Some Remarks on Fuzzy Hausdorf spaces

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Abstract: In this paper the concept of fuzzy Hausdorf space are studied. Several properties and examples of fuzzy Hausdorf space are given to illustrate the concept and relations between fuzzy hausdorf space and some fuzzy topological spaces are discussed.

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1. Introduction

The theory of fuzzy sets was initiated by L.A.Zadeh in his classical paper [14] in the year 1965 as an attempt to develop a mathematically precise framework in which to treat systems or phenomena which cannot themselves be characterized precisely. The potential of fuzzy notion was realized by the researchers and has successfully been applied for investigations in all the branches of Science and Technology. The paper of C.L.Chang [2] in 1968 paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. The concept of fuzzy Hausdorf spaces have been studied in Rekha Srivastava, S.N.Lal and Arun K. Srivastava in [4]. In this paper, we introduce the concept of Hausdorf spaces in fuzzy setting and investigate several characterizations of fuzzy Hausdorf spaces and the relations of fuzzy Hausdorf spaces and some fuzzy topological spaces are studied.

2. Preliminaries

Now we introduce some basic notions and results used in the sequel. In this work by $(X,T)$ or simply by $X$, we will denote a fuzzy topological space due to Chang [2].

Definition 2.1 ([2]). Let $\lambda$ and $\mu$ be any two fuzzy sets in a fuzzy topological space $(X,T)$. Then we define:

\[ \lambda \lor \mu : X \to [0,1] \text{ as follows: } (\lambda \lor \mu)(x) = \max \{ \lambda(x), \mu(x) \}; \]

\[ \lambda \land \mu : X \to [0,1] \text{ as follows: } (\lambda \land \mu)(x) = \min \{ \lambda(x), \mu(x) \}; \]

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• $\mu = X^c \Leftrightarrow \mu(x) = 1 - \lambda(x)$.

For a family $\{\lambda_i/i \in I\}$ of fuzzy sets in $(X,T)$, the union $\psi = \bigvee_i \lambda_i$ and intersection $\delta = \bigwedge_i \lambda_i$ are defined respectively as $\psi(x) = \sup_i \{\lambda_i(x), x \in X\}$ and $\delta(x) = \inf_i \{\lambda_i(x), x \in X\}$.

Definition 2.2 ([1]). Let $(X,T)$ be a fuzzy topological space. For a fuzzy set $\lambda$ of $X$, the interior and the closure of $\lambda$ are defined respectively as $\text{int}(\lambda) = \bigvee \{\mu/\mu \leq \lambda, \mu \in T\}$ and $\text{cl}(\lambda) = \bigwedge \{\mu/\lambda \leq \mu, 1 - \mu \in T\}$.

Definition 2.3 ([12]). A fuzzy set $\lambda$ in a fuzzy topological space $(X,T)$ is called fuzzy dense if there exists no fuzzy closed set $\mu$ in $(X,T)$ such that $\lambda < \mu < 1$. That is, $\text{cl}(\lambda) = 1$.

Definition 2.4 ([7]). A fuzzy set $\lambda$ in a fuzzy topological space $(X,T)$ is called fuzzy nowhere dense if there exists no non-zero fuzzy open set $\mu$ in $(X,T)$ such that $\mu < \text{cl}(\lambda)$. That is, $\text{int}(\lambda) = 0$.

Definition 2.5 ([8]). Let $(X,T)$ be a fuzzy topological space. A fuzzy set $\lambda$ in $(X,T)$ is called fuzzy first category set if $\lambda = \bigvee_{i=1}^{\infty} \lambda_i$, where $\lambda_i$'s are fuzzy nowhere dense sets in $(X,T)$. A fuzzy set which is not fuzzy first category set is called a fuzzy second category set in $(X,T)$.

Definition 2.6 ([8]). A fuzzy topological space $(X,T)$ is called fuzzy first category if $1 = \bigvee_{i=1}^{\infty} (\lambda_i)$ where $\lambda_i$'s are fuzzy nowhere dense sets in $(X,T)$. A topological space which is not of fuzzy first category, is said to be of fuzzy second category.

Definition 2.7 ([9]). Let $(X,T)$ be a fuzzy topological space. Then $(X,T)$ is called a fuzzy Baire space if $\text{int}(\bigvee_{i=1}^{\infty} (\lambda_i)) = 0$, where $\lambda_i$'s are fuzzy nowhere dense sets in $(X,T)$.

Definition 2.8 ([10]). A fuzzy topological space $(X,T)$ is said to be a fuzzy Quasi-maximal space if for every fuzzy dense set $\lambda$ in $(X,T)$ with $\text{int}(\lambda) \neq 0$ (the null set), $\text{int}(\lambda)$ is also fuzzy dense in $(X,T)$.

Definition 2.9 ([10]). A fuzzy topological space $(X,T)$ is called a fuzzy submaximal space if for each fuzzy set $\lambda$ in $(X,T)$ such that $\text{cl}(\lambda) = 1$, then $\lambda \in T$ in $(X,T)$.

Definition 2.10 ([11]). A fuzzy topological space $(X,T)$ is called a fuzzy nodec space if every non-zero fuzzy nowhere dense set $\lambda$ is fuzzy closed in $(X,T)$. That is, if $\lambda$ is a fuzzy nowhere dense set in $(X,T)$, then $1 - \lambda \in T$.

3. Fuzzy Hausdorff Space

Developed by the concept of fuzzy hausdorff space studied in [4] we shall now define:

Definition 3.1. A fuzzy topological space $(X,T)$ is said to be a fuzzy Hausdorff space if whenever $\lambda, \mu$ in $(X,T)$ and $\lambda \neq \mu$ we can find the fuzzy open sets $\gamma$ and $\delta$ such that $\lambda \leq \gamma, \mu \leq \delta$ and $\gamma \wedge \delta = 0$.

Example 3.2. Let $X = \{a,b,c\}$. The fuzzy sets $\lambda$ and $\mu$ are defined on $X$ as follows: $\lambda : X \to [0,1]$ defined as $\lambda(a) = 1; \lambda(b) = 0; \lambda(c) = 0$.
$\mu : X \to [0,1]$ defined as $\mu(a) = 0; \mu(b) = 1; \mu(c) = 1$. Then $T = \{0, \lambda, \mu, 1\}$ is a fuzzy topology on $X$. Now consider the following fuzzy sets defined on $X$ as follows:
\begin{align*}
\alpha : X \to [0,1] \text{ defined as } \alpha(a) &= 0.5; \alpha(b) = 0; \alpha(c) = 0.
\beta : X \to [0,1] \text{ defined as } \beta(a) = 0; \beta(b) = 0.4; \beta(c) = 0.3.
\delta : X \to [0,1] \text{ defined as } \delta(a) = 0; \delta(b) = 0; \delta(c) = 0.1.
\end{align*}
The fuzzy sets $\alpha$ and $\beta$ in $(X,T)$ and $\alpha \neq \beta$ then $\alpha \leq \lambda$ and $\beta \leq \mu$ implies that $\lambda \wedge \mu = 0$, where the fuzzy sets $\lambda$ and $\mu$ are fuzzy open sets in $(X,T)$. The fuzzy sets $\alpha$ and $\delta$ in $(X,T)$ and $\alpha \neq \delta$ then $\alpha \leq \lambda$ and $\delta \leq \mu$ implies that $\lambda \wedge \mu = 0$, where the fuzzy sets $\lambda$ and $\mu$ are fuzzy open sets in $(X,T)$. The fuzzy sets $\alpha$ and $1 - \lambda$ in $(X,T)$ and $\alpha \neq 1 - \lambda$ then $\alpha \leq \lambda$ and $1 - \lambda \leq \mu$ implies that $\lambda \wedge \mu = 0$, where the fuzzy sets $\lambda$ and $\mu$ are fuzzy open sets in $(X,T)$. The fuzzy sets $\alpha$ and $1 - \mu$ in $(X,T)$ and $\beta \neq 1 - \mu$ then $\beta \leq \mu$ and $1 - \mu \leq \lambda$ implies that $\lambda \wedge \mu = 0$, where the fuzzy sets $\lambda$ and $\mu$ are fuzzy open sets in $(X,T)$. The fuzzy sets $\delta$ and $1 - \mu$ in $(X,T)$ and $\delta \neq 1 - \mu$ then $\delta \leq \mu$ and $1 - \mu \leq \lambda$ implies that $\lambda \wedge \mu = 0$, where the fuzzy sets $\lambda$ and $\mu$ are fuzzy open sets in $(X,T)$. The fuzzy sets $\lambda$ and $\mu$ are fuzzy open sets in $(X,T)$. The fuzzy sets $\lambda$ and $\mu$ are fuzzy open sets in $(X,T)$. Therefore any two distinct fuzzy sets in $(X,T)$ which contains subsequence two disjoint fuzzy open sets in $(X,T)$. Hence $(X,T)$ is a fuzzy Hausdorff space.

**Example 3.3.** Let $X = \{a, b, c\}$. The fuzzy sets $\lambda, \mu, \gamma$ and $\delta$ are defined on $X$ as follows:

- $\lambda : X \rightarrow [0, 1]$ defined as $\lambda(a) = 0.5; \lambda(b) = 0$.
- $\mu : X \rightarrow [0, 1]$ defined as $\mu(a) = 0; \mu(b) = 0.4$.
- $\gamma : X \rightarrow [0, 1]$ defined as $\gamma(a) = 1; \gamma(b) = 0$.
- $\delta : X \rightarrow [0, 1]$ defined as $\delta(a) = 0; \delta(b) = 1$.

Then $T = \{0, \lambda, \mu, \gamma, \delta, (\lambda \vee \mu), (\lambda \wedge \delta), (\mu \vee \gamma), 1\}$ is a fuzzy topology on $X$. The fuzzy sets $1 - \lambda$ and $1 - \gamma$ in $(X,T)$ and $1 - \lambda \neq 1 - \gamma$ then $1 - \lambda \leq (\lambda \vee \delta)$ and $1 - \gamma \leq \delta$ implies that $(\lambda \vee \delta) \wedge \delta \neq 0$, where the fuzzy sets $\lambda \vee \delta$ and $\delta$ are fuzzy open sets in $(X,T)$. Hence $(X,T)$ is not a fuzzy Hausdorff space.

**Proposition 3.4.** The distinct fuzzy sets $\lambda \leq \alpha$ and $\mu \leq \beta$, where $\alpha, \beta \in T$ in a fuzzy Hausdorff space, then $\alpha \wedge \beta$ is fuzzy nowhere dense set in $(X,T)$.

**Proof.** The distinct fuzzy sets $\lambda \leq \alpha$ and $\mu \leq \beta$, where $\alpha, \beta \in T$ then by definition of fuzzy Hausdorff space $\alpha \wedge \beta = 0$ implies that $\text{int cl}(\alpha \wedge \beta) = 0$. Hence $\alpha \wedge \beta$ is fuzzy nowhere dense set in $(X,T)$.

**Theorem 3.5.** A fuzzy set $\lambda$ is fuzzy nowhere dense in a fuzzy topological space $(X,T)$ then $\text{int}(\lambda) = 0$ in $(X,T)$.

**Proposition 3.6.** The distinct fuzzy sets $\lambda \leq \alpha$ and $\mu \leq \beta$, where $\alpha, \beta \in T$ in a fuzzy Hausdorff space, then $\text{int}(\alpha \wedge \beta) = 0$ in $(X,T)$.

**Proof.** The distinct fuzzy sets $\lambda \leq \alpha$ and $\mu \leq \beta$, where $\alpha, \beta \in T$. By Proposition 3.4, $\alpha \wedge \beta$ is fuzzy nowhere dense set and by Theorem 3.5, $\alpha \wedge \beta$ is empty interior. That is $\text{int}(\alpha \wedge \beta) = 0$.

**Proposition 3.7.** The distinct fuzzy sets $\lambda \leq \alpha$ and $\mu \leq \beta$, where $\alpha, \beta \in T$ in a fuzzy Hausdorff space, then $\alpha \wedge \beta$ is not a fuzzy regular open set in $(X,T)$.

**Proof.** The distinct fuzzy sets $\lambda \leq \alpha$ and $\mu \leq \beta$, where $\alpha, \beta \in T$ in a fuzzy Hausdorff space, then by Proposition 3.4, $\text{intcl}(\alpha \wedge \beta) = 0$ implies that $\text{intcl}(\alpha \wedge \beta) = 0 \not= \lambda$. Hence $\alpha \wedge \beta$ is not a fuzzy regular open set in $(X,T)$.

**Theorem 3.8.** A fuzzy set $\lambda$ is fuzzy nowhere dense in a fuzzy topological space $(X,T)$ then $\lambda$ is fuzzy semi-closed in $(X,T)$.

**Lemma 3.9 ([1]).** In a fuzzy topological space $(X,T)$,

1. every fuzzy open set is fuzzy $\alpha$-open.
2. every fuzzy $\alpha$-open set is both fuzzy semiopen and fuzzy preopen.
3. every fuzzy semiopen set is fuzzy semi-preopen.
(4). every fuzzy preopen set is fuzzy semi-preopen.

**Proposition 3.10.** The distinct fuzzy sets \( \lambda \leq \alpha \) and \( \mu \leq \beta \), where \( \alpha, \beta \in T \) in a fuzzy Hausdorff space, then \( 1 - (\alpha \land \beta) \) in \((X, T)\) is

(1). is fuzzy semiopen.

(2). is fuzzy semi-preopen.

**Proof.** The distinct fuzzy sets \( \lambda \leq \alpha \) and \( \mu \leq \beta \), where \( \alpha, \beta \in T \) in a fuzzy Hausdorff space, by Proposition 3.10, \( (\alpha \land \beta) \) is fuzzy semi-closed. Then \( 1 - (\alpha \land \beta) \) is fuzzy open. Now by Lemma 3.9 (3), \( 1 - (\alpha \land \beta) \) is fuzzy semi-open, and by Lemma 3.9 (4), \( 1 - (\alpha \land \beta) \) fuzzy semi-preopen. \( \square \)

**Proposition 3.11.** The distinct fuzzy sets \( \lambda \leq \alpha \) and \( \mu \leq \beta \), where \( \alpha, \beta \in T \) in a fuzzy Hausdorff space, then \( \alpha \land \beta \) is not a fuzzy regular-closed set in \((X, T)\).

**Proof.** The distinct fuzzy sets \( \lambda \leq \alpha \) and \( \mu \leq \beta \), where \( \alpha, \beta \in T \) in a fuzzy Hausdorff space, then by Proposition 3.6, \( \text{int}(\alpha \land \beta) = 0 \) implies that \( \text{clint}(\alpha \land \beta) = 0 \neq \lambda \). Hense \( \alpha \land \beta \) is not a fuzzy regular-closed in \((X, T)\). \( \square \)

**Proposition 3.12.** The distinct fuzzy sets \( \lambda \leq \alpha \) and \( \mu \leq \beta \), where \( \alpha, \beta \in T \) in a fuzzy Hausdorff space, then \( \alpha \land \beta \) is fuzzy semi-closed set in \((X, T)\).

**Proof.** The distinct fuzzy sets \( \lambda \leq \alpha \) and \( \mu \leq \beta \), where \( \alpha, \beta \in T \) in a fuzzy Hausdorff space, then by Proposition 3.4, \( \text{int}(\alpha \land \beta) = 0 \) implies that \( \text{clint}(\alpha \land \beta) = 0 \leq \lambda \land \beta \). Hense \( \alpha \land \beta \) is fuzzy semi-closed in \((X, T)\). \( \square \)

**Proposition 3.13.** The distinct fuzzy sets \( \lambda \leq \alpha \) and \( \mu \leq \beta \), where \( \alpha, \beta \in T \) in a fuzzy Hausdorff space, then \( \alpha \land \beta \) is fuzzy \( \alpha \)-closed set in \((X, T)\).

**Proof.** The distinct fuzzy sets \( \lambda \leq \alpha \) and \( \mu \leq \beta \), where \( \alpha, \beta \in T \) in a fuzzy Hausdorff space, then by Proposition 3.6, \( \text{int}(\alpha \land \beta) = 0 \) implies that \( \text{clint}(\alpha \land \beta) = 0 \leq \lambda \land \beta \). Hense \( \alpha \land \beta \) is fuzzy \( \alpha \)-closed in \((X, T)\). \( \square \)

**Proposition 3.14.** The distinct fuzzy sets \( \lambda \leq \alpha \) and \( \mu \leq \beta \), where \( \alpha, \beta \in T \) in a fuzzy Hausdorff space, then \( \alpha \land \beta \) is fuzzy \( \alpha \)-pre-open set in \((X, T)\).

**Proof.** The distinct fuzzy sets \( \lambda \leq \alpha \) and \( \mu \leq \beta \), where \( \alpha, \beta \in T \) in a fuzzy Hausdorff space, then by Proposition 3.6, \( \text{int}(\alpha \land \beta) = 0 \) implies that \( \text{clint}(\alpha \land \beta) = 0 \leq \lambda \). Hense \( \alpha \land \beta \) is fuzzy \( \alpha \)-pre-open in \((X, T)\). \( \square \)

**Proposition 3.15.** The distinct fuzzy sets \( \lambda \leq \alpha \) and \( \mu \leq \beta \), where \( \alpha, \beta \in T \) in a fuzzy Hausdorff space, then \( \alpha \land \beta \) is fuzzy semi-pre-closed set in \((X, T)\).

**Proof.** The distinct fuzzy sets \( \lambda \leq \alpha \) and \( \mu \leq \beta \), where \( \alpha, \beta \in T \) in a fuzzy Hausdorff space, then by Proposition 3.6, \( \text{int}(\alpha \land \beta) = 0 \) implies that \( \text{clint}(\alpha \land \beta) = \text{int}(0) = 0 \leq \lambda \). Hense \( \alpha \land \beta \) is fuzzy semi-pre-closed in \((X, T)\). \( \square \)

**Lemma 3.16 ([1]).** In a fuzzy topological space \((X, T)\),

(1). every fuzzy closed set is fuzzy \( Fg \)-closed.

(2). every fuzzy semi-closed set is fuzzy \( Fsg \)-closed.

(3). every fuzzy \( \alpha \)-closed set is fuzzy \( F\alpha \)-closed.

(4). every fuzzy pre-closed set is fuzzy \( Fpg \)-closed.
Proposition 3.17. The distinct fuzzy sets $\lambda \leq \alpha$ and $\mu \leq \beta$, where $\alpha, \beta \in T$ in a fuzzy Hausdorff space, then $(\alpha \wedge \beta)$ is
(1). fuzzy semi-closed in $(X, T)$.
(2). $F_{sg}$-closed.
(3). $F_{ag}$-closed.
(4). $F_{pg}$-closed.
(5). $F_{spg}$-closed.

Proof. The distinct fuzzy sets $\lambda \leq \alpha$ and $\mu \leq \beta$, where $\alpha, \beta \in T$ in a fuzzy Hausdorff space, then
(1). by Proposition 3.12, $(\alpha \wedge \beta)$ is fuzzy semi-closed in $(X, T)$.
(2). Now $(\alpha \wedge \beta)$ is fuzzy semi-closed in $(X, T)$ by Lemma 3.16 (2), $(\alpha \wedge \beta)$ is $F_{sg}$-closed.
(3). by Proposition 3.13, $(\alpha \wedge \beta)$ is fuzzy $\alpha$-closed and by Lemma 3.16 (3), $(\alpha \wedge \beta)$ $F_{ag}$-closed.
(4). by Proposition 3.14, $(\alpha \wedge \beta)$ is fuzzy pre-closed and by Lemma 3.16 (4), $(\alpha \wedge \beta)$ $F_{pg}$-closed.
(5). by Proposition 3.15, $(\alpha \wedge \beta)$ is fuzzy semi-pre-closed and by Lemma 3.16 (5), $(\alpha \wedge \beta)$ $F_{spg}$-closed.

Proposition 3.18. The distinct fuzzy sets $\lambda \leq \alpha$ and $\mu \leq \beta$, where $\alpha, \beta \in T$ in a fuzzy Hausdorff space, then $\text{cl int}[1 - (\alpha \wedge \beta)] = 1$.

Proof. The distinct fuzzy sets $\lambda \leq \alpha$ and $\mu \leq \beta$, where $\alpha, \beta \in T$ in a fuzzy Hausdorff space. Then $\alpha \wedge \beta = 0$ implies that $\text{cl}(\alpha \wedge \beta) = 0$ implies that $1 - \text{cl}(\alpha \wedge \beta) = 1$ implies that $\text{int}(1 - \alpha \wedge \beta) = 1$ therefore $\text{cl int}(1 - \alpha \wedge \beta) = 1$. 

4. Fuzzy Hausdorff Space and Some Fuzzy Topological Spaces

Proposition 4.1. A fuzzy Hausdorff space, then the space is a fuzzy Quasi-maximal space.

Proof. Let $(X, T)$ be a fuzzy Hausdorff space, The distinct fuzzy sets $\lambda \leq \alpha$ and $\mu \leq \beta$ in $(X, T)$, and $\alpha, \beta \in T$, By Proposition 3.6, $\text{int}(\alpha \wedge \beta) = 0$. Now $\text{cl}(1 - \alpha \wedge \beta) = 1$ and $\text{int}(1 - \alpha \wedge \beta) \neq 0$, then $\text{cl int}(1 - \alpha \wedge \beta)$ implies that $\text{cl int}[1 - (\alpha \wedge \beta)] = \text{cl int}[1 - 0] = \text{cl int}(1) = 1$. Therefore the fuzzy dense set $1 - \alpha \wedge \beta$ empty interior and interior of $1 - \alpha \wedge \beta$ is fuzzy dense in $(X, T)$. Hence $(X, T)$ is fuzzy Quasi-maximal space.

Proposition 4.2. A fuzzy Hausdorff space need not be fuzzy Quasi-regular space.

Proof. In Example 3.2, $(X, T)$ is fuzzy Hausdorff space and the fuzzy open set $\lambda$ and $\mu$ is non-empty set in $(X, T)$, then $\text{cl}(\lambda) \nsubseteq \mu$ and $\text{cl}(\mu) \nsubseteq \lambda$. Hence $(X, T)$ is not of fuzzy Quasi-regular space.

Proposition 4.3. A fuzzy submaximal space need not be fuzzy Hausdorff space. Consider the following example.

Example 4.4. Let $X = \{a, b, c\}$. The fuzzy sets $\lambda, \mu$ and $\gamma$ are defined on $X$ as follows:
$\lambda : X \rightarrow [0, 1]$ defined as $\lambda(a) = 0.9; \lambda(b) = 0.8; \lambda(c) = 0.8$
$\mu : X \rightarrow [0, 1]$ defined as $\mu(a) = 0.9; \mu(b) = 0.8; \mu(c) = 0.7$
$\gamma : X \rightarrow [0, 1]$ defined as $\gamma(a) = 0.8; \gamma(b) = 0.7; \gamma(c) = 0.7$
Then $T = \{0, \lambda, \mu, \gamma, 1\}$ is a fuzzy topology on $X$. The fuzzy dense sets $\lambda, \mu$ and $\gamma$ in $(X, T)$ are fuzzy open. Hence $(X, T)$ is fuzzy submaximal space but not a fuzzy Hausdorff space. Since $\mu \leq \lambda, \gamma \leq \mu$ and $\lambda \neq \mu$ but $\lambda \wedge \mu \neq 0$ same as $\gamma \leq \lambda, 1 - \lambda \leq \gamma$ and $\lambda \neq \gamma$ but $\lambda \wedge \gamma \neq 0$.

**Proposition 4.5.** A fuzzy Hyperconnected space need not be fuzzy Hausdorff space.

**Proof.** In Example 3.3, the fuzzy open sets in $\lambda, \mu, \gamma$ in $(X, T)$ are fuzzy dense in $(X, T)$, hence $(X, T)$ is fuzzy Hyperconnected space but not of fuzzy Hausdorff space (by Example 3.3).

References