International Journal of Mathematics And its Applications
Volume 5, Issue 4–F (2017), 963–970.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Autometrized Implicative Almost Distributive Lattices

Research Article

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Abstract: In this paper,we introduce two binary operations + and − on an IADL and we obtain few results concerning the operation defined.We introduce a binary operation * on IADL L and prove that (L, *) is a metric space. We introduce the concept of Autometrized Implicative Almost Distributive Lattices(AIADLs)as extension of Autometrized algebra in the class of Almost Distributive Lattices and also regular autometrized IADL. We discuss some theorems of AIADL L and show that * on L is not a group operation.

Keywords: Almost Distributive Lattice, Implicative Algebras, Implicative Almost Distributive lattice, Autometrized Algebra, Regular Autometrized Algebra, Autometrized Implicative Almost Distributive and Regular Autometrized Implicative Almost Distributive Lattice.

1. Introduction

To establish an alternative logic for knowledge representation and reasoning, Xu [8] proposed a logical algebra-lattice implication algebra in 1993 by combining algebraic lattice and implication algebra. In lattice implication algebra, the lattice is defined to describe uncertainties, especially for the in comparability and the implication is designed to describe the ways of human's reasoning. Venkateswarlu Kulluru and Berhanu Bekele [6] introduced binary operation + and − on an implicative algebra and obtained certain properties with these operations. Further they proved that any implicative algebra is a metric space. Also they proved that every implicative algebra can be made into a regular Autometrized Algebra of Swamy (1964). The concept of an Almost Distributive Lattice(ADL) was introduced in 1981 by U.M.Swamy and G.C.Rao [5] as a common abstraction to most of the existing ring theoretic and lattice theoretic generalization of Boolean algebra. Before this paper, we introduced the concept of an Implicative Almost Distributive Lattice (IADL) [1] as a generalization of an implicative algebra in the class of ADLs. We proved some properties and equivalence condition in an implicative almost distributive lattice and characterization of Implicative Almost Distributive lattice. In this paper, we introduce two binary operations + and − on an IADL and we obtain few results concerning the operation defined. We introduce a binary operation $*$ on an IADL L and proved that $(L, *)$ is a metric space. We introduce the concept of an Autometrized Implicative Almost Distributive Lattices(AIADLs)as extension of Autometrized algebra in the class of Almost Distributive Lattices and also regular autometrized IADL. We discuss some theorems of AIADL L and show that $*$ on L is not a group operation. In the following, we give some important definitions and results that will be useful in the subsequent section.

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2. Premilinaries

Definition 2.1 ([5]). *An algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (ADL) with 0 if it satisfies the following axioms:*

(1). $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$.

(2). $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

(3). $(x \vee y) \wedge y = y$.

(4). $(x \vee y) \wedge x = x$.

(5). $x \vee (x \wedge y) = x$.

(6). $0 \wedge x = 0$, *for all* $x, y, z \in L$.

If $(L, \vee, \wedge, 0)$ is an ADL, for any $x, y \in L$, define $x \leq y$ if and only if $x = x \wedge y$ or equivalently $x \vee y = y$, then $\leq$ is a partial ordering on L.

Theorem 2.2 ([5]). *Let L be an ADL. For any $x, y, z \in L$, we have the following.*

(1). $x \vee y = x \Leftrightarrow x \wedge y = y$.

(2). $x \vee y = y \Leftrightarrow x \wedge y = x$.

(3). $x \wedge y = y \wedge x = x$ *whenever* $x \leqslant y$.

(4). $\wedge$ *is associative.*

(5). $x \wedge y \wedge z = y \wedge x \wedge z$.

(6). $(x \vee y) \wedge z = (y \vee x) \wedge z$.

(7). $x \wedge x = x = x \vee x$.

(8). $x \wedge 0 = 0$ *and* $0 \vee x = x = x \vee 0$.

(9). If $x \leqslant z$ *and* $y \leqslant z$, *then* $x \wedge y = y \wedge x$ *and* $x \vee y = y \vee x$.

Definition 2.3 ([5]). *An element m in an ADL L is called maximal if for any $x \in L, m \leq x$ implies $m = x$.*

Definition 2.4 ([5]). *An algebra $(L, \rightarrow, \prime, 0, 1)$ of type $(2, 1, 0, 0)$ is called implicative algebra if it satisfies the following conditions:*

(1). $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

(2). $1 \rightarrow x = x$.

(3). $x \rightarrow 1 = 1$.

(4). $x \rightarrow y = y' \rightarrow x'$.

(5). $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$.

(6). $0' = 1$, *for* $x, y, z \in L$.

Definition 2.5 ([6]). *A relation $\leq$ on an implicative algebra L is defined as follows: $x \leq y \Leftrightarrow x \to y = 1$ for all $x, y \in L$.*

Definition 2.6 ([1]). *Let $(L, \vee, \wedge, 0, m)$ be an ADL with 0 and maximal element m. Then an algebra $(L, \vee, \wedge, \to, ', 0, m)$ of type $(2, 2, 2, 1, 0, 0)$ is called Implicative Almost Distributive Lattice (IADL) if it satisfies the following conditions:*

(1). $x \vee y = (x \to y) \to y$.

(2). $x \wedge y = [(x \to y) \to x']'$.

(3). $x \to (y \to z) = y \to (x \to z)$.

(4). $m \to x = x$.

(5). $x \to m = m$.

(6). $x \to y = y' \to x'$.

(7). $0' = m$, *for all* $x, y, z \in L$.

Definition 2.7.

(1). A semigroup M is a nonempty set equipped with a binary operation, which is required (only) to be associative

(2). A monoid is a semigroup with an identity element.

Define the relation $\leq$ in an IADL by $x \leq y \Leftrightarrow x \to y = m$.

Theorem 2.8 ([1]). *In an IADL $(L, \vee, \wedge, \to, ', 0, m)$, the following conditions hold:*

(1). $[(x \to y) \to y] \wedge m = [(y \to x) \to x] \wedge m$.

(2). $[((x \to y) \to x')'] \wedge m = [((y \to x) \to y')'] \wedge m$.

(3). $x \to x = m$.

(4). $m' = 0$.

(5). $(x')' = x$ *(is called involution).*

(6). $x' = x \to 0$.

(7). $0 \to x = m$.

(8). $x \to y = m = y \to x$ *implies* $x = y$.

(9). If $x \to y = m$ *and* $y \to z = m$, *then* $x \to z = m$

(10). $x \leq y$ *if and only if* $z \to x \leq z \to y$ *and* $y \to z \leq x \to z$.

(11). $(x \vee y)' = x' \wedge y'$.

(12). $(x \wedge y)' = x' \vee y'$, *for all* $x, y, z \in L$.

(13). $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.

(14). $(x \wedge y) \to z = (x \to z) \vee (y \to z)$.

(15). $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

(16). $x \to (y \vee z) = (x \to y) \vee (x \to z)$.

Definition 2.9 ([6]). *A system* $\boldsymbol{A} = (A, +, \leq, *)$ *is called an Autometrized Algebra if and only if*

(1). $(A, +, 0)$ *is a commutative monoid.*

(2). $(A, \leq)$ *is an ordered set and for all* $x, y, z \in L$*:* $x \leq y$ *implies* $x + z \leq y + z$,

(3). $* : A \times A \to A$ *is a mapping such that for all* $x, y, z \in A$

(a). $x * y \geq 0$ *and* $x * y = 0$ *if and only if* $x = y$.

(b). $x * y = y * x$.

(c). $x * z \leq x * y + y * z$.

Definition 2.10 ([6]). *An Autometrized Algebra* A *is said to be regular if* $x * 0 = x$ *for all* $x \in A$.

Theorem 2.11 ([5]). *Let* L *be an implicative algebra. Then* $x * 0 = x$ *for all* $x \in L$.

Theorem 2.12 ([6]). *Any implicative algebra* L *is a regular Autometrized Algebra.*

3. Autometrized Implicative Almost Distributive Lattices(AIADLs)

In this section first we introduce two binary operation on an Implicative Almost Distributive Lattice(IADL) namely $+$ and $-$ and obtain few results concerning the operation defined. Also we obtain some geometric properties of IADL that used in the later discussion. We begin with the following. Let L be an IADL. Define $+$ and $-$ on L as follows. For $x, y \in L$, $x + y = x' \to y$ and $x - y = (x \to y)'$. Then we have the following in IADL.

Lemma 3.1. *Let L be an IADL. Then (L, +, 0) is a commutative monoid.*

Proof. Let L be an IADL and $x, y, z \in L$.

(1).
$$\begin{aligned}(x + y) + z &= (x' \to y)' \to z \\ &= z' \to (x' \to y) \\ &= x' \to (z' \to y) \\ &= x' \to (y' \to z) \\ &= x + (y + z).\end{aligned}$$

Therefore $+$ is associative on L.

(2). $x + 0 = x' \to 0 = 0' \to (x')' = m \to x = x$ and $0 + x = x$. Therefore 0 is identity element for L.

(3). $x + y = x' \to y = y' \to x = y + x$. Therefore $+$ is commutative on L. Hence (L,+,0) is a commutative monoid. □

The following lemma is an easy consequence of definition of $+$ and $-$.

Lemma 3.2. *Let L be an IADL. Then the following conditions hold:*

(1). $x - x = 0$.

(2). $x - 0 = x$.

(3). $(x-y)\vee 0 = x-y$.

(4). $x-y\leq 0 \Leftrightarrow x\leq y$.

(5). $(x\vee y)-z=(x-z)\vee(y-z)$.

(6). $x\vee y=(x-y)+y$.

(7). $x\wedge y = x-(x-y)$.

(8). $x\leq y+z\Leftrightarrow x-y\leq z$.

(9). $0-x=0$.

(10). $x-(x+y)=(x-y)-x$.

(11). $(x-y)+(y-z)\geq x-z$.

(12). $x=(x\vee 0)+(x\wedge 0)$.

(13). $x-(y+z)=(x-z)-y=(x-y)-z$.

(14). $x\geq y\Rightarrow (x-y)+y=x$.

(15). $(a-(x\wedge y))+b=((a-x)+b)\vee((a-y)+b)$, *for* $x,y,z,a,b\in L$.

Proof. Let L be an IADL and $x,y,z,a,b\in L$. By using definition of $+$ and $-$ above and some properties of IADL L, we prove the theorem as follows:

(1). $x-x=(x\to x)'=m'=0$.

(2). $x-0=(x\to 0)'=x$.

(3). $(x-y)\vee 0=((x-y)\to 0)\to 0=((x-y)')'=x-y$.

(4). Assume $x-y\leq 0$. Then $(x\to y)'\leq 0\Rightarrow 0'\leq x\to y$. Since m is maximal element of L, we have $m\leq x\to y$ implies $x\to y=m$. Thus $x\leq y$. Conversely, assume $x\leq y$. Now $(x-y)\to 0=(x\to y)'\to 0=m'\to 0=0\to 0=m$. Therefore $x-y\leq 0$.

(5). $(x\vee y)-z=((x\vee y)\to z)'=((x\to z)\wedge(y\to z))'=(x\to z)'\vee(y\to z)'=(x-z)\vee(y-z)$.

(6). $(x-y)+y=((x\to y)')'\to y=(x\to y)\to y=x\vee y$.

(7). $x-(x-y)=[(x\to(x\to y)']'=x\wedge y$.

(8). $x\leq y+z\Rightarrow x\leq y'\to z\Rightarrow x\to(y'\to z)=m\Rightarrow z'\to(x\to y)=m\Rightarrow (x\to y)'\to z=m\Rightarrow x-y\leq z$. Conversely, $x-y\leq z\Rightarrow (x-y)\to z=m\Rightarrow (x\to y)'\to z=m\Rightarrow z'\to(x\to y)=m\Rightarrow x\to(y'\to z)=m$. Therefore $x\leq y+z$.

(9). $0-x=(0\to x)'=m'=0$.

(10). $x-(x+y)=[x\to(x'\to y)]'=[x'\to(x\to y)]'=[(x\to y)'\to x]'=(x-y)-x$

(11). $(x\to z)'\to[(x\to y)\to(y\to z)']=(x\to y)\to[(x\to z)'\to(y\to z)']=(x\to y)\to[(y\to z)\to(x\to z)]=(z\to y)\to[(x\to y)\to(x\to y)]=m$.Therefore $x-z\leq(x-y)+(y-z)$.

(12). $(x \vee 0) + (x \wedge 0) = x + 0 = x$.

(13). $(x - z) - y = [(x \to z)' \to y]' = [x \to (y' \to z)]' = x - (y + z)$.

(14). Assume $y \leq x$. $(x - y) + y = (x \to y) \to y = x \vee y = x$.

(15). $[a - (x \wedge y)] + b = (a \to (x \wedge y))' + b = ((a \to x) \wedge (a \to y)) \to b = ((a \to x) \to b) \vee ((a \to y) \to b) = ((a - x) + b) \vee ((a - y) + b)$. □

Now we are in a position to introduce the concept of a metric space on an Implicative Almost Distributive Lattice.

Definition 3.3. *Let L be an IADL. Define a map $* : L \times L \to L$ by $x * y = (x - y) + (y - x)$ for all $x, y \in L$.*

Then we have the following.

Lemma 3.4. *Let L be an IADL and $x, y \in L$. Then $x * y = (x \to y) \to (y \to x)' = 0$ implies $x = y$.*

Proof. Let $x, y \in L$. Assume $(x \to y) \to (y \to x)' = 0$. Using conditions in the definition of IADL and our assumption we have $x \to y = m \to (x \to y) = (x \to y)' \to m' = (x \to y)' \to 0 = (x \to y)' \to ((x \to y) \to (y \to x)') = (y \to x) \to ((x \to y)' \to (x \to y)') = m$. Therefore $x \to y = m$. Thus $x \leq y$. Similarly $y \to x = m \to (y \to x) = (y \to x)' \to m' = (y \to x)' \to 0 = (y \to x)' \to ((x \to y) \to (y \to x)' = (x \to y) \to ((y \to x)' \to (y \to x)') = (x \to y) \to m = m$. Therefore $y \to x = m$. Thus $y \leq x$. Hence x = y. □

Theorem 3.5. *Let L be an IADL. Then $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in L$.*

Proof. Let $x, y, z \in L$. By using definition of $+$ and IADL we have, $(x + z) \to (y + z) = (x' \to z) \to (y' \to z) = y' \to ((x' \to z) \to z) \wedge m) = y' \to (z \to x') \to x') \wedge m) = (z \to x') \to ((y' \to x') = m$. Therefore $(x + z) \to (y + z) = m$. Hence $x + z \leq y + z$. □

Corollary 3.6. *Let L be an IADL. If $x \leq y$ and $z \leq w$ then $x + z \leq y + w$.*

Proof. Let $x, y, z, w \in L$. Assume $x \leq y$ and $z \leq w$. By theorem 3.5, $x + z \leq y + z$ and $y + z \leq y + w$. This implies $x + z \leq y + w$. □

Now we have the following.

Theorem 3.7. *Let L be an IADL. Then $(L, *)$ is a mertic space.*

Proof. Let L be an IADL and $x, y, z \in L$. We need to show

(1). $x * y \geq 0$ for all $x, y \in L$.

(2). $x * y = 0 \Leftrightarrow x = y$.

(3). $x * y = y * x$.

(4). $x * z \leq (x * y) + (y * z)$.

Then

(1). Clearly $(x \to y)' \geq 0$.Thus $x * y = (x \to y) \to (y \to x)' \geq 0$.

(2). Assume $x * y = 0$. Then by Lemma 3.4, we have x = y. Conversely assume x = y. Then $x * y = x * x = (x \to x) \to (x \to x)' = 0$.

(3). Since $+$ is commutative in L,we have $x * y = y * x$.

(4). $(x * y) + (y * z) = (x - y) + (y - x) + (y - z) + (z - y) = (x - y) + (y - z) + (z - y) + (y - x) = (x - z) + (z - x)$ (by Lemma 3.2)$\geq x * z$. Thus $*$ is a metric on L. Hence $(L, *)$ is a metric space. □

Now we have the following notion.

Definition 3.8. *An algebra $(L, +, *, \vee, \wedge, \rightarrow, ', 0, m)$ of type $(2, 2, 2, 2, 2, 1, 0, 0)$ is an Autometrized Implicative Almost Distributive Lattice(AIADL) if it satisfies the following conditions:*

(1). $(L, \vee, \wedge, \rightarrow, ', 0, m)$ is an Implicative Almost Distributive Lattice.

(2). $(L, +, 0)$ is a commutative monoid.

(3). $(L, *)$ is a metric space. That is $* : L \times L \rightarrow L$ is a mapping that satisfies the following: for $x, y, z \in L$.

(a). $*(x, y) \geq 0$ and $*(x, y) = 0 \Leftrightarrow x = y$.

(b). $*(x, y) = *(y, x)$.

(c). $*(x, z) \leq *(x, y) + *(y, z)$, thus $*$ is a metric on L and $(L, *)$ is called a metric space.

Definition 3.9. *An Autometrized IADL L is said to be Regular Autometrized IADL if $x * 0 = x$, for all $x \in L$.*

Theorem 3.10. *Let L be an IADL. Then $x * 0 = x$ for all $x \in L$.*

Proof. Let $x \in L$. $x * 0 = (x - 0) + (0 - x) = (x \rightarrow 0) \rightarrow (0 \rightarrow x)' = x' \rightarrow 0 = x$. Therefore $x * 0 = x$, $\forall x \in L$. □

Theorem 3.11. *Any IADL L is a Regular Autometrized IADL.*

Proof. Assume L is an IADL and $x, y, z \in L$. We need to prove $(L, +, *, \vee, \wedge, \rightarrow, ', 0, m)$ is a Regular Autometrized IADL. Then

(1). By assumption, L is an IADL.

(2). By Lemma 3.1, $(L, +, 0)$ is a commutative moniod

(3). By Theorem 3.7, $(L, *)$ is a metric space and

(4). By Theorem 3.10, $x * 0 = x$ for all $x \in L$. Therefore L is Regular Autometrized IADL. □

We end this section by looking at the following.

Example 3.12. *Let $L = \{0, x, y, z, m\}$ be a chain defined by $0 < x < y < z < m$. Define $', \rightarrow$ and $*$ in the tables below. $x \vee y = max\ \{x, y\}$, $x \wedge y = min\{x, y\}$. Then*

(1). $(L, \vee, \wedge, \rightarrow, ', 0, m)$ is an IADL.

(2). since $x + 0 = x$, $x + y = y + x$ and $(x + y) + z = x + (y + z)$, $(L, +, 0)$ is a commutative monoid.

(3). Clearly $x * y = y * x$ for any $x, y \in L$.

(4). $x * x = 0$ and $x * z \leq (x * y) + (y * z)$ for any $x, y, z \in L$. Therefore $(L, +, *, \vee, \wedge, \rightarrow, ', 0, m)$ is AIADL.

a	a'
0	m
x	z
y	y
z	x
m	0

$\to$	0	x	y	z	m
0	m	m	m	m	m
x	z	m	m	m	m
y	y	z	m	m	m
z	x	y	z	m	m
m	0	x	y	z	c

$*$	0	x	y	z	m
0	0	x	y	z	m
x	x	0	x	y	z
y	y	x	0	x	y
z	z	y	x	0	x
m	m	c	y	x	0

Remark 3.13. *$*$ is not a group operation on an AIADL L. Since $x * x = 0$ for all $x \in L$, every element is the inverse of itself and $x * 0 = 0 * x = x$ for all $x \in L$ implies 0 is the identity element of L but $(x * y) * z = y \neq 0 = x * (y * z)$. This implies * is not associative on L. Hence * is not group operation.*

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