IFα**g Closed Sets in Intuitionistic Fuzzy Topological Spaces

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Abstract: In this paper, we introduce and study new classes of sets called IFα**g closed set in intuitionistic fuzzy topological spaces and IFα**g open set in intuitionistic fuzzy topological spaces. We focus upon the some of their basic properties.

Keywords: Intuitionistic fuzzy topology, IFα**g closed set in intuitionistic fuzzy topological spaces, IFα**g open set in intuitionistic fuzzy topological spaces.

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1. Introduction

The concept of fuzzy sets was introduced by Zadeh [11] and later Atanassov [1] generalized this idea to intuitionistic fuzzy sets using the notion of fuzzy sets. On the other hand Coker [2] introduced intuitionistic fuzzy topological spaces using the notion of intuitionistic fuzzy sets. In this paper we introduce IFα**g closed set in intuitionistic fuzzy topological spaces and IF**g open set in intuitionistic fuzzy topological spaces.

2. Preliminaries

Definition 2.1 ([1]). Let X be a non empty fixed set. An intuitionistic fuzzy set (IFS in short) A in X is an object having the form A = \{\langle x, \mu_A(x), \nu_A(x) \rangle / x \in X \} where the functions \mu_A(x) : X \rightarrow [0,1] and \nu_A(x) : X \rightarrow [0,1] denote the degree of membership (namely \mu_A(x)) and the degree of non-membership (namely \nu_A(x)) of each element x \in X to the set A, respectively, and 0 \leq \mu_A(x) + \nu_A(x) \leq 1 for each x \in X. We denote the set of all intuitionistic fuzzy sets in X, by IFS (X).

Definition 2.2 ([1]). Let A and B be IFSs of the form A = \{\langle x, \mu_A(x), \nu_A(x) \rangle / x \in X \} and B = \{\langle x, \mu_B(x), \nu_B(x) \rangle / x \in X \}. Then

(a). A \subseteq B if and only if \mu_A(x) \leq \mu_B(x) and \nu_A(x) \geq \nu_B(x) for all x \in X.

(b). A = B if and only if A \subseteq B and B \subseteq A.

(c). A^c = \{\langle x, \nu_A(x), \mu_A(x) \rangle / x \in X \}.

(d). A \cap B = \{\langle x, \mu_A(x) \land \mu_B(x), \nu_A(x) \lor \nu_B(x) \rangle / x \in X \}.

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(e). \( A \cup B = \{ (x, \mu_A(x) \lor \mu_B(x), \nu_A(x) \land \nu_B(x)) / x \in X \} \).

For the sake of simplicity, we shall use the notation \( A = \langle x, \mu_A, \nu_A \rangle \) instead of \( A = \{ (x, \mu_A(x), \nu_A(x)) / x \in X \} \). Also for the sake of simplicity, we shall use the notation \( A = \langle x, (\mu_A, \mu_B), (\nu_A, \nu_B) \rangle \) instead of \( A = \{ (x, (A/\mu_A, B/\mu_B), (A/\nu_A, B/\nu_B)) \} \).

The intuitionistic fuzzy sets 0ₜ = \{ (x, 0, 1) / x \in X \} and 1ₜ = \{ (x, 1, 0) / x \in X \} are respectively the empty set and the whole set of \( X \).

**Definition 2.3** ([2]). An intuitionistic fuzzy topology (IFT in short) on \( X \) is a family \( \tau \) of IFSs in \( X \) satisfying the following axioms.

(a). \( 0, 1 \in \tau \).

(b). \( G_1 \cap G_2 \in \tau \), for any \( G_1, G_2 \in \tau \).

(c). \( \bigcup_{i\in J} G_i \in \tau \) for any family \( \{ G_i / i \in J \} \subseteq \tau \).

In this case the pair \( (X, \tau) \) is called an intuitionistic fuzzy topological space (IFTS in short) and any IFS in \( \tau \) is known as an intuitionistic fuzzy open set (IFOS in short) in \( X \). The complement \( A^c \) of an IFOS \( A \) in an IFTS \( (X, \tau) \) is called an intuitionistic fuzzy closed set (IFCS in short) in \( X \).

**Definition 2.4.** An IFS \( A \) of an IFTS \( (X, \tau) \) is an

(a). Intuitionistic fuzzy interior of \( A \) [2] if \( \text{int}(A) = \bigcup \{ G / G \text{ is an IFOS in } X \text{ and } G \subseteq A \} \).

(b). Intuitionistic fuzzy closure of \( A \) [2] if \( \text{cl}(A) = \bigcap \{ K / K \text{ is an IFCS in } X \text{ and } A \subseteq K \} \).

(c). Intuitionistic fuzzy semi closed set [4] (IFSCS in short) if \( \text{int}(\text{cl}(A)) \subseteq A \).

(d). Intuitionistic fuzzy pre open set [4] (IFPOS in short) if \( A \subseteq \text{int}(\text{cl}(A)) \).

(e). Intuitionistic fuzzy pre closed set [4] (IFPCS in short) if \( \text{cl}(\text{int}(A)) \subseteq A \).

(f). Intuitionistic fuzzy pre open set [4] (IFPOS in short) if \( A \subseteq \text{int}(\text{cl}(A)) \).

(g). Intuitionistic fuzzy a-open set [4] (IF α OS in short) if \( A \subseteq \text{int}(\text{cl}(A)) \).

(h). Intuitionistic fuzzy a-closed set [4] (IF α CS in short) if \( \text{cl}(\text{int}(A)) \subseteq A \).

(i). Intuitionistic fuzzy γ-open set [5] (IF γ OS in short) if \( A \subseteq \text{int}(\text{cl}(A)) \cup \text{cl}(\text{int}(A)) \).

(j). Intuitionistic fuzzy γ-closed set [5] (IF γ CS in short) if \( \text{cl}(\text{int}(A)) \cap \text{cl}(\text{int}(A)) \subseteq A \).

(k). Intuitionistic fuzzy semi pre open set [4] (IFSPPOS in short) if there exists an IFPOS \( B \) such that \( B \subseteq A \subseteq \text{cl}(B) \).

(l). Intuitionistic fuzzy semi pre closed set [4] (IFSPCS in short) if there exists an IFPCS \( B \) such that \( \text{int}(B) \subseteq A \subseteq B \).

(m). Intuitionistic fuzzy regular open set [9] (IFROS in short) if \( A = \text{int}(\text{cl}(A)) \).

(n). Intuitionistic fuzzy regular closed set [9] (IFRCS in short) if \( A = \text{cl}(\text{int}(A)) \).

(o). Intuitionistic fuzzy generalized closed set [9] (IFGCS in short) if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is an IFOS in \( X \).

(p). Semi closure of \( A \) [7] (\( \text{scl}(A) \) in short) is defined as \( \text{scl}(A) = \bigcap \{ K / K \text{ is an IFSCS in } X \text{ and } A \subseteq K \} \).

(q). Semi interior of \( A \) [7] (\( \text{sint}(A) \) in short) is defined as \( \text{sint}(A) = \bigcup \{ K / K \text{ is an IFPOS in } X \text{ and } K \subseteq A \} \).
(r). Intuitionistic fuzzy generalized semi closed set [10] (IFGSCS in short) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq UR$ and $U$ is an IFOS in $X$.

(s). Regular open [7] if $A = \text{int}(\text{cl}(A))$.

(t). $\pi$ open [7] if $A$ is the union of regular open sets.

Note that for any IFS $A$ in $(X, \tau)$, we have $\text{cl}(A^c) = (\text{int}(A))^c$ and $\text{int}(A^c) = (\text{cl}(A))^c$.

3. $IF\alpha^{**}g$ Closed Sets in Intuitionistic Fuzzy Topological Spaces

In this section we introduce intuitionistic fuzzy $a^{**}g$ closed sets and study some of their properties.

**Definition 3.1** ([6]). An IFS $A$ is said to be an intuitionistic fuzzy $a^{**}g$ closed set (IF$a^{**}g$ CS in short) in $(X, \tau)$ if $a\text{cl}(A) \subseteq \text{cl}(\text{int}(U))$ whenever, $A \subseteq U$ and $U$ is an IFOS in $X$. The family of all IF$a^{**}g$ CSs of an IFTS $(X, \tau)$ is denoted by $IF\alpha^{**}g$ CS$(X)$.

**Example 3.2.** Let $X = \{a, b\}$ and let $\tau = \{0_-, \{x\}_-, \{x, y\}, \{y\}_+, \{y, z\}_+, \{z\}_+\}$ be an IFT on $X$. Then $\mu_{0_+}(a) = 0.7$, $\mu_{0_+}(b) = 0.6$, $\nu_{0_+}(a) = 0.1$ and $\nu_{0_+}(b) = 0.2$. Let us consider the IFS $A = \{x, (0.1, 0.7), (0.1, 0.2)\}$. Then $\mu_0(A) = 0.1$, $\mu_{0_+}(A) = 0.1$, $\nu_{0_+}(A) = 0.1$ and $\nu_0(A) = 0.1$. Hence, $A$ is an IF$g$ CS in $X$.

**Theorem 3.3.** Every IFCS in $(X, \tau)$ is an IF$g$ CS in $(X, \tau)$ but not conversely.

**Proof.** Assume that $A$ is an IFCS in $(X, \tau)$. Let us consider an IFS $A \subseteq U$ and $U$ be an IFOS in $(X, \tau)$. Since $a\text{cl}(A) \subseteq \text{cl}(\text{int}(U))$ and $A \subseteq U$, $A$ is an IFCS in $X$. Therefore, $A$ is an IF$g$ CS in $X$.

**Example 3.4.** Let $X = \{a, b\}$ and let $\tau = \{0_-, \{x\}_-, \{x, y\}, \{y\}_+, \{y, z\}_+, \{z\}_+\}$ be an IFT on $X$. Then $\mu_{0_+}(a) = 0.7$, $\mu_{0_+}(b) = 0.6$, $\nu_{0_+}(a) = 0.1$ and $\nu_{0_+}(b) = 0.2$. Let us consider the IFS $A = \{x, (0.1, 0.7), (0.1, 0.2)\}$. Clearly $A \subseteq 1_+$ and $\text{cl}(A) = \{x, (0.5, 0.6), (0.3, 0.1)\} \subseteq 1_+$. Hence, $A$ is an IF$g$ CS. But $A$ is not an IFCS in $(X, \tau)$.

**Theorem 3.5.** Every IFRCS in $(X, \tau)$ is an IF$g$ CS in $(X, \tau)$ but not conversely.

**Proof.** Let $A$ be an IFRCS in $(X, \tau)$. By definition, $A = \text{cl}(\text{int}(A))$. This implies $\text{cl}(A) = A$. That is $A$ is an IFCS in $X$. Therefore, $A$ is an IF$g$ CS in $X$.

**Example 3.6.** Let $X = \{a, b\}$ and let $\tau = \{0_-, \{x\}_-, \{x, y\}, \{y\}_+, \{y, z\}_+, \{z\}_+\}$ be an IFT on $X$. Then the IFS $A = \{x, (0.1, 0.7), (0.1, 0.2)\}$ is an IFRCS in $X$ but not an IFRCS in $X$.

**Theorem 3.7.** Every IF$\alpha CS$ in $(X, \tau)$ is an IF$g$ CS in $(X, \tau)$ but not conversely.

**Proof.** Let us consider an IFS $A \subseteq U$ and $U$ be an IFOS in $(X, \tau)$. Since $\text{cl}(\text{int}(A)) \subseteq A$, then $\text{int}(\text{cl}(A)) \subseteq A$. This implies $\text{cl}(A) \subseteq A = \text{int}(\text{cl}(A))$. Since, $A$ is an IFRCS in $X$. Hence, $A$ is an IF$g$ CS in $X$.

**Example 3.8.** Let $X = \{a, b\}$ and let $\tau = \{0_-, G_1, G_2, 1_+\}$, where $G_1 = \{x, (0.4, 0.2), (0.6, 0.7)\}$, $G_2 = \{x, (0.8, 0.8), (0.2, 0.2)\}$. Then the IFS $A = \{x, (0.4, 0.4), (0.5, 0.6)\}$ is an IF$g$ CS in $X$. But $A$ is not an IF$\alpha CS$ in $X$ because $\text{cl}(\text{int}(A)) = \{x, (0.6, 0.7), (0.4, 0.2)\} \subseteq A$.

**Theorem 3.9.** Every IFGCS in $(X, \tau)$ is an IF$g$ CS in $(X, \tau)$ but its converse may not be true in general.
Let $\alpha \in \mathbb{C}$.

Example 3.10. Let $X = \{a, b\}$ and let $\tau = \{0, G, 1\}$ be an IFT on $X$, where $G = \langle x, (0.1, 0.7), (0.2, 0) \rangle$. Then the IFS $A = \langle x, (0.1, 0), (0.3, 0.8) \rangle$ is an IFP CS but not an IFGCS in $X$, since $\text{cl}(A) \notin \text{cl}(U)$ even though $A \subseteq G$ and $G$ is an IFOS in $X$.

Theorem 3.11. Every IFPCS in $(X, \tau)$ is an IFGCS in $(X, \tau)$ but its converse may not be true in general.

Proof. Assume that $A$ is an IFPCS in $(X, \tau)$. Let an IFS $A \subseteq U$ and $U$ be an IFOS in $(X, \tau)$. By hypothesis, $\text{cl}(A) \subseteq U$. Clearly $\text{cl}(A) \subseteq \text{cl}(U)$. Whenever, $A \subseteq U$ and $U$ is an IFOS in $X$. Hence $A$ is an IFPCS in $X$.

Example 3.12. Let $X = \{a, b\}$ and let $\tau = \{0, G, 1\}$ be an IFT on $X$, where $G = \langle x, (0.2, 0.3), (0.4, 0.5) \rangle$. Then the IFS $A = \langle x, (0.1, 0), (0.3, 0), (0.5, 0.6) \rangle$ is an IFGCS in $X$ but $A$ is not an IFGCS in $X$, since $\text{cl}(A) \notin \text{cl}(U)$ even though $A \subseteq G$ and $G$ is an IFOS in $X$.

Remark 3.13. An IFPCS in $(X, \tau)$ is need not be an IFPCS in $(X, \tau)$.

Example 3.14. Let $X = \{a, b\}$ and $G = \langle x, (0, 0.8), (0.4, 0.1) \rangle$ and let $\tau = \{0, G, 1\}$ be an IFT on $X$. Then the IFS $A = \langle x, (0, 0.2), (0.5, 0.4) \rangle$ is an IFGCS in $X$ but $A$ is not an IFCS in $X$, since $\text{cl}(A) \notin \text{cl}(U)$ even though $A \subseteq G$ and $G$ is an IFOS in $(X, \tau)$.

Remark 3.15. An IFPCS in $X$ is closedness is independent of an IFP closedness.

Example 3.16. Let $X = \{a, b\}$ and $G = \langle x, (0, 0.8), (0.4, 0.1) \rangle$ and let $\tau = \{0, G, 1\}$ be an IFT on $X$. Then the IFS $A = \langle x, (0, 0.2), (0.6, 0.6) \rangle$ is an IFPCS but is not an IFPCS in $X$, as $\text{cl}(A) \notin \text{cl}(U)$ even though $A \subseteq G$ and $G$ is an IFT in $X$.

Example 3.17. Let $X = \{a, b\}$ and $G = \langle x, (0.2, 0.2), (0.5, 0.7) \rangle$ and let $\tau = \{0, G, 1\}$ be an IFT on $X$. Then the IFS $A = \langle x, (0.3, 0.2), (0.5, 0.4) \rangle$ is an IFPCS but not an IFPCS in $X$, as $\text{cl}(A) \notin A$.

Remark 3.18. An IFPCS is closedness is independent of an IFP closedness.

Example 3.19. Let $X = \{a, b\}$ and $G = \langle x, (0.1, 0.9), (0.6, 0.1) \rangle$ and let $\tau = \{0, G, 1\}$ be an IFT on $X$. Consider an IFS $A = \langle x, (0, 0.4), (0.7, 0.6) \rangle$ in $X$. Since $\text{cl}(A) \notin \text{cl}(U)$ even though $A \subseteq G$ and $G$ is an IFT in $X$, $A$ is not an IFPCS in $X$. But $A$ is an IFTS in $(X, \tau)$.

Example 3.20. Let $X = \{a, b\}$ and $G = \langle x, (0, 0.7), (0.3, 0.2) \rangle$ and let $\tau = \{0, G, 1\}$ be an IFT on $X$. Then the IFS $A = \langle x, (0.7, 0.8), (0.1, 0.2) \rangle$ is an IFPCS but not an IFTS in $X$ since $\text{int}(\text{cl}(A)) = 1 \notin A$.

Remark 3.21. An IFPCS is closedness is independent of an IFTS closedness.

Example 3.22. Let $X = \{a, b\}$ and $G = \langle x, (0, 0.3), (0.5, 0.6) \rangle$ and let $\tau = \{0, G, 1\}$ be an IFT on $X$. Then the IFS $A = \langle x, (0.2, 0.3), (0.5, 0.6) \rangle$ is an IFCS but not an IFCS in $X$, as $\text{cl}(A) \notin \text{cl}(U)$ even though $A \subseteq G$ and $G$ is an IFT in $X$.

Example 3.23. Let $X = \{a, b\}$ and $G = \langle x, (0.4, 0), (0.4, 0.8) \rangle$ and let $\tau = \{0, G, 1\}$ be an IFT on $X$. Then the IFS $A = \langle x, (0.6, 0.7), (0.1, 0) \rangle$ is an IFPCS but not an IFT in $X$, as $\text{cl}(A) \cap \text{int}(\text{cl}(A)) = A$. 

}\vspace{0.2cm}
Remark 3.24. The union of two IFα∗∗g C sets is an IFα∗∗g CS in $(X, \tau)$. 

Example 3.25. Let $X = \{a, b\}$ and $G = \langle x, (0.4, 0), (0.1, 1) \rangle$ and let $\tau = \{0_-, G, 1_\}$ be an IFT on $X$. Then the IFS $A = \langle x, (0.6, 0), (0.2, 1) \rangle$, $B = \langle x, (0.2, 1), (0.5, 0) \rangle$ are IFα∗∗g CS. Now $A \cup B = \langle x, (0.6, 1), (0.2, 0) \rangle$. Since $\alphacl(A \cup B) \subseteq int(cl(U))$, $A \cup B$ is an IFα∗∗g CS in $X$.

Theorem 3.26. Let $(X, \tau)$ be an IFTS. If $A$ is an IFS of $X$ then $\alphacl(A) = (aint(A))^\circ$.

Proof. By Theorem, $\alphacl(A) = A \cup cl(int(cl(A)))$. Replacing $A$ by $A^c$, we get $\alphacl(A^c) = A^c \cup cl(int(cl(A^c)))$. That is $\alphacl(A^c) = A^c \cup cl(int(A))^\circ$. This implies $\alphacl(A^c) = A^c \cup (int(cl(A))^\circ)$. That is $\alphacl(A^c) = (A \cap int(cl(A)))^\circ = (aint(A))^\circ$. 

Remark 3.27. The intersection of any two IFα∗∗g CS is not an IFα∗∗g CS in general as seen from the following example.

Example 3.28. Let $X = \{a, b\}$ and $G = \langle x, (0.5, 0), (0.1, 1) \rangle$ and let $\tau = \{0_-, G, 1_\}$ be an IFT on $X$. Then the IFS $A = \langle x, (0.2, 1), (0.7, 0) \rangle$, $B = \langle x, (0.6, 0), (0.3, 1) \rangle$ are IFα∗∗g CS. Now $A \cap B = \langle x, (0.2, 0), (0.7, 1) \rangle$. Since $\alphacl(A \cap B) \not\subseteq int(cl(U))$ even though $A \cap B \subseteq G$ and $G$ is an IFOS in $X$, $A \cap B$ is not an IFα∗∗g CS in $X$.

Theorem 3.29. Let $(X, \tau)$ be an IFTS. Then for every $A \in \text{IFα∗∗g CS}(X)$ and for every $B \in \text{IFS}(X)$, $A \subseteq B \subseteq \alphacl(A)$ implies $B \in \text{IFα∗∗g CS}$. 

Proof. Let an IFS $B \subseteq U$ and $U$ be an IFOS in $X$. Since $A \subseteq B$, $A \subseteq U$ and $A$ is an IFα∗∗g CS, $\alphacl(A) \subseteq int(cl(U))$. By hypothesis, $B \subseteq \alphacl(A)$, $\alphacl(B) \subseteq \alphacl(A) \subseteq int(cl(U))$. Therefore, $\alphacl(B) \subseteq int(cl(U))$. Hence B is an IFα∗∗g CS of X.

Theorem 3.30. If $A$ is an IFOS in $(X, \tau)$ and an IFα∗∗g CS in $(X, \tau)$, then $A$ is an IFαCS in $X$.

Proof. Let $A$ be an IFOS in $X$. Since $A \subseteq A$, by hypothesis, $\alphacl(A) \subseteq A = int(cl(A))$ by Theorem 3.5. But $A \subseteq \alphacl(A)$. Therefore, $\alphacl(A) = A$. Hence A is an IFαCS of X.

Theorem 3.31. The union of IFα∗∗g CS $A$ and $B$ is an IFα∗∗g CS in $(X, \tau)$, if $A$ and $B$ are IFCS in $(X, \tau)$.

Proof. Since $A$ and $B$ are IFCS in $X$, $cl(A) = A$ and $cl(B) = B$. Assume that $A$ and $B$ are IFα∗∗g CS in $(X, \tau)$. Let $A \cup B \subseteq U$ and $U$ be IFOS in $X$. Then $cl(int(cl(A \cup B))) = cl(int(A \cup B)) \subseteq cl(A \cup B) = A \cup B \subseteq U$. That is $\alphacl(A \cup B) \subseteq int(cl(U))$. Therefore, the union of $A$ and $B$ is an IFα∗∗g CS in $(X, \tau)$.

Theorem 3.32. Let $(X, \tau)$ be an IFTS and $A$ be an IFS in $X$. Then $A$ is an IFα∗∗g CS if and only if $\overline{A \overline{F}}$ implies $\alphacl(A) \overline{F}$ for every IFCS $F$ of $X$. 

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Proof. Necessity: Let $F$ be an IFCS in $X$ and let $A \subseteq F^c$. Then $A \subseteq F^c$ is an IFOS in $X$. Therefore $acl(A) \subseteq F^c$, by hypothesis. Hence $acl(A) \exists F$.

Sufficiency: Let $F$ be an IFCS in $X$ and let $A$ be an IFS in $X$. Then by hypothesis, $A \exists F^c$ implies $acl(A) \exists F$. Then $acl(A) \subseteq F^c$ whenever $A \subseteq F^c$ and $F^c$ is an IFOS in $X$. Hence $A$ is an $IFa^{**}g$ CS in $X$. 

\[ \square \]

Theorem 3.33. Let $(X, \tau)$ be an IFTS. Then $IFaO(X) = IFaC(X)$ if and only if every IFS in $(X, \tau)$ is an $IFa^{**}g$ CS in $X$.

Proof. Necessity: Suppose that $IFaO(X) = IFaC(X)$. Let $A \subseteq U$ and $U$ be an IFOS in $X$. This implies $acl(A) \subseteq acl(U)$. Since $U$ is an IFOS, $U$ is an $IFaOS$ in $X$. Since by hypothesis $U$ is an $IFaCS$ in $X$, $acl(U) = U$. This implies $acl(A) \subseteq int(cl(U))$. Therefore, $A$ is an $IFa^{**}g$ CS of $X$.

Sufficiency: Suppose that every IFS in $(X, \tau)$ is an $IFa^{**}g$ CS in $X$. Let $U \in IFO(X)$, then $U \in IFaO(X)$. Since $U \subseteq U$ and $U$ is IFOS in $X$. By hypothesis, $acl(A) \subseteq acl(U)$. But clearly $U \subseteq acl(U)$. Hence $U = acl(U)$. That is $U \in IFaC(X)$.

Hence $IFaO(X) \subseteq IFaC(X)$. Let $A \in IFaC(X)$ then $A^c$ is an $IFaOS$ in $X$. But $IFaO(X) \subseteq IFaC(X)$. Therefore, $A^c \in IFaC(X)$. Hence, $A \in IFaO(X)$. This implies, $IFaC(X) \subseteq IFaO(X)$. Thus $IFaO(X) = IFaC(X)$. 

\[ \square \]

Theorem 3.34. If $A$ is an IFOS and an $IFa^{**}g$ CS in $(X, \tau)$, then

1. $A$ is an IFROS in $X$
2. $A$ is an IFRCS in $X$.

Proof.

1. Let $A$ be an IFOS and an $IFa^{**}g$ CS in $X$. Then $acl(A) \subseteq A$. This implies, $cl(int(cl(A))) \subseteq A$. That is $int(cl(A)) \subseteq A$.

   Since $A$ is an IFOS, $A$ is an IFPOS in $X$. Hence $A \subseteq int(cl(A))$. Therefore $A = int(cl(A))$. Hence, $A$ is an IFROS in $X$.

2. Let $A$ be an IFOS and an $IFaGCS$ in $X$. Then $cl(int(cl(A))) \subseteq A$. That is $cl(int(A)) \subseteq A$.

   Since $A$ is an IFOS, $A$ is an IFSOS in $X$. Hence, $A \subseteq cl(int(A))$. Therefore $A = cl(int(A))$. Hence, $A$ is an IFRCS in $X$. 

\[ \square \]

4. $IFa^{**}g$ Open Sets in Intuitionistic Fuzzy Topological Spaces

In this section, we have introduced intuitionistic fuzzy $a^{**}g$ open sets and studied some of their properties.

Definition 4.1. An IFS $A$ is said to be an intuitionistic fuzzy $a^{**}g$ open set ($IFa^{**}g$ OS in short) in $(X, \tau)$, if the complement $A^c$ is an $IFa^{**}g$ CS in $X$. The family of all $IFa^{**}g$ OS of an IFTS $(X, \tau)$ is denoted by $IFa^{**}gO(X)$.

Theorem 4.2. For any IFTS $(X, \tau)$, every IFOS is an $IFa^{**}g$ OS but not conversely.

Proof. Let $A$ be an IFOS in $X$. Then $A^c$ is an IFS in $X$. By Theorem 3.3, $A^c$ is an $IFa^{**}g$ CS in $X$. Hence, $A$ is an $IFa^{**}g$ OS in $X$.

\[ \square \]

Example 4.3. Let $X = \{a, b\}$ and let $\tau = \{\emptyset, G, 1\}$ be an IFT on $X$, where $G = \{x, (0.3, 0.1), (0.5, 0.6)\}$. Then the IFS $A = \{x, (0.4, 0.5), (0.4, 0.3)\}$ is an $IFa^{**}g$ OS in $X$, but $A$ is not an IFOS in $X$.

Theorem 4.4. For any IFTS $(X, \tau)$, every IFROS is an $IFa^{**}g$ OS but not conversely.

Proof. Let $A$ be an IFROS in $X$. Then $A^c$ is an IFRCS in $X$. By Theorem 3.5, $A^c$ is an $IFa^{**}g$ CS in $X$. Hence, $A$ is an $IFa^{**}g$ OS in $X$.

\[ \square \]
Example 4.5. Let \( X = \{a, b\} \) and let \( \tau = \{0, G, 1\} \) be an IFT on \( X \), where \( G = \langle x, (0.1, 0.2), (0.5, 0.6) \rangle \). Then the IFS \( A = \langle x, (0.7, 0.6), (0.1, 0.1) \rangle \) is an IFA**g OS but not an IFROS in \( X \).

Theorem 4.6. In any IFTS \((X, \tau)\), every IFA**g OS is an IFA**g OS but not conversely.

Proof. Let \( A \) be an IFA*OS in \( X \). Then \( A^c \) is an IFA*CS in \( X \). By Theorem 3.7, \( A^c \) is an IFA**g CS in \( X \). Hence, \( A \) is an IFA**g OS in \( X \).

Example 4.7. Let \( X = \{a, b\} \) and let \( \tau = \{0, G, 1, 2\} \) be an IFT on \( X \), where \( G_1 = \langle x, (0.4, 0.2), (0.6, 0.7) \rangle \) and \( G_2 = \langle x, (0.8, 0.8), (0.2, 0.2) \rangle \). Then the IFS \( A = \langle x, (0.5, 0.6), (0.4, 0.4) \rangle \) is an IFA**g OS in \( X \) but \( A \) is not an IFA*OS in \( X \).

Theorem 4.8. For any IFTS \((X, \tau)\), every IFGOS is an IFA**g OS but its converse may not be true in general.

Proof. Let \( A \) be an IFGOS in \( X \). Then \( A^c \) is an IFGCS in \( X \). By Theorem 3.9, \( A^c \) is an IFA**g CS in \( X \). Hence, \( A \) is an IFA**g OS in \( X \).

Example 4.9. Let \( X = \{a, b\} \) and let \( \tau = \{0, G, 1\} \) be an IFT on \( X \), where \( G = \langle x, (0.1, 0.7), (0.2, 0.2) \rangle \). Then the IFS \( A = \langle x, (0.3, 0.8), (0.1, 0.1) \rangle \) is an IFA**g OS in \( X \) but \( A \) is not an IFGOS in \( X \), as \( \text{cl}(A^c) \not\subseteq G \) even though \( A^c \subseteq G \) and \( G \) is an IFOS in \((X, \tau)\).

Remark 4.10. An IFA**g OS in \((X, \tau)\) is not an IFGSOS in \((X, \tau)\).

Example 4.11. Let \( X = \{a, b\} \) and let \( \tau = \{0, G, 1, 2\} \) be an IFT on \( X \), where \( G = \langle x, (0.2, 0.3), (0.4, 0.5) \rangle \). Then the IFS \( A = \langle x, (0.5, 0.6), (0.1, 0.1) \rangle \) is an IFA**g OS in \( X \) but \( A \) is not an IFGSOS in \( X \), as \( \text{cl}(A^c) \not\subseteq G \) even though \( A^c \subseteq G \) and \( G \) is an IFOS in \( X \).

Theorem 4.12. For any IFTS \((X, \tau)\), every IFA**g OS is an IFGPOS but its converse may not be true in general.

Proof. Let \( A \) be an IFA**g OS in \( X \). Then \( A^c \) is an IFA**g CS in \( X \). By Theorem 3.13, \( A^c \) is an IFGPCS in \( X \). Hence, \( A \) is an IFGPOS in \( X \).

Example 4.13. Let \( X = \{a, b\} \) and let \( \tau = \{0, G, 1\} \) be an IFT on \( X \), where \( G = \langle x, (0.9, 0.8), (0.4, 0.1) \rangle \). Then the IFS \( A = \langle x, (0.6, 0.6), (0.0, 0.1) \rangle \) is an IFGPOS in \( X \) but \( A \) is not an IFA**g OS in \( X \), as \( \text{cl}(A^c) \not\subseteq G \) even though \( A^c \subseteq G \) and \( G \) is an IFOS in \( X \).

Theorem 4.14. Let \((X, \tau)\) be an IFTS. If \( A \) is an IFS of \( X \) then the following properties are equivalent:

(i) \( A \in \text{IFA**} gO(X) \).

(ii) \( V \subseteq \text{int}(\text{cl}(\text{int}(A))) \) whenever \( V \subseteq A \) and \( V \) is an IFCS in \( X \).

(iii) There exists IFOS \( G_1 \) such that \( G_1 \subseteq V \subseteq \text{int}(\text{cl}(G)) \), where \( G = \text{int}(A) \), \( V \subseteq A \) and \( V \) is an IFCS in \( X \).

Proof. (i) \( \Rightarrow \) (ii) Let \( A \in \text{IFA**} gO(X) \). Then \( A^c \) is an IFA**g CS in \( X \). Therefore \( \text{cl}(A^c) \subseteq U \) whenever \( A^c \subseteq U \) and \( U \) is an IFOS in \( X \). That is \( \text{cl}(\text{int}(\text{cl}(A^c))) \subseteq U \). Taking complement on both sides, we get \( \text{int}(\text{cl}(\text{int}(A^c)))^{\circ} = \text{int}(\text{cl}(\text{int}(A^c))) \subseteq U \). This implies \( U^c \subseteq \text{int}(\text{cl}(A^c)) \) whenever \( U^c \subseteq A \) and \( U \) is an IFCS in \( X \). Replacing \( U^c \) by \( V \), \( V \subseteq \text{int}(\text{cl}(A)) \) whenever \( V \subseteq A \) and \( V \) is an IFCS in \( X \).

(ii) \( \Rightarrow \) (iii) Let \( V \subseteq \text{int}(\text{cl}(A)) \) whenever \( V \subseteq A \) and \( V \) is an IFCS in \( X \). Hence \( \text{int}(V) \subseteq V \subseteq \text{int}(\text{cl}(A)) \). Then there exist IFOS \( G_1 \) in \( X \) such that \( G_1 \subseteq V \subseteq \text{int}(\text{cl}(G)) \) where \( G = \text{int}(A) \) and \( G_1 = \text{int}(V) \).
(iii) \(\Rightarrow\) (i) Suppose that there exists IFOS \(G_1\) such that \(G_1 \subseteq V \subseteq \text{int}(\text{cl}(G))\) where \(G = \text{int}(A)\), \(V \subseteq A\) and \(V\) is an IFCS in \(X\). It is clear that \((\text{int}(\text{cl}(G)))^\circ \subseteq V^\circ\). That is \((\text{int}(\text{int}(A)))^\circ \subseteq V^\circ\). This implies that \(\text{cl}(\text{int}(A)) = \text{cl}(\text{int}(\text{cl}(A))) \subseteq V^\circ\). Therefore \(A^\circ \subseteq V^\circ\) is IFOS in \(X\). Hence, \(\alpha cl(A^\circ) \subseteq V^\circ\). That is \(A^\circ\) is an \(IF\alpha^*g\) CS in \(X\). Which implies \(A \in IF\alpha^*gO(X)\).

\[\square\]

**Theorem 4.15.** Let \((X, \tau)\) be an IFTS. Then for every \(A \in IF\alpha^*gO(X)\) and for every \(B \in IF\alpha^*gO(X)\), \(\alpha int(A) \subseteq B \subseteq A\) implies \(B \in IF\alpha^*gO(X)\).

**Proof.** By hypothesis, \(\alpha int(A) \subseteq B \subseteq A\). Taking complement on both sides, we get \(A^\circ \subseteq B^\circ \subseteq (\alpha int(A))^\circ\). Let \(B^\circ \subseteq U\) and \(U\) be an IFOS in \(X\). Since \(A^\circ \subseteq B^\circ\), \(A^\circ \subseteq U\). Since \(A^\circ\) is an \(IF\alpha GCS\), \(\alpha cl(A^\circ) \subseteq U\). Also \(B^\circ \subseteq (\alpha int(A))^\circ = \alpha cl(A^\circ)\).

Therefore \(\alpha cl(B^\circ) \subseteq \alpha cl(A^\circ) \subseteq U\). Hence, \(B^\circ\) is an \(IF\alpha GCS\) in \(X\). This implies that \(B\) is an \(IF\alpha^*gOS\) of \(X\). That is \(B \in IF\alpha^*gO(X)\).

\[\square\]

**Remark 4.16.** The union of any two \(IF\alpha^*g\) OS in \((X, \tau)\) is an \(IF\alpha^*g\) OS in \((X, \tau)\).

**Example 4.17.** Let \(X = \{a, b\}\) and let \(G = \{(x, (0,4,0), (0,1,1))\}\). Then \(\tau = \{0, 1\}\) is an IFT on \(X\) and the IFS \(A = \{x, (0,2,1), (0,6,0), B = \{x, (0,5,0), (0,2,1)\}\}\) are \(IF\alpha^*g\) OS in \(X\) but \(A \cup B = \{(x, (0,1,0), (0,2,0))\}\) is an \(IF\alpha^*gOS\) in \(X\).

**Theorem 4.18.** An IFS \(A\) of an IFTS \((X, \tau)\) is an \(IF\alpha^*g\) OS if and only if \(F \subseteq \alpha int(A)\) whenever \(F \subseteq A\) and \(F\) is an IFCS in \(X\).

**Proof.** Necessity: Suppose \(A\) is an \(IF\alpha^*g\) OS in \(X\). Let \(F\) be an IFCS in \(X\) and \(F \subseteq A\). Then \(F^\circ\) is an IFOS in \(X\) such that \(A^\circ \subseteq F^\circ\). Since \(A^\circ\) is an \(IF\alpha^*g\) CS, we have \(\alpha cl(A^\circ) \subseteq F^\circ\). Hence \((\alpha int(A))^\circ \subseteq F^\circ\). Therefore \(F \subseteq \alpha int(A)\).

Sufficiency: Let \(A\) be an IFS in \(X\) and let \(F \subseteq \alpha int(A)\) whenever \(F\) is an IFCS in \(X\) and \(F \subseteq A\). Then \(A^\circ \subseteq F^\circ\) and \(F^\circ\) is an IFOS. By hypothesis, \((\alpha int(A))^\circ \subseteq F^\circ\), which implies \(\alpha cl(A^\circ) \subseteq F^\circ\). Therefore \(A^\circ\) is an \(IF\alpha^*g\) OS of \(X\). Hence \(A\) is an \(IF\alpha^*g\) OS of \(X\).

\[\square\]

**References**

