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# On Polynomial Solutions of Quadratic Diophantine Equation

**Research Article**

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**Abstract:** Let $P := P(t)$ be a polynomial in $Z[X]$. In this paper, we consider the polynomial solutions of Diophantine Equation $D : x^2 - 6y^2 - 10x + 24y = 0$. We also obtain some formulae and recurrence relations on the Polynomial solution $(x_n, y_n)$ of $D$.

**Keywords:** Diophantine equation, Polynomial solution, Pells equation, Continued fraction expansion.


## 1. Introduction

A Diophantine equation is an indeterminate polynomial equation that allows the variables to be integers only. Diophantine problems have fewer equations than unknown variables and involve finding integers that work correctly for all equations. They are named after the Hellenistic Mathematician Diophantus of Alexandria. The mathematical study of Diophantine problems is called Diophantine Analysis. The formulation of general theories of Diophantine equations was an achievement of the twentieth century. There are Diophantine equations which possess no solutions, finite number of solutions or infinite number of solutions. Among the various Diophantine equations, the Pythagorean equation and Pell's equation are bestowed with importance. A Pythagorean equation is a quadratic Diophantine equation $x^2 + y^2 = z^2$. The equation $x^2 - Dy^2 = N$ with given integers $D$ and $N$ and unknowns $x, y$ is called Pell's equation. If $D$ is a square or negative, it can have only a finite number of solutions. The generalized Pell's equation $x^2 - Dy^2 = 1$ was solved by Lagrange in terms of simple Continued fraction. If $(x_0, y_0)$ represents the fundamental solution, a sequence of solutions can be derived from this by using the equality $x_n + \sqrt{D}\, y_n = \left(x_0 + \sqrt{D}\, y_0\right)^n$. In this communication, yet another interesting quadratic Diophantine equation given by $x^2 - 6y^2 - 10x + 24y = 0$ is considered for finding all possible solutions by various methods.

## 2. The Diophantine Equation $x^2 - 6y^2 - 10x + 24y = 0$

In [11–15], we considered some specific Pell (also Diophantine) equations and their integer solutions. In the present paper, we consider the integer solutions of Diophantine equation

$$D : x^2 - 6y^2 - 10x + 24y = 0 \tag{1}$$

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over $\boldsymbol{Z}$. Note that it is very difficult to solve $D$ in its present form, that is, we cannot determine how many integer solutions $D$ has and what they are. So we have to transform $D$ into an appropriate Diophantine equation which can be easily solved. To get this let

$$T:\begin{cases} x=u+h \\ y=v+k \end{cases} \tag{2}$$

be a translation for some $h$ and $k$. In this case $\{h,k\}$ is called the base of $T$ and denote it by $T\,[h;k]=\{h,k\}$. If we apply $T$ to $D$, then we get

$$T\,(D)=\tilde{D}:(u+h)^2-6(v+k)^2-10\,(u+h)+24\,(v+k)=0 \tag{3}$$

In (3), we consider $u(2h+2-12)$ and $v\,(-12k+24)$. So we get $h=5$ and $k=2$. Consequently for $x=u+5$ and $y=v+2$, we have the Diophantine equation

$$\tilde{D}:\ u^2-6v^2=1 \tag{4}$$

which is a Pell equation. Now we try to find all integer solutions $(u_n,v_n)$ of $\tilde{D}$ and then we can retransfer all results from $\tilde{D}$ to $D$ by using the inverse of $T$.

**Theorem 2.1.** *Let $\tilde{D}$ be the Diophantine equation in (4). Then*

*(1). The continued fraction expansion of $\sqrt{6}$ is $\sqrt{6}=[2;\overline{2,\ 4}]$.*

*(2). The fundamental solution of $\tilde{D}$ is $(u_1,v_1)=(5,2)$.*

*(3). Define the sequence $\{(u_n,v_n)\}$, where*

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix}=\begin{pmatrix} 5 & 12 \\ 2 & 5 \end{pmatrix}^n\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5}$$

*for $n\geq 1$. Then $(u_n,v_n)$ is a solution of $\tilde{D}$.*

*(4). The solutions $(u_n,v_n)$ satisfy $u_n=5u_{n-1}+12v_{n-1}$ and $v_n=2u_{n-1}+5v_{n-1}$ for $n\geq 2$.*

*(5). The solutions $(u_n,v_n)$ satisfy the recurrence relations $u_n=9\,(u_{n-1}+u_{n-2})-u_{n-3}$ and $v_n=9\,(v_{n-1}+v_{n-2})-v_{n-3}$ for $n\geq 4$.*

*(6). The $n^{th}$ solution $(u_n,v_n)$ can be given by*

$$\frac{u_n}{v_n}=[2;2,\ 4,\cdots,\ 2,\ 4,\ 2]\,,\ \textit{for } n\geq 1. \tag{6}$$

*Proof.*

(1). The continued fraction expansion of $\sqrt{6}$ is

$$\begin{aligned}
\sqrt{6}&=2+\left(\sqrt{6}\ -3+1\right)\\
\sqrt{6}&=2+\frac{1}{\frac{\sqrt{6}+2}{2}}\\
&=2+\frac{1}{2+\frac{\sqrt{6}-2}{2}}\\
&=2+\frac{1}{2+\frac{1}{\sqrt{6}+2}}\\
&=2+\frac{1}{2+\frac{1}{4+\sqrt{6}-2}}
\end{aligned}$$

so $\sqrt{6}=\left[2;\overline{2,\ 4}\right]$.

(2). It is easily seen that $(u_1, v_1) = (5, 2)$ is the fundamental solution of $\tilde{D}$ since $5^2 - 6(2)^2 = 1$.

(3). We prove it by Mathematical induction. Let $n = 1$. Then by (5), we get $(u_1, v_1) = (5, 2)$ which is the fundamental solution and so is a solution of $\tilde{D}$. Let us assume that the Diophantine equation in (4) is satisfied for $n-1$, that is, $\tilde{D}:\ u_{n-1}^2 - 6v_{n-1}^2 = 1$. We want to show that this equation is also satisfied for $n$. Applying (5), we find that

$$\begin{aligned}
\begin{pmatrix} u_n \\ v_n \end{pmatrix} &= \begin{pmatrix} 5 & 12 \\ 2 & 5 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} u_n \\ v_n \end{pmatrix} &= \begin{pmatrix} 5 & 12 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 5 & 12 \\ 2 & 5 \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} u_n \\ v_n \end{pmatrix} &= \begin{pmatrix} 5 & 12 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} u_{n-1} \\ v_{n-1} \end{pmatrix}
\end{aligned} \tag{7}$$

Hence we conclude that

$$\begin{aligned}
u_n^2 - 6v_n^2 &= (5u_{n-1} + 12v_{n-1})^2 - 6(2u_{n-1} + 5v_{n-1})^2 \\
&= u_{n-1}^2 - 6v_{n-1}^2 = 1.
\end{aligned}$$

So $(u_n, v_n)$ is also a solution $\tilde{D}$.

(4). From (7), we find that $u_n = 5u_{n-1} + 12v_{n-1}$ and $v_n = 2u_{n-1} + 5v_{n-1}$ for $n \geq 2$.

(5). We only prove that $u_n$ satisfy the recurrence relation. For $n = 4$, we get $u_1 = 5$, $u_2 = 49$, $u_3 = 477$, $u_4 = 4729$. Hence

$$\begin{aligned}
u_4 &= 9\,(u_3 + u_2) - u_1 \\
&= 9\,(477 + 49) - 5
\end{aligned}$$

So $u_n = 9\,(u_{n-1} + u_{n-2}) - u_{n-3}$ is satisfied for $n = 4$. Let us assume that this relation is satisfied for $n-1$, that is,

$$u_{n-1} = 9\,(u_{n-2} + u_{n-3}) - u_{n-4} \tag{8}$$

Then applying the previous assertion, (7) and (8), we conclude that $u_n = 9\,(u_{n-1} + u_{n-2}) - u_{n-3}$ for $n \geq 4$

(6). Note that $\frac{u_1}{v_1} = [2; 2] = 2 + \frac{1}{2} = \frac{5}{2}$ which is the fundamental solution. Let us assume that $(u_n, v_n)$ is a solution of $\tilde{D}$, that is, $u_n^2 - 6v_n^2 = 1$. Then by (6), we derive

$$\begin{aligned}
\frac{u_{n+1}}{v_{n+1}} &= 2 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{\cdots + 4 + \frac{1}{2}}}}}} \\
&= 2 + \cfrac{1}{2 + \cfrac{1}{2 + 2 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{\cdots + 4 + \frac{1}{2}}}}}} \\
\frac{u_{n+1}}{v_{n+1}} &= 2 + \cfrac{1}{2 + \cfrac{1}{2 + \frac{u_n}{v_n}}} \\
&= \frac{5u_n + 12v_n}{2u_n + 5v_n}
\end{aligned}$$

So $(u_{n+1}, v_{n+1})$ is also a solution of $\tilde{D}$ since $u_{n+1}^2 - 6v_{n+1}^2 = (5u_n + 12v_n)^2 - 6(2u_n + 5v_n)^2 = u_n^2 - 6v_n^2 = 1$. □

**Corollary 2.2.** *The base of the transformation $T$ in (2) is the fundamental solution of $\tilde{D}$, that is $T\left[h;k\right] = \{h,k\} = \{u_1, v_1\}$.*

*Proof.* We proved that $(u_1, v_1) = (5, 2)$ is the fundamental solution of $\tilde{D}$. Also we showed that $h = 5$ and $k = 2$. So the base of $T$ is $T\left[h,k\right] = \{5, 2\}$ as we claimed. We saw as above that the Diophantine equation $D$ could be transformed into the Diophantine equation $\tilde{D}$ via the transformation $T$. Also we showed that $x = u + 5$ and $y = v + 2$. So we can retransfer all results from $\tilde{D}$ to $D$ by using the inverse of $T$. Thus we can give the following main theorem. □

**Theorem 2.3.** *Let $D$ be the Diophantine equation in (1). Then*

*(1). The fundamental solution of $D$ is $(x_1, y_1) = (10, 4)$.*

*(2). Define the sequence $\{(x_n, y_n)\}_{n\geq 1} = \{(u_n + 5, v_n + 2)\}$, where $\{(u_n, v_n)\}$ defined in (5). Then $(x_n, y_n)$ is a solution of $D$. So it has infinitely many integer solutions $(x_n, y_n) \in \boldsymbol{Z} \times \boldsymbol{Z}$.*

*(3). The solutions $(x_n, y_n)$ satisfy*

$$x_n = 5x_{n-1} + 12y_{n-1} - 44$$
$$y_n = 2x_{n-1} + 5y_{n-1} - 18$$

*(4). The solutions $(x_n, y_n)$ satisfy the recurrence relations*

$$x_n = 9\left(x_{n-1} + x_{n-2}\right) - x_{n-3} - 80$$
$$y_n = 9\left(y_{n-1} + y_{n-2}\right) - y_{n-3} - 32$$

*for $n \geq 4$.*

*Proof.*

(1). It is easily seen that $(x_1, y_1) = (10, 4)$ is the fundamental solution of $D$ since $10^2 - 6(4)^2 - 10\,(10) + 24\,(4) = 0$.

(2). We prove it by Mathematical induction. Let $n = 1$. Then $(x_1, y_1) = (u_1 + 5, v_1 + 2) = (10, 4)$ which is the fundamental solution and so is a solution of $D$. Let us assume that the Diophantine equation in (1) is satisfied for $n - 1$, that is, $(u_{n-1} + 5)^2 - 6\left(v_{n-1}^2 + 2\right) - 10\,(u_{n-1} + 5) + 24\,(v_{n-1} + 2) = 0$. We want to show that this equation is also satisfied for $n$.

$$\begin{aligned} x^2 - 6y^2 - 10x + 24y &= (u_n + 5)^2 - 6(v_n + 2)^2 - 10\,(u_n + 5) + 24(v_n + 2) \\ &= u_n^2 - 6v_n^2 - 1 \\ &= 0 \quad (u_n \text{ and } v_n \text{ solutions of } \tilde{D}). \end{aligned}$$

So $(x_n, y_n) = (u_n + 5, v_n + 2)$ is also a solution $D$.

(3). From (7), $u_n = 5u_{n-1} + 12v_{n-1}$. Adding 5 on both sides, $u_n + 5 = 5u_{n-1} + 12v_{n-1} + 5$. We know that $x_{n-1} = u_{n-1} + 5$ and $y_{n-1} = v_{n-1} + 2$. Therefore, $u_{n-1} = x_{n-1} - 5$ and $v_{n-1} = y_{n-1} - 2$.

$$u_n + 5 = 5u_{n-1} + 12v_{n-1} + 5$$
$$x_n = 5\left(x_{n-1} - 5\right) + 12\left(y_{n-1} - 2\right) + 5$$

We get,

$$x_n = 5x_{n-1} + 12y_{n-1} - 44 \tag{9}$$

Similarly,

$$y_n = 2x_{n-1} + 5y_{n-1} - 18 \tag{10}$$

(4). We prove that $x_n$ satisfy the recurrence relation. For $n = 4$, we get $x_1 = 10$, $x_2 = 54$, $x_3 = 490$, $x_4 = 4806$. Hence

$$\begin{aligned} x_4 &= 9\left(x_3 + x_2\right) - x_1 - 80 \\ &= 9\left(490 + 54\right) - 10 - 80 \end{aligned}$$

So $x_n = 9\left(x_{n-1} + x_{n-2}\right) - x_{n-3} - 80$ is satisfied for $n = 4$. Let us assume that this relation is satisfied for $n - 1$, that is,

$$x_{n-1} = 9\left(x_{n-2} + x_{n-3}\right) - x_{n-4} - 80 \tag{11}$$

Then applying the previous assertion, (9) and (11), we conclude that $x_n = 9\left(x_{n-1} + x_{n-2}\right) - x_{n-3} - 80$ for $n \geq 4$. Now prove that $y_n$ satisfy the recurrence relation. For $n = 4$, we get $y_1 = 4$, $y_2 = 22$, $y_3 = 200$, $y_4 = 1962$. Hence

$$\begin{aligned} y_4 &= 9\left(y_3 + y_2\right) - y_1 - 32 \\ &= 9\left(200 + 22\right) - 4 - 32 \end{aligned}$$

So $y_n = 9\left(y_{n-1} + y_{n-2}\right) - y_{n-3} - 32$ is satisfied for $n = 4$. Let us assume that this relation is satisfied for $n - 1$, that is,

$$y_{n-1} = 9\left(y_{n-2} + y_{n-3}\right) - y_{n-4} - 32 \tag{12}$$

Then applying the previous assertion, (10) and (12), we conclude that $y_n = 9\left(y_{n-1} + y_{n-2}\right) - y_{n-3} - 32$, for $n \geq 4$. $\square$

## 3. Conclusion

Diophantine equations are rich in variety. There is no universal method for finding all possible solution (if it exists) for Diophantine equations. The method looks to be simple but it is very difficult for reaching the solutions.

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