(Z,Y)A(X) = 0.$$

Theorem 3.6. *On PKCT-Riemannian manifold,*

$$(D_z{}`F)(\overline{X},\overline{Y}) - (D_z{}`F)(X,Y) - g(Z,X)A(Y) + g(Z,Y)A(X) = 0. \qquad (28)$$

It is known [5]

$$(D_z{}`F)(X,Y) = A(X)g(Y,Z) - A(Y)g(X,Z) \qquad (29)$$

Barring X and Y

$$(D_z{}`F)(\overline{X},\overline{Y}) = A(\overline{X})g(\overline{Y},Z) - A(\overline{Y})g(\overline{X},Z) \qquad (30)$$

$$(D_z`F)(\overline{X}, \overline{Y}) = 0 \tag{31}$$

Using (15), it comes

$$(D_z`F)(\overline{X}, \overline{Y}) = A(K(\overline{X}, \overline{Y}, Z)) = 0 \tag{32}$$

From (28)

$$(D_z`F)(X, Y) + A(Y)g(Z, X) - A(X)g(Z, Y) = 0 \tag{33}$$

$$(D_z`F)(X, Y) = A(X)g(Z, Y) - A(Y)g(Z, X)$$

Using Definition 1.1 (c), it comes

$$(D_z`F)(X, Y) = -A(X)g(\overline{Z}, \overline{Y}) + A(Y)g(\overline{Z}, \overline{X}) \tag{34}$$

Using (19), it comes

$$(D_z`F)(X, Y) = A(Y)(D_z`F)(X, T) - A(X)(D_z`F)(Y, T). \tag{35}$$

Theorem 3.7. *On PKCT-Riemannian manifold,*

$$(D_z`F)(X, Y) = A(Y)(D_z`F)(X, T) - A(X)(D_z`F)(Y, T).$$

Definition 3.8. *On PKCT-Riemannian manifold structure $\{F, T, A\}$ is said to be normal if*

$$\underset{O}{N}(X, Y) = 0 \tag{36}$$

Where $\underset{O}{N}(X, Y) = \underset{F}{N}(X, Y) + dA(X, Y)T = 0$.

$$\begin{aligned}
\underset{O}{N}(X, Y) &= [\overline{X}, \overline{Y}] + \overline{\overline{[X, Y]}} - \overline{[\overline{X}, Y]} - \overline{[X, \overline{Y}]} + \{XA(Y) - YA(X) - A(X, Y)\}T \\
&= D_{\overline{x}}\overline{Y} - D_{\overline{Y}}\overline{X} + [X, Y] - A([X, Y])T - \overline{D_{\overline{X}}Y} + \overline{D_Y\overline{X}} - \overline{D_X\overline{Y}} + \overline{D_{\overline{Y}}X} + \{XA(Y) - YA(X) - A([X, Y])\}T \\
&= (D_{\overline{x}}F)(Y) + F(D_{\overline{x}}Y) - (D_{\overline{y}}F)(X) - F(D_{\overline{y}}X) + D_xY - D_yX \\
&- A(D_xY)T + A(D_yX)T - \overline{D_{\overline{x}}Y} + \overline{(D_yF)(X)} + \overline{\overline{D_yX}} - \overline{(D_xF)(Y)} - \overline{\overline{D_xY}} + \overline{D_{\overline{y}}X} \\
&+ (D_xA)(Y)T + A(D_xY)T - (D_yA)(X)T - A(D_yX)T - A(D_xY)T + A(D_yX)T, \\
&= (D_{\overline{x}}F)(Y) + \overline{D_{\overline{x}}Y} - (D_{\overline{y}}F)(x) - (\overline{D_{\overline{y}}X}) + D_xY - D_yX - A(D_xY)T \\
&+ A(D_yX)T - \overline{D_{\overline{x}}Y} + \overline{(D_yF)(X)} + D_yX - A(D_yX)T - \overline{(D_xF)(Y)} - D_xY + A(D_xY)T \\
&+ \overline{D_{\overline{y}}X} + (D_xA)(Y)T + A(D_xY)T - (D_yA)(X)T - A(D_yX)T - A(D_xY)T + A(D_yX)T),
\end{aligned}$$

So,

$$\underset{O}{N}(X, Y) = (D_{\overline{x}}F)(Y) - (D_{\overline{y}}F)(X) + \overline{(D_yF)(X)} - \overline{(D_xF)(Y)} + \{(D_xA)(Y) - (D_yA)(X)\}.T \tag{37}$$

Differentiating covariantly the equation $\overline{\overline{Y}} = F\overline{Y}$ and using (1.1) and (16), it comes

$$\overline{(D_xF)(Y)} = -(D_xF)\overline{Y} - (D_xA)(Y)T + A(Y)(\overline{X}). \tag{38}$$

Using (37) and (38), it is seen that

$$\underset{O}{N}(X,Y)=0 \Leftrightarrow (D_{\overline{x}}F)(Y)-(D_{\overline{y}}F)(X)-(D_yF)(\overline{X})-(D_yA)(X)T+A(X)(\overline{Y})+(D_xF)(\overline{Y})$$
$$+(D_xA)(Y)T-A(Y)\overline{X}+(D_xA)(Y)T-(D_yA)(X)T=0,$$
$$\Leftrightarrow (D_{\overline{x}}F)(Y)-(D_{\overline{y}}F)(X)+(D_xF)(\overline{Y})-(D_yF)(\overline{X})-A(Y)(\overline{X})+A(X)(\overline{Y})$$
$$+2((D_xA)(Y)-(D_yA)(X))T=0.$$

From (2), it comes $\underset{O}{N}(X,Y)=0$ if and only if

$$(D_{\overline{x}}F)(Y)-(D_{\overline{y}}F)(X)+(D_xF)(\overline{Y})-(D_yF)(\overline{X})-A(Y)(\overline{X})+A(X)(\overline{Y})+4`F(X,Y)T=0, \tag{39}$$

which is equivalent to

$$g((D_{\overline{x}}F)Y,Z)-g((D_{\overline{y}}F)X,Z)+g((D_xF)\overline{Y},Z)-g((D_yF)\overline{X},Z)$$
$$-A(Y)g((\overline{X},Z)+A(X)g(\overline{Y},Z)+4'F(X,Y)g(T,Z)=0,$$
$$\text{or}\quad (D'_{\overline{x}}F)(Y,Z)+(D'_{\overline{y}}F)(Z,X)+(D'_xF)(\overline{Y},X)-(D'_yF)(\overline{X},Z)$$
$$-A(Y)g(Z,\overline{X})+A(X)g(Z,\overline{Y})+4'F(X,Y)A(Z)=0. \tag{40}$$

Using (6), (40) becomes

$$(d'F)((\overline{X},Y,Z)-(D'_yF)(Z,\overline{X})-(D'_zF)(\overline{X},Y)+(d'F)(X,\overline{Y},Z)-(D'_xF)(\overline{Y},Z)-(D'_zF)(X,\overline{Y})+(D'_xF)(\overline{Y},Z)$$
$$-(D'_yF)(\overline{X},Z)-A(Y)`F(X,Z)+A(X)`F(Y,Z)+4`F(X,Y)A(Z)=0$$
$$\text{or}\quad (d`F)((\overline{X},Y,Z)+(d'F)(X,\overline{Y},Z)-(D'_zF)(\overline{X},Y)-(D'_zF)(X,\overline{Y}))$$
$$-A(Y)`F(X,Z)+A(X)`F(Y,Z)+4`F(X,Y)A(Z)=0. \tag{41}$$

Since on a APST-Riemannian manifold, it comes $(d`F)=0$, the above Equation is equivalent to

$$(D'_zF)(\overline{X},Y)+(D'_zF)(X,\overline{Y})=-A(Y)'F(X,Z)+A(X)'F(Y,Z)-4'F(X,Y)A(Z) \tag{42}$$

Theorem 3.9. *PKCT-Riemannian structure is normal if* (42) *holds.*

4. Conclusion

It is concluded from (6) that an almost para contact metric manifold on which 'F is closed is called Para Quasi-Sasakian type Manifold (PQST) manifold and from (16) an almost para contact metric type Riemannian manifold is called Para K-Contact Type (PKCT)-Riemannian manifold. If (42) holds then PKCT Riemannian structure (F,T,A,g) is normal. These results can be helpful for further researches.

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