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Equitable Edge Coloring of Some Join Graphs

Research Article

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Abstract: The notion of equitable coloring was introduced by Meyer in 1973. Let $G\,(V,E)$ be a graph. For $k-$proper edge coloring f of graph G, if $||E_i|-|E_j||\leq 1$, $i,j=0,1,2,\cdots k-1$, where $E_i\,(G)$ is the set of edges of color i in G, then f is called a $k-$equitable edge coloring of graph G, and $\chi'_e\,(G)=\min\{k|$ there is a k equitable edge-coloring of graph G$\}$ is called the equitable edge chromatic number of G. In this paper, we obtain the equitable edge chromatic number of the join graph of $P_l\vee K_{m,n}$ and $P_m\vee K_{1,n,n}$.

MSC: Primary 05C15; Secondary 05C76.

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1. Introduction

Graph coloring is an important research problem [2, 6, 11]. In this paper we only consider simple graphs. We will use the standard notation of graph theory and definitions not given here may be found in [8]. The proper edge coloring that uses colors from a set of k colors is a $k-$edge coloring. Thus a $k-$coloring of a graph G can be described as a function $c:E\,(G)\rightarrow\{1,2,\cdots,k\}$ such that $c\,(e)\neq e\,(f)$ for every two adjacent edges e and f in G. A graph G is $k-$edge colorable if there exists a $k-$edge coloring of G. We are often interested in edge coloring of graphs using a minimum number of colors. The chromatic number (or chromatic index) $\chi'\,(G)$ of a graph G is the minimum positive integer k for which G is $k-$edge colorable. Since every edge coloring of a graph G must assign distinct colors to adjacent edges, for which vertex v of G it follows that $\deg v$ colors must be used to color the edges incident with v in G. Therefore,

$$\chi'\,(G)\geq\Delta\,(G)$$

for every nonempty graph G. While $\Delta\,(G)$ is a rather obvious lower bound for the chromatic index of a nonempty graph G, the Russian graph theorist Vadim G. Vizing [9] established a remarkable upper bound for the chromatic index of a graph. Vizing's theorem, published in 1964, must be considered the major theorem in the area of edge colorings. Vizing's theorem was rediscovered in 1966 by Ram Prakash Gupta[4].

Definition 1.1. *For $k-$proper edge coloring f of graph G, if $||E_i|-|E_j||\leq 1$, $i,j=0,1,2,\cdots k-1$, where $E_i\,(G)$ is the set of edges of color i in G, then f is called a $k-$equitable edge coloring of graph G, and*

$$\chi'_e\,(G)=\min\{k|\ \text{there is a } k\text{-equitable edge-coloring of graph } G\}$$

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is called the equitable edge chromatic number of G.

Definition 1.2 ([1]). *The join graph $G \vee H$ of disjoint graphs G and H is defined as follows:*

$$V(G \vee H) = V(G) \cup V(H)$$
$$E(G \vee H) = E(G) \cup E(H) \cup \{uv | u \in V(G), v \in V(H)\}$$

Lemma 1.3 ([1]). *For any simple graph $G(V,E)$; $\chi'_e \geq \Delta(G)$. For any simple graph G and H, $\chi'_e(G) = \chi'(G)$ [7], and if $H \subseteq G$, then $\chi'(H) \leq \chi'(G)$ [1, 12], where $\chi'_e(G)$ is the proper edge chromatc number of G. So Lemma 1.4 and Lemma 1.5 are obtained*

Lemma 1.4. *For any simple graph G and H, if H is a subgraph of G, then $\chi'_e(H) \leq \chi'_e(G)$.*

Lemma 1.5. *For any complete graph K_p with order p,*

$$\chi'_e(K_p) = \begin{cases} p, & p \equiv 1 \,(mod\ 2), \\ p-1, & p \equiv 0 \,(mod\ 2), \end{cases}$$

Lemma 1.6 ([1, 10]). *Let G be a simple graph, if $G[V_\Delta]$ does not contain cycle, then $\chi'_e(G) = \Delta(G)$. Where $V(G[V_\Delta]) = V_\Delta = \{v | d(v) = \Delta(G), v \in V(G)\}$, $E(G[V_\Delta]) = \{uv | u, v \in V_\Delta, uv \in E(G)\}$,*

Lemma 1.7 ([5]). *For a finite simple graph G, $\chi'_e(G) = \chi'(G)$.*

2. Main Results

Theorem 2.1. *For any positive integer l, m and n, then*

$$\chi'_e(P_l \vee K_{m,n}) = \Delta(P_l \vee K_{m,n}) = \begin{cases} 3 & if\ \ l = m = n = 1 \\ m+n & if\ \ l = m = 1, n > 1 \\ m+n & if\ \ l = 1, m > 1, n > 1, n > m \\ m+n+2 & if\ l, m, n > 1, n > m, m+n > l \\ l+n & if\ \ l, m, n > 1, n > m, l \geq m+n \end{cases}$$

Proof. Let $V(P_l) = \{w_k | k = 1, 2, \cdots, l\}$ and $V(K_{m,n}) = \{u_i | i = 1, 2, \cdots, m\} \cup \{v_j | j = 1, 2, \cdots, n\}$. Let $E(P_l) = \{w_k w_{k+1} | k = 1, 2, \cdots, l-1\}$ and $E(K_{m,n}) = \bigcup_{i=1}^{m} \{u_i v_j | j = 1, 2, \cdots, n\}$. By the definition of join graph,

$$V(P_l \vee K_{m,n}) = V(P_l) \cup V(K_{m,n}) \quad \text{and}$$
$$E(P_l \vee K_{m,n}) = E(P_l) \cup E(K_{m,n}) \cup \bigcup_{k=1}^{l} \{w_k u_i : 1 \leq i \leq m\} \cup \bigcup_{k=1}^{l} \{w_k v_j : 1 \leq j \leq n\}$$

Let f be a mapping from $E(P_l \vee K_{m,n})$ as follows:

Case 1: If $l = m = n = 1$, $f(w_1 u_1) = 1$; $f(w_1 v_1) = 2$; $f(u_1 v_1) = 3$. Obviously, the f is 3-EEC of $\chi'_e(P_l \vee K_{m,n})$, for $l = m = n = 1$.

Case 2: If $l = m = 1$, $n > 1$, $f(w_1 u_1) = 1$; $f(w_1 v_j) = j+1$, $1 \leq j \leq n$; $f(u_1 v_j) = j+2$, $1 \leq j \leq n-1$; $f(u_1 v_n) = 2$.

Case 3: If $l = 1$, $m > 1$, $n > 1$, $f(u_{2i-1}v_j) = j + i - 1$, $1 \le j \le n$, $1 \le i \le \lceil \frac{m}{2} \rceil$. For $1 \le j \le n$, $1 \le i \le \lfloor \frac{m}{2} \rfloor$

$$f(u_{2i}v_j) = \begin{cases} n + j + i - 1\,(mod\ m+n) & \text{if } n + j + i - 1 \not\equiv 0\,(mod\ m+n) \\ m + n\,(mod\ m+n) & \text{if } n + j + i - 1 \equiv 0\,(mod\ m+n); \end{cases}$$

$$f(w_1u_{2i}) = i + 1,\ 1 \le i \le \left\lfloor \frac{m}{2} \right\rfloor;$$

$$f(w_1u_{2i-1}) = n + i,\ 1 \le i \le \left\lceil \frac{m}{2} \right\rceil;$$

$$f(w_1v_j) = m + j - 1,\ 1 \le j \le \left\lceil \frac{n}{2} \right\rceil;$$

$$f\left(w_1v_{\lceil \frac{n}{2} \rceil + j + 1}\right) = n + \left\lceil \frac{m}{2} \right\rceil + j,\ 1 \le j \le \left\lfloor \frac{n}{2} \right\rfloor - 1;$$

$$f\left(w_1v_{\lceil \frac{n}{2} \rceil + 1}\right) = 1;$$

To prove Case (ii) and Case (iii), $\chi'_e(P_l \vee K_{m,n}) \le m + n$. We have $\chi'_e(P_l \vee K_{m,n}) \ge \Delta(P_l \vee K_{m,n}) \ge m + n$, $\chi'_e(P_l \vee K_{m,n}) \ge m + n$ by Lemma 1.3. Hence $\chi'_e(P_l \vee K_{m,n}) = m + n$. The conclusion is true.

Case 4: If $l, m, n > 1$, $n > m$, $m + n > l$, $f(w_ku_i) = j + k - 1$, $1 \le i \le m$, $1 \le k \le l$. For $1 \le j \le n$, $1 \le k \le l$,

$$f(w_kv_j) = \begin{cases} m + j + k - 1\,(mod\ m+n) & \text{if } m + j + k - 1 \not\equiv 0\,(mod\ m+n) \\ m + n\,(mod\ m+n) & \text{if } m + j + k - 1 \equiv 0\,(mod\ m+n); \end{cases}$$

For $1 \le i \le m - 1$, $3 \le j \le n$,

$$f(w_kv_j) = \begin{cases} m + j + i(n-2) - 5\,(mod\ m+n) & \text{if } m + j + i(n-2) - 5 \not\equiv 0\,(mod\ m+n) \\ m + n\,(mod\ m+n) & \text{if } m + j + i(n-2) - 5 \equiv 0\,(mod\ m+n); \end{cases}$$

$$f(u_mv_j) = m + j,\ 1 \le j \le n - 3;$$

$$f(u_mv_n) = 2;$$

$$f(u_iv_j) = m + n + 1,\ i = j,\ 1 \le i \le m,\ 1 \le j \le n;$$

$$f(u_iv_{j+1}) = m + n + 2,\ i = j,\ 1 \le i \le m,\ 1 \le j \le n;$$

$$f(w_{2k-1}w_{2k}) = m + n + 1,\ 1 \le k \le \left\lceil \frac{l}{2} \right\rceil;$$

$$f(w_{2k}w_{2k+1}) = m + n + 2,\ 1 \le k \le \left\lfloor \frac{l}{2} \right\rfloor;$$

To prove $\chi'_e(P_l \vee K_{m,n}) \le m+n+2$. We have $\chi'_e(P_l \vee K_{m,n}) \ge \Delta(P_l \vee K_{m,n}) \ge m+n+2$, $\chi'_e(P_l \vee K_{m,n}) \ge m+n+2$ by Lemma 1.3. Hence $\chi'_e(P_l \vee K_{m,n}) = m + n + 2$. The conclusion is true.

Case 5: If $l, m, n > 1$, $n > m$, $l \ge m + n$. For $1 \le k \le l - 1$,

$$f(w_kw_{k+1}) = \begin{cases} k\,(mod\ n) & \text{if } k \not\equiv 0\,(mod\ n) \\ n\,(mod\ n) & \text{if } k \equiv 0\,(mod\ n); \end{cases}$$

For $1 \le i \le m$, $1 \le k \le l$,

$$f(w_ku_i) = \begin{cases} i + k + 2\,(mod\ l+n) & \text{if } i + k + 2 \not\equiv 0\,(mod\ l+n) \\ l + n\,(mod\ l+n) & \text{if } i + k + 2 \equiv 0\,(mod\ l+n); \end{cases}$$

For $1 \leq k \leq l$, $1 \leq j \leq n$,

$$f(w_k v_j) = \begin{cases} m+k+j+2\,(mod\ l+n) & \text{if } m+k+j+2 \not\equiv 0\,(mod\ l+n) \\ l+n\,(mod\ l+n) & \text{if } m+k+j+2 \equiv 0\,(mod\ l+n); \end{cases}$$

For $1 \leq i \leq m$, $1 \leq j \leq n$,

$$f(u_i v_j) = \begin{cases} i+j-1\,(mod\ n) & \text{if } i+j-1 \not\equiv 0\,(mod\ n) \\ n\,(mod\ l+n) & \text{if } i+j-1 \equiv 0\,(mod\ l+n); \end{cases}$$

To prove $\chi'_e(P_l \vee K_{m,n}) \leq l+n$. We have $\chi'_e(P_l \vee K_{m,n}) \geq \Delta(P_l \vee K_{m,n}) \geq l+n$, $\chi'_e(P_l \vee K_{m,n}) \geq l+n$ by Lemma 1.3. Hence $\chi'_e(P_l \vee K_{m,n}) = l+n$. The conclusion is true. □

Theorem 2.2. *For any positive integer m and n, then*

$$\chi'_e(P_m \vee K_{1,n,n}) = \Delta(P_m \vee K_{1,n,n}) = \begin{cases} 2n+1 & \text{if } m=1 \\ 2n+2 & \text{if } m=2 \\ 2n+3 & \text{if } 2<m\leq n+3 \\ m+n & \text{if } m>n+3. \end{cases}$$

Proof. Let $V(P_m) = \{u_i | i=1,2,\cdots,m\}$ and $V(K_{1,n,n}) = \{v_0\} \cup \{v_{2j-1} | j=1,2,\cdots,n\} \cup \{v_{2j} | j=1,2,\cdots,n\}$. Let $E(P_m) = \{u_i u_{i+1} | i=1,2,\cdots,m-1\}$ and $E(K_{1,n,n}) = \{v_0 v_{2j-1} | j=1,2,\cdots,n\} \cup \{v_{2j-1} v_{2j} | j=1,2,\cdots,n\}$. By the definition of join graph,

$$\begin{aligned} V(P_m \vee K_{1,n,n}) &= V(P_m) \cup V(K_{1,n,n}) \text{ and} \\ E(P_m \vee K_{1,n,n}) &= E(P_m) \cup E(K_{1,n,n}) \cup \bigcup_{i=1}^{m} \{u_i v_j : 0 \leq j \leq 2n\} \end{aligned}$$

Let f be a mapping from $E(P_m \circ K_{1,n,n})$ as follows:

Case 1: For $m=1$,

$f(u_1 v_j) = j, 0 \leq j \leq 2n$; $f(v_0 v_{2j-1}) = 2n+2j+1\,(mod\ 2n+1), 1 \leq j \leq n$; $f(v_{2j-1} v_{2j}) = 2n+2j+2\,(mod\ 2n+1), 1 \leq j \leq n$; Obviously, the f is $2n+1$-EEC of $\chi'_e(P_m \vee K_{1,n,n})$.

Case 2: For $m=2$,

$f(u_i v_j) = i+j-1\,(mod\ 2n), i=1,2, 0 \leq j \leq 2n$; $f(u_1 u_2) = 2n+1$; $f(v_0 v_{2j-1}) = 2n+2j+3\,(mod\ 2n+2), 1 \leq j \leq n$; $f(v_{2j-1} v_{2j}) = 2n+2j+4\,(mod\ 2n+2), 1 \leq j \leq n$; To prove $\chi'_e(P_m \vee K_{1,n,n}) \leq 2n+2$. We have $\chi'_e(P_m \vee K_{1,n,n}) \geq \Delta(P_m \vee K_{1,n,n}) \geq 2n+2$, $\chi'_e(P_m \vee K_{1,n,n}) \geq 2n+2$ by Lemma 1.3. Hence $\chi'_e(P_m \vee K_{1,n,n}) = 2n+2$.

Case 3: For $2 < m \leq n+3$,

$f(u_i v_j) = i+j-1\,(mod\ 2n+3), 1 \leq i \leq m, 0 \leq j \leq 2n$; $f(u_i u_{i+1}) = 2n+i+1\,(mod\ 2n+3), 1 \leq i \leq m-1$; $f(v_0 v_{2j-1}) = 2n+2j\,(mod\ 2n+3), 1 \leq j \leq n$; $f(v_{2j-1} v_{2j}) = 2n+2j-1\,(mod\ 2n+3), 1 \leq j \leq n$; To prove $\chi'_e(P_m \vee K_{1,n,n}) \leq 2n+3$. We have $\chi'_e(P_m \vee K_{1,n,n}) \geq \Delta(P_m \vee K_{1,n,n}) \geq 2n+3$, $\chi'_e(P_m \vee K_{1,n,n}) \geq 2n+3$ by Lemma 1.3. Hence $\chi'_e(P_m \vee K_{1,n,n}) = 2n+3$.

Case 4: For $m > n + 3$,

$f(u_i v_j) = i + j - 1\,(mod\ m + n), 1 \le i \le m, 0 \le j \le 2n$; $f(u_i u_{i+1}) = 2n + i + 1\,(mod\ m + n), 1 \le i \le m - 1$;

$f(v_0 v_{2j-1}) = m + 2j - 1\,(mod\ m + n), 1 \le j \le n$; $f(v_{2j-1} v_{2j}) = m + 2j\,(mod\ m + n), 1 \le j \le n$;

To prove $\chi'_e(P_m \vee K_{1,n,n}) \le m + n$. We have $\chi'_e(P_m \vee K_{1,n,n}) \ge \Delta(P_m \vee K_{1,n,n}) \ge m + n$, $\chi'_e(P_m \vee K_{1,n,n}) \ge m + n$ by Lemma 1.3. Hence $\chi'_e(P_m \vee K_{1,n,n}) = m + n$. The conclusion is true. □

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