Cordiality in the Context of Duplication in Helm and Closed Helm

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Abstract: Let $G = (V(G), E(G))$ be a graph and let $f : V(G) \rightarrow \{0, 1\}$ be a mapping from the set of vertices to $\{0, 1\}$ and for each edge $uv \in E$ assign the label $|f(u) - f(v)|$. If the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1 and the number of edges labled with 0 and the number of edges labeled with 1 differ by at most 1, then $f$ is called a cordial labeling. We discuss cordial labeling of graphs obtained from duplication of certain graph elements in helm and closed helm.

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1. Introduction

We begin with simple, finite, undirected graph $G = (V(G), E(G))$ where $V(G)$ and $E(G)$ denotes the vertex set and the edge set respectively. For a finite set $A$, $|A|$ denotes the number of elements of $A$. For all other terminology we follow Gross [2]. We provide some useful definitions for the present work.

\textbf{Definition 1.1.} The graph labeling is an assignment of numbers to the vertices or edges or both subject to certain condition(s).

A detailed survey of various graph labeling is explained in Gallian [1].

\textbf{Definition 1.2.} For a graph $G = (V(G), E(G))$, a mapping $f : V(G) \rightarrow \{0, 1\}$ is called a binary vertex labeling of $G$ and $f(v)$ is called the label of the vertex $v$ of $G$ under $f$. For an edge $e = uv$, the induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ defined as $f^*(uv) = |f(u) - f(v)|$. Let $v_f(0), v_f(1)$ be the number of vertices of $G$ having labels 0 and 1 respectively under $f$ and let $e_f(0), e_f(1)$ be the number of edges having labels 0 and 1 respectively under $f^*$.

\textbf{Definition 1.3.} Duplication of a vertex $v$ of a graph $G$ produces a new graph $G'$ by adding a new vertex $v'$ such that $N(v') = N(v)$. In other words a vertex $v'$ is said to be duplication of $v$ if all the vertices which are adjacent to $v$ in $G$ are also adjacent to $v'$ in $G'$.
Definition 1.4. Duplication of an edge $e = uv$ of a graph $G$ produces a new graph $G'$ by adding an edge $e' = u'v'$ such that $N(u') = N(u) \cup \{v'\} - \{v\}$ and $N(v') = N(v) \cup \{u'\} - \{u\}$.

Definition 1.5. The wheel $W_n$, is join of the graphs $C_n$ and $K_1$, i.e $W_n = C_n + K_1$. Here vertices corresponding to $C_n$ are called rim vertices and $C_n$ is called rim of $W_n$ while, the vertex corresponding to $K_1$ is called the apex vertex, edges joining the apex vertex and a rim vertex is called spoke.

Definition 1.6 ([1]). The helm $H_n$, is the graph obtained from the wheel $W_n$ by adding a pendant edge at each rim vertex. Each pendent edges are called outer spoke.

Definition 1.7 ([1]). The closed helm $CH_n$, is the graph obtained from a helm by joining each pendent vertex to form a cycle, here vertices corresponding to this cycle are called outer rim vertices and vertices corresponding to wheel except the apex vertex are called inner rim vertices.

Definition 1.8. A binary vertex labeling $f$ of a graph $G$ is called a cordial labeling if $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. A graph $G$ is said to be cordial if it admits cordial labeling.

The concept of cordial labeling was introduced by Cahit [3] in which he proved that the wheel $W_n$ is cordial if and only if $n \not\equiv 3 \pmod 4$. Vaidya and Dani [4] proved that the graphs obtained by duplication of an arbitrary edge of a cycle and a wheel admit a cordial labeling. Prajapati and Gajjar [5] proved that complement of wheel graph and complement of cycle graph are cordial if $n \not\equiv 4 \pmod 8$ or $n \not\equiv 7 \pmod 8$. Prajapati and Gajjar [6] proved that cordial labeling in the context of duplication of cycle graph and path graph. In this paper, for every natural number $n$ the set $\{1, 2, ..., n\}$ will be denoted by $[n]$.

2. Main Results

Theorem 2.1. The graph obtained by duplicating all the vertices except the apex vertex of the helm $H_n$ is cordial.

Proof. Let $V(H_n) = \{w\} \cup \{u_i, v_i/1 \leq i \leq n\}$ and $E(H_n) = \{wu_i, u_iv_i/1 \leq i \leq n\} \cup \{u_iu_{i+1}/1 \leq i \leq n-1\}$. Let $G$ be the graph obtained by duplicating all the vertices except the apex vertex in $H_n$. Let $u_1', u_2', ..., u_n', v_1', v_2', ..., v_n'$ be the new vertices of $G$ by duplicating $u_1, u_2, ..., u_n, v_1, v_2, ..., v_n$ respectively. Then $V(G) = \{w\} \cup \{u_i, v_i, u_i'/1 \leq i \leq n\}$ and $E(G) = \{u_nu_1, u_nu_1, u_nu_0\} \cup \{u_iu_{i+1}, u_i'u_{i+1}, u_i'u_{i+1}/1 \leq i \leq n-1\} \cup \{wu_i, wu_i', u_iv_i, u_iv_i', u_iv_i'/1 \leq i \leq n\}$. Therefore $|V(G)| = 4n + 1$ and $|E(G)| = 8n$. Define a vertex labeling $f : V(G) \rightarrow \{0, 1\}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = w; \\ 1 & \text{if } x \in \{u_i, v_i\}, i \in [n]; \\ 0 & \text{if } x \in \{u_i', v_i'\}, i \in [n]. \end{cases}$$

Thus $v_f(1) = 2n + 1$ and $v_f(0) = 2n$. The induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 1 & \text{if } e \in \{u_i,v_i, wu_i'\}, i \in [n]; \\ 1 & \text{if } e \in \{u_i',u_{i+1}, u_i'u_{i+1}'\}, i \in [n-1]; \\ 0 & \text{if } e \in \{wu_i, v_i'u_i', u_iv_i'\}, i \in [n]; \\ 0 & \text{if } e = u_iu_{i+1}, i \in [n-1]; \\ 0 & \text{if } e = u_nu_1; \\ 1 & \text{if } e \in \{u_n'u_1, u_nu_1'\}. \end{cases}$$
Thus \( e_f(1) = 4n \) and \( e_f(0) = 4n \). Therefore \( f \) satisfies the conditions \(|v_f(1) - v_f(0)| \leq 1\) and \(|e_f(1) - e_f(0)| \leq 1\). So, \( f \) admits cordial labeling on \( G \). Hence \( G \) is cordial.

\[ \text{Theorem 2.2. The graph obtained by duplicating all the vertices of the helm \( H_n \) is cordial.} \]

\[ \text{Proof. Let} \ V(H_n) = \{ w \} \cup \{ u_i, v_i/1 \leq i \leq n \} \text{ and } E(H_n) = \{ wu_i, u_i v_i/1 \leq i \leq n \} \cup \{ u_i u_{i+1}/1 \leq i \leq n-1 \}. \text{ Let } G \text{ be the graph obtained by duplicating all the vertices in } H_n. \text{ Let } w', u'_1, u'_2, ..., u'_n, v'_1, v'_2, ..., v'_n \text{ be the new vertices of } G \text{ by duplicating } w, u_1, u_2, ..., u_n, v_1, v_2, ..., v_n \text{ respectively. Then } V(G) = \{ w, w' \} \cup \{ u_i, v_i, u'_i, v'_i/1 \leq i \leq n \} \text{ and } E(G) = \{ u_n u_1, u'_n u_1, u_n u'_1 \} \cup \{ u_i u_{i+1}, u'_i u'_{i+1}, u_i u'_i u'_{i+1}/1 \leq i \leq n-1 \} \cup \{ wu_i, wv_i, u_i v_i, u_i v'_i, w'u_i u'_i/1 \leq i \leq n \}. \text{ Therefore } |V(G)| = 4n + 2 \text{ and } |E(G)| = 9n. \text{ Define a vertex labeling } f : V(G) \to \{ 0, 1 \} \text{ as follows:}
\]

\[
f(x) = \begin{cases} 
1 & \text{if } x = w' \\
1 & \text{if } x = u_i, i \in [n] \\
\frac{1+(-1)^{i+1}}{2} & \text{if } x = v'_i, i \in [n] \\
0 & \text{if } x = w \\
\frac{1+(-1)^i}{2} & \text{if } x = u_i, i \in [n] \\
0 & \text{if } x = u'_i, i \in [n]. 
\end{cases}
\]

Thus even \( v_f(1) = 2n + 1 \) and \( v_f(0) = 2n + 1 \). The induced edge labeling \( f^* : E(G) \to \{ 0, 1 \} \) is \( f^*(uv) = |f(u) - f(v)| \), for every edge \( e = uv \in E \). Therefore

\[
f^*(e) = \begin{cases} 
1 & \text{if } e = wu_i, i \in [n] \\
\frac{1+(-1)^{i+1}}{2} & \text{if } e = u_i v_i, i \in [n] \\
1 & \text{if } e \in \{ u'_i u_{i+1}, u'_i u'_{i+1} \}, i \in [n-1] \\
0 & \text{if } e = u_i u_{i+1}, i \in [n-1] \\
0 & \text{if } e \in \{ w'u_i, w'v'_i \}, i \in [n] \\
\frac{1+(-1)^i}{2} & \text{if } e \in \{ u'_i v'_i, u'v'_{i+1} \}, i \in [n] \\
0 & \text{if } e = u_n u_1 \\
1 & \text{if } e \in \{ u'_n u_1, u_n u'_1 \}. 
\end{cases}
\]

Thus \( e_f(1) = 9n - \frac{(1+(-1)^{n+1})}{2} \) and \( e_f(0) = 9n + \frac{(1+(-1)^{n+1})}{2} \). Therefore \( f \) satisfies the conditions \(|v_f(1) - v_f(0)| \leq 1\) and \(|e_f(1) - e_f(0)| \leq 1\). So, \( f \) admits cordial labeling on \( G \). Hence \( G \) is cordial.

\[ \text{Theorem 2.3. The graph obtained by duplicating all the edges other than spoke edges of the helm \( H_n \) is cordial.} \]

\[ \text{Proof. Let} \ V(H_n) = \{ w \} \cup \{ u_i, v_i/1 \leq i \leq n \} \text{ and } E(H_n) = \{ k_i = wu_i, m_i = u_i v_i/1 \leq i \leq n \} \cup \{ l_i = u_i u_{i+1}/1 \leq i \leq n-1 \} \cup \{ l_n = u_n u_1 \}. \text{ Let } G \text{ be the graph obtained by duplicating all the edges other than spoke edges in } H_n. \text{ For each } i \in 1, 2, ..., n, \text{ let } l'_i = a_i b_i \text{ and } m'_i = c_i d_i \text{ be the new edges of } G \text{ by duplicating } l_i \text{ and } m_i \text{ respectively. Then } V(G) = \{ w \} \cup \{ u_i, v_i, a_i, b_i, c_i, d_i/1 \leq i \leq n \} \text{ and } E(G) = \{ wu_i, wv_i, wc, a_i b_i, c_i d_i, a_i v_i, w_a, w_b, w_{b}/1 \leq i \leq n \} \cup \{ u_n u_1, c_n u_1, a_n b_1, u_n v_1, a_n d_1, u_{n-1} u_1, b_{n-1} u_{n-1} \} \cup \{ u_i u_{i+1}, c_i u_{i+1}, a_i v_{i+1}, b_i v_{i+1}, u_i a_{i+1}/1 \leq i \leq n-1 \} \cup \{ b_i u_{i+2}/1 \leq i \leq n-2 \}. \text{ Therefore } |V(G)| = 6n + 1 \text{ and } |E(G)| = 14n. \text{ Define a vertex labeling } f : V(G) \to \{ 0, 1 \} \text{ as follows:}
\]

\[
f(x) = \begin{cases} 
1 & \text{if } x = w \\
1 & \text{if } x \in \{ u_i, v_i, c_i \}, i \in [n] \\
0 & \text{if } x \in \{ a_i, b_i, d_i \}, i \in [n]. 
\end{cases}
\]
Thus \( v_f(1) = 3n + 1 \) and \( v_f(0) = 3n \). The induced edge labeling \( f^* : E(G) \to \{0, 1\} \) is \( f^*(uv) = |f(u) - f(v)| \), for every edge \( e = uv \in E \). Therefore

\[
f^*(e) = \begin{cases} 
1 & \text{if } e \in \{c_i d_i, a_i v_i, w_{i1}, w_{i2}\}, \ i \in [n]; \\
0 & \text{if } e \in \{w_{i1}, u_i v_i, w_{i2}, a_i b_i\}, \ i \in [n]; \\
0 & \text{if } e \in \{u_i u_{i+1}, c_i u_{i+1}, u_i c_i\}, \ i \in [n - 1]; \\
1 & \text{if } e \in \{b_i v_{i+1}, u_i a_{i+1}\}, \ i \in [n - 1]; \\
1 & \text{if } e = b_i v_{i+2}, \ i \in [n - 2]; \\
1 & \text{if } e \in \{b_{i-1} u_1, b_n u_2, b_n v_1, u_n a_1\}; \\
0 & \text{if } e \in \{u_n u_1, c_n u_1, u_n c_1\}.
\end{cases}
\]

Thus \( e_f(1) = 7n \) and \( e_f(0) = 7n \). Therefore \( f \) satisfies the conditions \( |v_f(1) - v_f(0)| \leq 1 \) and \( |e_f(1) - e_f(0)| \leq 1 \). So, \( f \) admits cordial labeling on \( G \). Hence \( G \) is cordial.

**Theorem 2.4.** The graph obtained by duplicating all the vertices of the closed helm \( CH_n \) is cordial.

**Proof.** Let \( V(CH_n) = \{w\} \cup \{u_i, v_i/1 \leq i \leq n\} \) and \( E(CH_n) = \{w u_i, u_i v_i/1 \leq i \leq n\} \cup \{u_n u_1, v_n v_1\} \cup \{u_i u_{i+1}, v_i v_{i+1}/1 \leq i \leq n - 1\} \). Let \( G \) be the graph obtained by duplicating all vertices of \( CH_n \). Let \( w', u'_1, u'_2, \ldots, v'_1, v'_2, \ldots, v'_n \) be the new vertices of \( G \) by duplicating \( w, u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n \) respectively. Then \( V(G) = \{w, w'\} \cup \{u_i, v_i, v'_i/1 \leq i \leq n\} \) and \( E(G) = \{u_n u_1, v_n v_1, v'_n v_1, u'_n u_1, u_n u_1\} \cup \{w u_i, u_i v_i, w' u_i, v'_i v_i, u'_i u_i, u_n v_i/1 \leq i \leq n\} \cup \{u_i u_{i+1}, v_i v_{i+1}, v'_i v_{i+1}, u'_i u_{i+1}, u_n u_{i+1}/1 \leq i \leq n - 1\} \). Therefore \( |V(G)| = 4n + 2 \) and \( |E(G)| = 12n \). Define a vertex labeling \( f : V(G) \to \{0, 1\} \) as follows:

\[
f(x) = \begin{cases} 
1 & \text{if } x = w'; \\
1 & \text{if } x \in \{u_i, v'_i\}, \ i \in [n]; \\
0 & \text{if } x = w; \\
0 & \text{if } x \in \{v_i, u'_i\}, \ i \in [n].
\end{cases}
\]

Thus \( v_f(1) = 2n + 1 \) and \( v_f(0) = 2n + 1 \). The induced edge labeling \( f^* : E(G) \to \{0, 1\} \) is \( f^*(uv) = |f(u) - f(v)| \), for every edge \( e = uv \in E \). Therefore

\[
f^*(e) = \begin{cases} 
1 & \text{if } e \in \{u_i v_i, w_{i1}\}, \ i \in [n]; \\
1 & \text{if } e \in \{v'_i v_{i+1}, u_i u_{i+1}, u'_i u_{i+1}, v_i v'_{i+1}\}, \ i \in [n - 1]; \\
0 & \text{if } e \in \{u_i u_{i+1}, v_i v_{i+1}\}, \ i \in [n - 1]; \\
0 & \text{if } e \in \{w' u_i, w u'_i, u_i v'_i, u'_i v_i\}, \ i \in [n]; \\
0 & \text{if } e \in \{u_n u_1, v_n v_1\}; \\
1 & \text{if } e \in \{v'_n v_1, v_n v'_1, a_n u_1, u_n u_1\}.
\end{cases}
\]

Thus \( e_f(1) = 6n \) and \( e_f(0) = 6n \). Therefore \( f \) satisfies the conditions \( |v_f(1) - v_f(0)| \leq 1 \) and \( |e_f(1) - e_f(0)| \leq 1 \). So, \( f \) admits cordial labeling on \( G \). Hence \( G \) is cordial.

**Theorem 2.5.** The graph obtained by duplicating all the outer rim vertices of the closed helm \( CH_n \) is cordial.

**Proof.** Let \( V(CH_n) = \{w\} \cup \{u_i, v_i/1 \leq i \leq n\} \) and \( E(CH_n) = \{w u_i, u_i v_i/1 \leq i \leq n\} \cup \{u_n u_1, v_n v_1\} \cup \{u_i u_{i+1}, v_i v_{i+1}/1 \leq i \leq n - 1\} \). Let \( G \) be the graph obtained by duplicating all the outer rim vertices in \( CH_n \). Let \( v'_1, v'_2, \ldots, v'_n \) be the new vertices of \( G \) by duplicating \( v_1, v_2, \ldots, v_n \) respectively. Then \( V(G) = \{w\} \cup \{u_i, v_i, v'_i/1 \leq i \leq n\} \) and \( E(G) = \{w u_i, u_i v_i, u_i v'_i/1 \leq i \leq n\} \cup \{u_i u_{i+1}, v_i v_{i+1}, v_i v'_{i+1}, u_i u'_{i+1}, u_i u_{i+1}/1 \leq i \leq n - 1\} \). Therefore
Thus $v_f(1) = \frac{3n + \left(1+(-1)^n\right)}{2}$ and $v_f(0) = \frac{3n + 1 + \left(1+(-1)^n\right)}{2}$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 
1 & \text{if } e \in \{wu, u,v_1\}, i \in [n]; \\
\frac{1+(-1)^i+1}{2} & \text{if } e = v'v_{i+1}, i \in [n-1]; \\
\frac{1+(-1)^i+1}{2} & \text{if } e = u_iu'_i, i \in [n]; \\
\frac{1+(-1)^i}{2} & \text{if } e = v_iu'_i, i \in [n-1]; \\
0 & \text{if } e \in \{v_iu_{i+1}, u_iu_{i+1}\}, i \in [n-1]; \\
0 & \text{if } e \in \{u_{n+1}, v_{n+1}\}; \\
\frac{1+(-1)^n}{2} & \text{if } e = v_nv'_n; \\
\frac{1+(-1)^n+1}{2} & \text{if } e = v'_nv_n.
\end{cases}$$

Thus $e_f(1) = \frac{7n + \left(1+(-1)^n+1\right)}{2}$ and $e_f(0) = \frac{7n - \left(1+(-1)^n+1\right)}{2}$. Therefore $f$ satisfies the conditions $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. So, $f$ admits cordial labeling on $G$. Hence $G$ is cordial.

**Theorem 2.6.** The graph obtained by duplicating all the vertices except the apex vertex of the closed helm $CH_n$ is cordial.

**Proof.** Let $V(CH_n) = \{w\} \cup \{u_i,v_i/1 \leq i \leq n\}$ and $E(CH_n) = \{wu_i, u_iu_i/1 \leq i \leq n\} \cup \{u_iu_{i+1}, v_iv_i+1/1 \leq i \leq n-1\}$. Let $G$ be the graph obtained by duplicating all the vertices except the apex vertex in $CH_n$. Let $u'_1, u'_2, ..., u'_n, v'_1, v'_2, ..., v'_n$ be the new vertices of $G$ by duplicating $u_1, u_2, ..., u_n, v_1, v_2, ..., v_n$ respectively. Then $V(G) = \{w\} \cup \{u_i, v_i, u'_i, v'_i/1 \leq i \leq n\}$ and $E(G) = \{wu_i, u_iu_i, u_iu'_i, v_iv_i, u'_i, v'_i/1 \leq i \leq n\} \cup \{u_iu_{i+1}, v_iv_{i+1}, u'_iu_{i+1}, u'_iv_{i+1}, u'_iu'_i+1, v'_iv'_i+1, u'_iu'_i+1, v'_iv'_i+1/1 \leq i \leq n-1\}$ and $E(G) = \{u_{n+1}, v_{n+1}, u'_n, v'_n, u'_n, v'_n, u'_n, v'_n\}$. Therefore $|V(G)| = 4n+1$ and $|E(G)| = 11n$. Define a vertex labeling $f : V(G) \to \{0,1\}$ as follows:

$$f(x) = \begin{cases} 
0 & \text{if } x = w; \\
1 & \text{if } x = u_i, i \in [n]; \\
0 & \text{if } x = v_i, i \in [n]; \\
\frac{1+(-1)^i+1}{2} & \text{if } x \in \{u'_i, v'_i\}, i \in [n].
\end{cases}$$

Thus $v_f(1) = 2n + \frac{1+(-1)^n+1}{2}$ and $v_f(0) = 2n + \frac{1+(-1)^n}{2}$. The induced edge labeling $f^* : E(G) \to \{0,1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore
Thus $e_f(1) = 11n + \frac{1 + (-1)^{n+1}}{2}$ and $e_f(0) = 11n - \frac{1 + (-1)^{n+1}}{2}$. Therefore $f$ satisfies the conditions $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. So, $f$ admits cordial labeling on $G$. Hence $G$ is cordial.

Theorem 2.7. The graph obtained by duplicating all the edges other than spoke edges of the closed helm $CH_n$ is cordial.

Proof. Let $V(CH_n) = \{v \} \cup \{u_i, v_i/1 \leq i \leq n\}$ and $E(CH_n) = \{j_i = wu_i, l_i = u_i, v_i/1 \leq i \leq n\} \cup \{k_i = u_i u_{i+1}, m_i = v_i v_{i+1} /1 \leq i \leq n - 1\} \cup \{k_n = u_n u_1, m_n = v_n v_1\}$. Let $G$ be the graph obtained by duplicating all the edges other than spoke edges in $CH_n$. For each $i \in 1, 2, ..., n$ let $k_i' = a_i b_i, \ell_i' = c_i d_i$ and $m_i' = e_i i_i$. Be the new edges of $G$ by duplicating $k_i, \ell_i$ and $m_i$, respectively. Then $V(G) = \{u \} \cup \{u_i, v_i, a_i, b_i, c_i, d_i, e_i, f_i/1 \leq i \leq n\}$ and $E(G) = \{wu_i, u_i v_i, c_i d_i, a_i b_i, v_i v_i, w_i b_i, w_i c_i, e_i f_i, e_i u_i /1 \leq i \leq n\} \cup \{b_i u_{i+2}, v_i f_{i+2} /1 \leq i \leq n - 2\} \cup \{u_i u_{i+1}, v_{i+1}, v_i e_i, v_i e_i, v_i a_i, d_i v_i, a_i d_i, u_i c_i, b_{i-1} u_1, b_{i+1}, v_{i-1} f_i, v_v f_2\}$. Therefore $|V(G)| = 10n + 1$ and $|E(G)| = 22n$. Define a vertex labeling $f : V(G) \rightarrow \{0, 1\}$ as follows:

$$f(x) = \begin{cases} 
1 & \text{if } x = w; \\
0 & \text{if } x \in \{u_i, v_i, e_i, f_i\}, i \in [n]; \\
1 & \text{if } x \in \{a_i, b_i, c_i, d_i\}, i \in [n].
\end{cases}$$

Thus $v_f(1) = 5n + 1$ and $v_f(0) = 5n$. The induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is $f^*(uv) = |f(u) - f(v)|$, for every edge $e = uv \in E$. Therefore

$$f^*(e) = \begin{cases} 
1 & \text{if } e \in \{wu_i, u_i v_i, c_i d_i\}, i \in [n]; \\
0 & \text{if } e \in \{u_i u_{i+1}, v_i v_{i+1}, b_i u_{i+2}, f_{i+2}\}, i \in [n - 2]; \\
1 & \text{if } e \in \{a_i b_i, w_i b_i, w_i c_i, e_i f_i, e_i u_i\}, i \in [n]; \\
0 & \text{if } e \in \{v_i e_i, v_i a_i, d_i v_i, a_i d_i, v_i c_i, b_{i-1} u_1, b_{i+1}, v_{i-1} f_i, v_v f_2\}; \\
1 & \text{if } e \in \{a_i b_i, a_i v_i, w_i a_i, w_i c_i, e_i f_i, e_i u_i\}, i \in [n]; \\
0 & \text{if } e \in \{u_i u_{i+1}, v_{i+1}, v_i e_i, v_i e_i, v_i a_i, d_i v_i, a_i d_i, u_i c_i, b_{i-1} u_1, b_{i+1}, v_{i-1} f_i, v_v f_2\}.
\end{cases}$$

Thus $e_f(1) = 11n$ and $e_f(0) = 11n$. Therefore $f$ satisfies the conditions $|v_f(1) - v_f(0)| \leq 1$ and $|e_f(1) - e_f(0)| \leq 1$. So, $f$ admits cordial labeling on $G$. Hence $G$ is cordial.
3. Conclusion

we have derived seven new results by investigating cordial labeling in the context of duplication in helm and closed helm.

More exploration is possible for other graph families and in the context of different graph labeling problems.

References