On Schur Complements in Range Quaternion Hermitian Matrices

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Abstract: It is established that under contain conditions a schur complement in a q-EP matrix is as well as q-EP matrix. As an application a decomposition of a partitioned matrix into a sum of q-EP matrices is given.

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1. Introduction

Throughout we shall deal with $n \times n$ quaternion matrices: Let $A^*$ denote the conjugate transpose of $A$. Any matrix $A \in H_{n \times n}$ is called q-EP. If $R(A) = R(A^*)$ and is called $q$-EP, if $A$ is q-EP and $rk(A) = r$, where $N(A)$, $R(A)$ and $rk(A)$ denote the null space, range space and rank of $A$ respectively. It is well known that sum and product of q-EP, Generalized Inverse Group Inverse and Reverse order law for q-EP and Bicomplex representation methods and application of q-EP matrices. In this section, Schur complements in a q-EP matrices.

Lemma 1.1. If $X$ and $Y$ are generalized inverse of $A$, then $CXB = CYB$ if and only if $N(A) \subseteq M(C)$ and $N(A^*) \subseteq N(B^*)$ or, equivalently if and only if

$$ C = CA^*A \text{ and } B = AA^*B \text{ for every } A^* $$

Throughout this paper, we are concerned with $n \times n$ quaternion matrices $M$ partitioned in the form

$$ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} $$

Where $A$ and $D$ are square matrices with respect to this partitioning a Schur complements of $A$ in $M$ is a matrix at the form $(M/A) = D - CA^*B$. For entries of Schur complements one may refer to [2, 3, 5]. On account of Lemma 1.1 it is obvious that under certain conditions $(M/A)$ is independent of the choice of $A^*$. However in the sequel we shall always assume that $(M/A)$ is given in terms of specific choice of $A^*$.

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In [9] necessary and sufficient conditions are derived for a matrix of the (2) with $B = 0$ and $C = 0$ to be q-EP. The results are
here extended for general matrices of the form (2). If a partitioned matrix of the form (2) is q-EP, then in general $(M/A)$ is
not q-EP. Here we determine necessary and sufficient conditions for $M/A$ to be q-EP. In particular, when \( rk(M) = rk(A) \)our
results include as special cases the results of paper [13]. In [5] we have given conditions for a sum of q-EP matrices to be
q-EP.

**Theorem 1.2.** Let $M$ be a matrix of the form (2) with $N(A) \subseteq N(C)$ and $N(M/A) \subseteq N(B)$, then the following are
equivalent.

(1). $M$ is a q-EP matrix

(2). $A$ and $M/A$ are q-EP, $N(A^*) \subseteq N(B^*)$ and $N((M/A)^*) \subseteq M(C^*)$;

(3). Both the matrices

\[
\begin{pmatrix}
A & 0 \\
C & M/A
\end{pmatrix}
\text{ and }
\begin{pmatrix}
A & B \\
0 & M/A
\end{pmatrix}
\]

are q-EP.

**Proof.**

(1)$\Rightarrow$(2) Let us consider the matrices

\[
p = \begin{pmatrix}
I & 0 \\
CA^* & I
\end{pmatrix},
Q = \begin{pmatrix}
I & B(M/A)^* \\
0 & I
\end{pmatrix},
L = \begin{pmatrix}
A & 0 \\
0 & M/A
\end{pmatrix}
\]

Clearly P and Q are non-singular. By assumption $N(A) \subseteq N(C)$ and $N(M/A) \subseteq N(B)$ and by using Lemma 1.1 it
is obvious that $M$ can be factorized as $M = PQL$. Hence \( rk(M) = rk(L) \) and \( N(M) = N(L) \). But M is q-EP, e.g.,
$N(M^*) = N(M) = N(L)$. Therefore by using Lemma 1.1 again $M^* = M^*L^{-1}L$ holds for every $L^-$. One choice of $L^-$ is

\[
L^- = \begin{pmatrix}
A^- & 0 \\
0 & (M/A)^-
\end{pmatrix},
\]

which gives

\[
M^* = \begin{pmatrix}
A^* & C^* \\
B^* & D^*
\end{pmatrix} = \begin{pmatrix}
A^* & C^* \\
B^* & D^*
\end{pmatrix}
\begin{pmatrix}
A^- & 0 \\
0 & (M/A)^-(M/A)
\end{pmatrix}
\]

$A^* = A^*A^-A$ implies $N(A^*) \supseteq N(A)$, and since \( rk(A^*) = rk(A) \) these imply $N(A^*) = N(A)$. Hence $A$ is q-EP. From $B^* =
B^*A^-A$ it follows that $N(B) \supseteq N(A) = N(A^*)$. After substituting $D = M/A + BA^nhand using $C^* = C^*(M/A)^-(M/A)$
in $D^* = D^*(M/A)^-(M/A)$ we get $(M/A)^* = (M/A)^*(M/A)^-(M/A)$. This implies that $N((M/A)^*) \supseteq N(M/A)$ and since

\[
\text{we get } N((M/A)^*) = N(M/A).
\]

Thus $(1)$ holds.

(1)$\Rightarrow$(2) Since $N(A) \subseteq N(C)$, $N(A^*) \subseteq N(B^*)$, $N(M/A) \subseteq N(B)$ and $N((M/A)^*) \subseteq N(C^*)$ hold according to the assumption. So $M^1$ is given by the formula

\[
M^1 = \begin{pmatrix}
A^1 + A^1B(M/A)^1CA^1 & -A^1B(M/A)^1 \\
-(M/A)^1CA^1 & (M/A)^1
\end{pmatrix}
\]
According to Lemma 1.1 the assumptions \( N(A) \subseteq N(C) \) and \( N(A^*) \subseteq N(B^*) \) imply that \( M/A \) is invariant for every choice of \( A^* \). Hence \( M/A = D - CA^1B \). Further, using \( C = M/A(M/A)^!C \) and \( B = AA^1B \), \( MM^! \) is reduced to the form

\[
M^!M = \begin{pmatrix}
AA^! & 0 \\
0 & (M/A)(M/A)^!
\end{pmatrix}
\]

The relations \( AA^1 = A^1A \) and \( (M/A)(M/A)^! = (M/A)^!(M/A) \) result \( MM^! = M^!M \), e.g., \( M \) is q-EP. Thus (1) holds. (2)⇒(3) By Corollary 8 in [9]

\[
\begin{pmatrix}
A & 0 \\
C & M/A
\end{pmatrix}
\]

is q-EP, iff \( A \) and \( (M/A) \) are q-EP, further \( N(A) \subseteq N(C) \) and \( N((M/A)^*) \subseteq N(C^*) \)

Is q-EP iff \( A \) and \( M/A \) are q-EP, further \( N(A^*) \subseteq N(B^*) \) and \( N(M/A) \subseteq N(B) \). This proves the equivalence of (2) and (3). The proof is complete.

\[
M = \begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[\square\]

**Theorem 1.3.** Let \( M \) be a matrix of the form (2) with \( N(A^*) \subseteq N(B^*) \) and \( N((M/A)^*) \subseteq N(C^*) \), then the following are equivalent.

(1). \( M \) is an q-EP matrix

(2). \( A \) and \( (M/A) \) are q-EP matrices.

(3). Both the matrices \( \begin{pmatrix}
A & 0 \\
C & M/A
\end{pmatrix} \) and \( \begin{pmatrix}
A & B \\
0 & M/A
\end{pmatrix} \) are q-EP.

Proof. Theorem 1.3 follows immediately from Theorem 1.2 and from the fact that \( M \) is q-EP iff \( M^* \) is q-EP. If and only if \( M^* \) is q-EP. \[\square\]

In this special case when \( B = C^* \) we get the following.

**Corollary 1.4.** Let \( M = \begin{pmatrix}
A & C^* \\
C & D
\end{pmatrix} \) with \( N(A) \subseteq N(C) \) and \( N(M/A) \subseteq N(C^*) \), then the following are equivalent.

(1). \( M \) is an q-EP matrix

(2). \( A \) and \( (M/A) \) are q-EP matrices.

(3). the matrix \( \begin{pmatrix}
A & 0 \\
C & M/A
\end{pmatrix} \) is q-EP.
**Remark 1.5.** The conditions that taken on $M$ in the previous theorems are essential. This is illustrated in the following example. Let

$$M = \begin{bmatrix}
1 & 1 & 1 & 1 + i + j + k \\
1 & 1 & 1 - i - j - k & 1 \\
1 & 1 + i + j + k & 1 & 1 \\
1 - i - j - k & 1 & 1 & 0
\end{bmatrix}$$

$M$ is symmetric and

$$B = C = \begin{bmatrix}
1 & 1 + i + j + k \\
1 - i - j - k & 1
\end{bmatrix}$$

$$(M/A) = D - CA^1 B = \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}$$

Clearly $A$ and $(M/A)$ are q-EP, $N(A) \subseteq N(C)$ and $N(A^*) \subseteq N(B^*)$, but $N(M/A) \subseteq N(B)$ and $N((M/A)^*) \nsubset N(C^*)$, further $(A \ 0)
\begin{bmatrix}
C & M/A
\end{bmatrix}$ and $(A \ B
\begin{bmatrix}
0 & M/A
\end{bmatrix}$) Or not q-EP. Thus Theorem 1.2 and 1.3 as well as Corollary 1.4 fail.

**Remark 1.6.** We conclude from Theorem 1.2 and Theorem 1.3 that for an q-EP matrix $M$ of the form equation (2) the following are equivalent

$$N(A) \subseteq N(C), N(M/A) \subseteq N(B)$$

$$N(A^*) \subseteq N(B^*), N((M/A)^*) \nsubset N(C^*)$$

However this fails if we omit the condition that $M$ is q-EP. For example Let

$$M = \begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}$$

$M$ is not q-EP. Here

$$A = \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}, \quad B = C^* = \begin{bmatrix}
1 & 0 \\
1 & 0
\end{bmatrix}$$

$A$ is q-EP, $N(A) \subseteq N(C)$ and $N(A^*) \subseteq N(B^*)$. Hence $(M/A)$ is independent of the choice of $A^-$ and so

$$(M/A) = D - CA^1 B = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}$$

$(M/A)$ is not q-EP, $N((M/A)^*) \nsubset N(C^*)$, but $N(A) \subseteq N(B)$. Thus Equation (4) holds, while Equation (4) fails.

** Remark 1.7.** It has been proved is [2] that for any matrix Aits Moore-Penrose inverse. $M^1$ is given by the formula Equation (??) iff both Equation (3) and Equation (4) holds. However it is clear by the previous Remark 1.6 that for an q-EP matrix formula (??) gives $M^1$ iff either (3) or (4) holds.
Theorem 1.8. Let $M$ be of the form Equation (2) with $rk(M) = rk(A) = r$. Then $M$ is an q-EP, matrix if and only if $A$ is q-EP, and $CA^1 = (A^1B)^*$.

Proof. Since $rk(M) = rk(A) = r$, we have by reason of the corollary of Theorem 1 in [3] that $N(A) \subseteq N(C)$, $N(A^*) \subseteq N(B^*)$, and $M/A = D - CA^1B = 0$. According to Theorem 1.1 these relation are equivalent $C = CA^1A$, $B = AA^1B$ and $D = CA^1B$. Let us consider the matrices

$$P = \begin{pmatrix} 1 & 0 \\ CA^1 & I \end{pmatrix}, \quad Q = \begin{pmatrix} I & A^1B \\ 0 & I \end{pmatrix}, \quad L = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$ 

$P$ and $Q$ are non-singular and by assumption $CA^1 = (A^1B)^*$ it holds $P = Q^*$. Therefore $M$ can be factorized as $M = PLP^*$. Since $A$ is q-EP, consequently $L$ is as well q-EP. Hence $N(L) = N(L^*)$ and so we have according to Lemma 3 of [1] that $N(M) = N(PLP^*) = N(P^*L^*) = N(M^*)$. This shows that $M$ is q-EP.

Conversely, let us assume that $M$ is q-EP. Since $M = PLQ$, one choice of $A^*$ is

$$M^- = Q^{-1} \begin{pmatrix} A^1 & 0 \\ 0 & 0 \end{pmatrix} P^{-1}.$$

We know that $N(M) = N(M^*)$, therefore by Lemma 1.1 $M^* = M^*M^-M$ holds, e.g

$$M^* = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \begin{pmatrix} A^1A & A^1B \\ 0 & 0 \end{pmatrix}$$

or equivalently, $A^* = A^*A^1A$ and $C^* = C^*A^1B$. From $A^* = A^*A^1A$ it follows $N(A^*) = N(A)$, i.e., $A$ is q-EP, and therefore $AA^1 = A^1A$ taking into account $C^* = C^*A^1B$, we have

$$CA^1 = B^*(A^1)^*(A^1A) = B^*(A^1AA^1)^* = B^*(A^1)^* = (A^1B)^* \tag*{□}$$

Corollary 1.9. Let $M$ of the form (2) with $A$ non-singular matrix and $rk(M) = rk(A)$. Then $M$ is q-EP if and only if $CA^1 = (A^1B)^*$.

Corollary 1.10. Let $M$ be an $n \times n$ matrix of rank $r$. Then $M$ is q-EP if and only if every principal submatrix of rank $r$ is q-EP,.

Proof. Suppose $M$ is an q-EP, matrix. Let $A$ be any principal submatrix of $M$ such that $rk(M) = rk(A) = r$. Then there exists a permutation matrix such that $\tilde{M} = PMP^T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $rk(A) = r$. According to Lemma 3 in [1], is q-EP.

Now, we conclude from Theorem 1.3 that $A$ q-EP as well. Since $A$ was arbitrary, it follows that very principal submatrix of rank $r$ is q-EP. The converse is obvious. \tag*{□}

Remark 1.11. Theorem 1.8 fails if we relax the condition on rank of $M$. 

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2. Application

We give conditions under which a partitioned matrix is decomposed into complementary summands of q-EP matrices. $M_1$ and $M_2$ are called complementary summand of $M$ if $M = M_1 + M_2$ and $rk(M) = rk(M_1) + rk(M_2)$.

**Theorem 2.1.** Let $M$ of the form (2) with $rk(M) = rk(A) = rk(M/A)$, where $(M/A) = D - CA^\dagger B$. If $A$ and $(M/A)$ are q-EP matrices such that $CA^\dagger = (A + B)^*$ and $B(M/A)^\dagger = ((M/A)^\dagger C^*)$ then $M$ can be decomposed into complementary summands of q-EP matrices.

**Proof.** Let us consider the matrices

$$M_1 = \begin{pmatrix} A & AA^\dagger B \\ CA^\dagger A & CA^\dagger B \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} 0 & (I - AA^\dagger)B \\ C(I - A^\dagger A) & M/A \end{pmatrix}$$

Taking into account that $N(A) \subseteq N(CA^\dagger A)$, $N(A^*) \subseteq N(AA^\dagger B)^*$ and

$$M_1/A = CA^\dagger B - ((CA^\dagger A) - (AA^\dagger B) = CA^\dagger B - CA^\dagger B = 0$$

we obtain by the corollary after Theorem 1 in [5], that $rk(M_1) = rk(A)$. Since $A$ is q-EP and $(CA^\dagger A)A^\dagger = CA^\dagger = (A^\dagger B)^* = (A^\dagger AA^\dagger B)^*$. We have from Theorem 1.8 that $M_1$ is q-EP. Since $rk(M) = rk(A) + rk(M/A)$, Theorem 1 of [5] gives $N(M/A) \subseteq N(I - AA^\dagger)B$, $N(M/A) \subseteq N((I - A^\dagger)C)^*$ and $(I - AA^\dagger)M(M/A)^\dagger C(I - A^\dagger A) = 0$. Thus by the corollary of the just applied Theorem 1.1 in [5], we have $rk(M_2) = rk(M/A)$. Further, using $AA^\dagger = A^\dagger A$, we obtain

$$(I - AA^\dagger)B(M/A)^\dagger = (I - AA^\dagger)((M/A)^\dagger)^*$$

$$= ((M/A)^\dagger C(I - AA^\dagger))^*$$

$$= ((M/A)^\dagger C(I - A^\dagger A))^*$$

Thus by Theorem 1.8, $M_2$ is also q-EP. Clearly $M = M_1 + M_2$, where both $M_1$ and $M_2$ are q-EP matrices and

$$rk(M) = rk(A) + rk(M/A) = rk(M_1) + rk(M_2).$$

Hence $M_1$ and $M_2$ are complementary summands of q-EP matrices.

**Remark 2.2.** Any matrix that is represented as the sum of complementary summands of q-EP matrices is itself q-EP. For if $M = \sum_{i=1}^k M_i$ such that each $M_i$ is q-EP and $rk(M) = \sum rk(M_i)$, then

$$N(M) = \bigcap_{i=1}^k N(M_i) = \bigcap_{i=1}^k N(M_i^*) = N(M_i^*).$$

References


