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On Ternary Quadratic Diophantine Equation $z^2 = 40x^2 + y^2$

Research Article

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Abstract: The Ternary Quadratic Diophantine equation $z^2 = 40x^2 + y^2$ is analyzed for its non-zero distinct integral points on it. A few interesting properties among the solutions are presented.

Keywords: Integral points, Ternary quadratic, Polygonal numbers, Pyramidal numbers and special numbers.

1. Introduction

Diophantine equations is an interesting concept, as it can be seen from [1-2]. For an extensive review of various problems one may refer [3-11]. In this context one may also see [12-23]. This communication concerns with yet another interesting ternary quadratic Diophantine equation $z^2 = 40x^2 + y^2$ for determining its infinitely many non-zero integral solutions. Also a few interesting properties among the solutions are presented.

Notation:

$t_{m,n}$: Polygonal number of rank n with sides m.

p_n^m : Pyramidal number of rank n with sides m.

$ct_{m,n}$: Centered Polygonal number of rank n with sides m.

P_n : Pronic number

g_n : Gnomonic number

2. Method of Analysis

The ternary quadratic equation to be solved for its non-zero. Integral solution is

$$z^2 = 40x^2 + y^2 \tag{1}$$

Assume

$$z = z(a, b) = a^2 + 40b^2 \tag{2}$$

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Where a, b are non-zero integer. Different patterns of solutions for (1) are given below.

Pattern I: Using equation (2) in (1), we get,

$$40x^2 + y^2 = \left(a^2 + 40b^2\right)^2 \tag{3}$$

Employing the method of factorization and comparing rational and irrational parts, we get

$$\left(y + i\sqrt{40}x\right)\left(y - i\sqrt{40}x\right) = \left(a + i\sqrt{40}b\right)^2\left(a - i\sqrt{40}b\right)^2$$

Comparing the real and imaginary parts on the above equation, we get

$$\left.\begin{aligned} x &= x\left(a,b\right) = 2ab \\ y &= y\left(a,b\right) = a^2 - 40b^2 \end{aligned}\right\} \tag{4}$$

Thus (2) and (4) represents the non-zero distinct integral solutions of (1).

Properties:

(1). $x\left(1,n\right) + y\left(1,n\right) + z\left(1,n\right) + 78t_{3,n} \equiv 0(mod\ 41)$

(2). $x\left(2,n\right) + y\left(n,2\right) + z\left(2,n\right) - 8t_{3,n} \equiv -8\left(mod\ 37\right)$

(3). $x\left(1,n\right) + y\left(n,1\right) + z\left(1,n\right) - 4t_{3,n} \equiv 0\left(mod\ 39\right)$

(4). $x\left(n,2\right) + y\left(2,n\right) + z\left(n,2\right) + 78t_{3,n} \equiv 35\left(mod\ 43\right)$

(5). $x\left(3,n\right) + y\left(3,n\right) + z\left(n,3\right) - 6n(n+1) \equiv -9\left(mod\ 45\right)$

Pattern II: Equation (1) can be written as

$$40x^2 + y^2 = z^2 * 1 \tag{5}$$

Assume,

$$z = z\left(a,b\right) = a^2 + 40b^2 \tag{6}$$

Where a, b are non-zero distinct integers. Write 1 as,

$$1 = \frac{\left(3 + i\sqrt{40}\right)\left(3 - i\sqrt{40}\right)}{49} \tag{7}$$

using (6) and (7) in (5), we get

$$40x^2 + y^2 = \frac{\left(3 + i\sqrt{40}\right)\left(3 - i\sqrt{40}\right)}{49}\left(a^2 + 40b^2\right)^2$$

By the method of factorization and comparing the rational and irrational parts from the above equation, we get

$$\left.\begin{aligned} x &= x\left(a,b\right) = \frac{1}{7}[a^2 - 40b^2 + 6ab] \\ y &= y\left(a,b\right) = \frac{1}{7}[3a^2 - 120b^2 - 80ab] \end{aligned}\right\} \tag{8}$$

As our interest is to find only integer solution, it is seen x and y are integer for suitable choices of a and b. Let us assume, $a = 7A$ and $b = 7B$ the corresponding non-zero distinct integer solution of (1) are found to be,

$$x = x\left(A,B\right) = 7A^2 - 280B^2 + 42AB$$

$$y = y\left(A,B\right) = 21A^2 - 840B^2 - 560AB$$

$$z = z\left(A,B\right) = 49A^2 + 1960B^2$$

Properties:

(1). $3x(A, A+1) - y(A, A+1) - 1372t_{3,a} \equiv 0.$

(2). $3x(A, (A+1)(A+2)) - y(A, (A+1)(A+2)) - 4116{P_a}^3 \equiv 0.$

(3). $3x(A,\ A(A+1)) - y(A, A(A+1)) - 1372\ {P_a}^5 \equiv 0.$

(4). $z(3A, A) - 2401t_{4,a} \equiv 0.$

(5). $6x(A, (A+1)(A+2)) - 2y(A, (A+1)(A+2)) - 8232{P_a}^3 \equiv 0.$

Pattern III: Write (1) as,

$$(z+y)(z-y) = 40x^2$$

Case 1: $\frac{z+y}{40x} = \frac{x}{z-y} = \frac{P}{Q}$. This is equivalent to the following two equations

$$-P40x + Qy + Qz = 0 \tag{9}$$

$$Qx + Py - Pz = 0 \tag{10}$$

Solving the equations (9) & (10), we obtain

$$\left.\begin{aligned} x &= x(P,Q) = -2PQ \\ y &= y(P,Q) = 40P^2 - Q^2 \\ z &= z(P,Q) = -40P^2 - Q^2 \end{aligned}\right\} \tag{11}$$

Thus (11) represents non-zero distinct integral solutions which satisfy equation (1).

Properties:

(1). $y(A, A) + z(A, A) + 2t_{4,a} \equiv 0.$

(2). $x(A, A+1) + 4t_{3,a} \equiv 0.$

(3). $x(A, A(A+1)) + 4{P_a}^5 \equiv 0.$

(4). $x(A, (A+1)(A+2)) + 12{P_a}^3 \equiv 0.$

(5). $y(A, A) - z(A, A) -\ 80t_{4,a} \equiv 0.$

Case 2: Equation (11) can also be rewritten as

$$\frac{z+y}{x} = \frac{40x}{z-y} = \frac{P}{Q}$$

On following the procedure as in case (1) the non-zero distinct solutions of (1) are given by

$$\left.\begin{aligned} x &= x(P,Q) = -2PQ \\ y &= y(P,Q) = P^2 - 40Q^2 \\ z &= z(P,Q) = -P^2 - 40Q^2 \end{aligned}\right\} \tag{12}$$

Thus (12) represents non-zero distinct integral solutions which satisfy equation (1).

Properties:

(1). $y(A,B) - z(A,B) - 2t_{4,a} \equiv 0.$

(2). $y(A,B) + z(A,B) + 80t_{4,a} \equiv 0.$

(3). $x(A,(A+1)(2A+1)) + 12{P_a}^4 \equiv 0.$

(4). $x(A,A(A+1)) + 4{P_a}^5 \equiv 0.$

(5). $x(A,2A-1) + 2t_{6,a} \equiv 0.$

Case 3: Equation (11) can also be rewritten as

$$\frac{z-y}{40x} = \frac{x}{z+y} = \frac{P}{Q}$$

On following the procedure as in case (2) the non-zero distinct solutions of (1) are given by

$$\left.\begin{aligned} x &= x(P,Q) = 2PQ \\ y &= y(P,Q) = 40P^2 - Q^2 \\ z &= z(P,Q) = 40P^2 + Q^2 \end{aligned}\right\} \quad (13)$$

Thus (13) represents non-zero distinct integral solutions which satisfy equation (1).

Properties:

(1). $x(A,(A+1)(A+2)) - 12{P_a}^3 \equiv 0.$

(2). $y(A,B) + z(A,B) - 80t_{4,a} \equiv 0.$

(3). $x(A,A(A+1)) - 4{P_a}^5 \equiv 0.$

(4). $x(A,(A+1)) - 4t_{3,a} \equiv 0.$

(5). $y(A,2A) - z(A,2A) + 8t_{4,a} \equiv 0.$

Case 4: Equation (11) can also be rewritten as

$$\frac{z-y}{x} = \frac{40x}{z+y} = \frac{P}{Q}$$

On following the procedure as in case (3) the non-zero distinct solutions of (1) are given by

$$\left.\begin{aligned} x &= x(P,Q) = 2PQ \\ y &= y(P,Q) = P^2 - 40Q^2 \\ z &= z(P,Q) = P^2 + 40Q^2 \end{aligned}\right\} \quad (14)$$

Thus (14) represents non-zero distinct integral solutions which satisfy equation (1).

Properties:

(1). $x(A,A+1) - z(A,A+1) + 78t_{3,a} \equiv 0 (mod\ 2).$

(2). $X\left(A,\left(A+1\right)\left(2A+1\right)\right)-12P_a^4\equiv 0$.

(3). $x\left(A,A+2\right)-z\left(A,A+2\right)+78t_{3,a}\equiv 0(mod\ 4)$.

(4). $x\left(A,2A\right)+y\left(A,2A\right)-155t_{4,a}\equiv 0$.

(5). $x\left(A,3A\right)-z\left(A,3A\right)+355t_{4,a}\equiv 0$.

Pattern IV: Equation (1) can be written as,

$$z^2-40x^2=y^2*1 \tag{15}$$

Write 1 as

$$1=\frac{\left(\sqrt{40}+2\right)\left(\sqrt{40}-2\right)}{36} \tag{16}$$

Assume,

$$y=a^2-40b^2 \tag{17}$$

Using (16) and (17) and in (15), We get

$$\left(z+\sqrt{40}x\right)\left(z-\sqrt{40}x\right)=\frac{\left(\sqrt{40}+2\right)\left(\sqrt{40}-2\right)}{36}\left(a^2-40b^2\right)^2$$

Applying the method of cross multiplication, on employing the method of factorization and equating the positive and negative factors, we get

$$x=x\left(a,b\right)=6a^2+240b^2+24ab$$
$$y=y\left(a,b\right)=36a^2-1440b^2$$
$$z=z\left(a,b\right)=12a^2+480b^2+480ab$$

Properties:

(1). $2x\left(A,2A-1\right)-z\left(A,2A-1\right)+432t_{6,a}\ \equiv 0$.

(2). $2x\left(A,\left(A+1\right)\left(A+2\right)\right)-z\left(A,\left(A+1\right)\left(A+2\right)\right)+2592{P_a}^3\equiv 0$.

(3). $2x(A,A\left(A+1\right)-z(A,A\left(A+1\right)+864{P_a}^5\equiv 0$.

(4). $y\left(6A,A\right)-144t_{6,a}\ \equiv 0$.

(5). $4x\left(A,A(A+1)\right)-2z\left(A,A\left(A+1\right)\right)+1728{P_a}^5\equiv 0$.

Note: In equation (16), 1 can be rewritten as the following different ways

(1). $1=\frac{\left(9+i\sqrt{40}\right)\left(9-i\sqrt{40}\right)}{121}$.

(2). $1=\frac{\left(1+i\sqrt{35}\right)\left(1-i\sqrt{35}\right)}{36}$.

(3). $1=\frac{\left(4+i\sqrt{33}\right)\left(4-i\sqrt{33}\right)}{49}$.

3. Conclusion

One may search for other patterns of solution and their corresponding properties.

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