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Inverse Complementary Tree Domination Number of Graphs

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Abstract: A non-empty set $D \subseteq V$ of a graph is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality taken over all the minimal dominating sets of G. A dominating set D is called a complementary tree dominating set if the induced subgraph $< V - D >$ is a tree. The complementary tree domination number $\gamma_{ctd}(G)$ of G is the minimum cardinality taken over all minimal complementary tree dominating sets of G. Let D be a minimum dominating set of G. If $V - D$ contains a dominating set D', then D' is called the inverse dominating set of G w.r.t to D. The inverse domination number $\gamma'(G)$ is the minimum cardinality taken over all the minimal inverse dominating sets of G. In this paper, we define the notion of inverse complementary tree domination in graphs. Some results on inverse complementary tree domination number are established, Nordhaus-Gaddum type results are also obtained for this new parameter.

MSC: 05C69

Keywords: Dominating set, complementary tree dominating set, inverse complementary dominating set.

1. Introduction

Kulli V.R. et al. [1] introduced the concept of inverse domination in graphs. Let $G(V, E)$ be a simple, finite, undirected, connected graph with p vertices and q edges. A non-empty set $D \subseteq V$ of a graph is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality taken over all the minimal dominating sets of G. A dominating set D is called a complementary tree dominating set if the induced subgraph $< V - D >$ is a tree. The complementary tree domination number $\gamma_{ctd}(G)$ of G is the minimum cardinality taken over all minimal complementary tree dominating sets of G. Let D be a minimum dominating set of G. If $V - D$ contains a dominating set D', then D' is called the inverse dominating set of G w.r.t to D. The inverse domination number $\gamma'(G)$ is the minimum cardinality taken over all the minimal inverse dominating sets of G.

The cartesian product of two graphs G_1 and G_2 is a graph denoted by $G_1 \times G_2$, whose vertex set is $V(G_1) \times V(G_2)$. Two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if $[u_1 = v_1$ and u_2 adj $v_2]$ or $[u_2 = v_2$ and u_1 adj $v_1]$. The n-cube Q_n is defined recursively by $Q_1 = K_2$ and $Q_n = K_2 \times Q_{n-1}$. The n-Book graph is defined as the graph cartesian product $S_{m+1} \times P_2$ where S_m is a star graph and P_2 is the path on two vertices.

The purpose of this paper is to introduce the concept of inverse complementary tree domination in graphs. Let $D \subseteq V$ be a minimum complementary tree dominating (ctd) set of G. If $V - D$ contains a ctd set D' of D, then D' is called an inverse

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ctd set with respect to D. The inverse complementary tree domination number $\gamma'_{ctd}(G)$ of G is the minimum number of vertices in an inverse ctd set of G. In this paper, bounds on $\gamma'_{ctd}(G)$ are obtained and their exact values for some standard graphs are found. Nordhaus-Gaddum type results are also obtained for this parameter.

2. Results and Bounds

Here the exact values of $\gamma'_{ctd}(G)$ for some standard graphs and are given.

Remark 2.1.

(1). For any complete bipartite graph $K_{m,n}$ with $m, n \geq 2$, $\gamma'_{ctd}(K_{m,n}) = max(m, n)$.

(2). For any wheel W_n, $n \geq 5$, $\gamma'_{ctd}(W_n) = n - 3$.

(3). For $n \geq 3$, $\gamma'_{ctd}(K_1 \circ P_n) = \lfloor \frac{n}{2} \rfloor$.

(4). $\gamma'_{ctd}(B_n) = \begin{cases} 2 & \text{if } n = 1 \\ n & \text{if } n \geq 2 \end{cases}$ *where B_n is a book graph.*

(5). $\gamma'_{ctd}(Q_3) = 3$.

(6). $\gamma'_{ctd}(P_3 \times P_3) = 3$.

(7). $\gamma'_{ctd}(C_3) = 1$, $\gamma'_{ctd}(C_4) = 2$, $\gamma'_{ctd}(K_4) = 2$, $\gamma'_{ctd}(K_4 - e) = 2$.

(8). Inverse ctd sets will not exist for C_n, K_n, $n \geq 5$.

Example 2.2. *Consider the graph given in figure 1.*

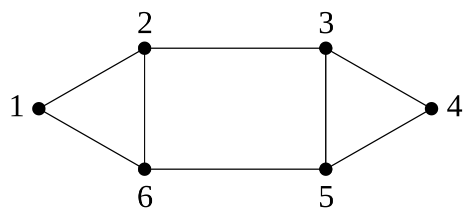

Figure 1.

$D = \{1, 3\}$ is a minimum ctd-set. $D' = \{2, 4\} \subseteq V(G) - D$ is a minimum inverse ctd-set and hence $\gamma_{ctd}(G) = 2 = \gamma'_{ctd}(G)$.

Proposition 2.3. *There will not exist inverse ctd sets for graphs with pendant vertices.*

Proof. Any ctd set of G contains all the pendant vertices of G. If $D' \subseteq V - D$ happens to be an inverse ctd set, then $V - D'$ will not be a tree, since it will contain isolated vertices. Hence the proposition follows. □

Hereafter, we consider the graphs G with $\delta(G) \geq 2$ and are not C_n, K_n, $n \geq 5$.

Theorem 2.4. *For any graph G, $\gamma'(G) \leq \gamma'_{ctd}(G)$.*

Proof. Since every inverse complementary tree dominating set of G is an inverse dominating set of G, we have $\gamma'(G) \leq \gamma'_{ctd}$. □

Theorem 2.5. *For a graph G, $\gamma_{ctd}(G) + \gamma'_{ctd}(G) \leq p$.*

Proof. We have, $\gamma(G)+\gamma'(G)\le p$ [1]. Since $\gamma(G)\le\gamma_{ctd}(G)$ and $\gamma'(G)\le\gamma'_{ctd}(G)$ and $\gamma(G)+\gamma'(G)\le p$ the results follows. This bound is attained, if $G\cong C_4$, K_4, K_4-e. □

Remark 2.6. *For any connected graph G*

$$\left\lceil\frac{p}{\Delta(G)+1}\right\rceil\le\gamma_{ctd}(G)$$

since $\gamma(G)\le\gamma_{ctd}(G)\le\gamma'_{ctd}(G)$

$$\left\lceil\frac{p}{\Delta(G)+1}\right\rceil\le\gamma'_{ctd}(G)$$

Equality holds if $G\cong C_4$ *and* K_4.

Theorem 2.7. *For any connected graph G*

$$\gamma'_{ctd}(G)\le\left\lceil\frac{\Delta(G)p}{\Delta(G)+1}\right\rceil$$

Proof. Since

$$\gamma_{ctd}(G)+\gamma'_{ctd}(G)\le p$$
$$\gamma'_{ctd}(G)\le p-\gamma_{ctd}(G)$$

But

$$\gamma'_{ctd}(G)\ge\left\lceil\frac{p}{\Delta(G)+1}\right\rceil$$
$$\gamma'_{ctd}(G)\le p-\left\lceil\frac{p}{\Delta(G)+1}\right\rceil=\left\lceil\frac{p\Delta(G)}{\Delta(G)+1}\right\rceil$$

Equality holds, if $G\cong C_4$. □

Remark 2.8. *Let D be a* γ_{ctd}*-set of G such that* $|D|=|V-D|$ *and* $<V-D>$ *is a dominating set. Then* $\gamma_{ctd}(G)=\gamma'_{ctd}(G)$.

Theorem 2.9. *For any* (p,q) *connected graph G,* $\gamma'_{ctd}(G)\le 2(q-p+1)$.

Proof. $\gamma_{ctd}(G)\ge 3p-2q-2$ by [3]. But, $\gamma_{ctd}(G)+\gamma'_{ctd}(G)\le p\Rightarrow p-\gamma_{ctd}(G)\le 2(q-p+1)$. This bound is attained, if $G\cong C_4$, since $\gamma'_{ctd}(C_4)=2=2(q-p+1)$. □

Theorem 2.10. *For any* (p,q) *connected graph G,* $\gamma'_{ctd}(G)\le p-2$.

Proof. If $\gamma'_{ctd}(G)=p-1$, then $\gamma_{ctd}(G)=1$. This occurs if and only if $G\cong T+K_1$, where T is a tree. Let $K_1=\{v\}$,

$$\gamma'_{ctd}(G)=p-1\Rightarrow \text{the set all the } p-1 \text{ vertices of } G \text{ other than } v \text{ is a ctd set.}$$
$$\Rightarrow G \text{ is a star on } p \text{ vertices.}$$

which is not possible. Hence $\gamma'_{ctd}(G)\le p-2$. □

In the following, the connected graph G for which $\gamma'_{ctd}(G)=p-2$ and $\gamma'_{ctd}(G)=1$ are characterized.

Theorem 2.11. $\gamma'_{ctd}(G)=p-2$ $(p\ge 5)$ *if and only if radius* $r(G)=1$ *and there exists a minimum connected ctd set D of cardinality 2 such that the vertices of D are the only central vertices of G.*

Proof. Let $r(G) = 1$ and there exist a minimum ctd set D with two adjacent vertices u, v such that u, v are the only central vertices. Then $V(G) - D = V(G) - \{u, v\}$ is a tree and any vertex w $(\neq u, v)$ is adjacent to both the vertices u and v and $V(G) - \{u, v\}$ is a minimum inverse ctd set. Hence, $\gamma'_{ctd}(G) = p - 2$.

Conversely, let $\gamma'_{ctd}(G) = p - 2$. Let D be a minimum ctd set. Then there exists a minimum inverse ctd set $D' \subseteq V - D$ with $p - 2$ vertices. Then $\gamma_{ctd}(G) \leq 2$. But, $\gamma_{ctd}(G) = 1$ if and only if $G \cong T + K_1$, where T is a tree, which is not possible. Therefore, $\gamma_{ctd}(G) = 2$ and D contains only two vertices say u, v. Also, $D' = V - D$ and $< V - D >$ is a tree. Since, $V - D'$ $(= D)$ is also a tree, the two vertices in D are adjacent.

If $r(G) \geq 2$, then either there exists no ctd set or any proper subset of $V - D$ will not be a ctd set. If $r(G) = 1$ and there exists exactly one central vertex, then also the above is true. If $r(G) = 1$ and there exist more than two central vertices, then $V - D$ will contain atleast one cycle. Therefore, $r(G) = 1$ and G contains exactly two central vertices. □

Theorem 2.12. *$\gamma'_{ctd}(G) = 1$ if and only if $G \cong K_2 + mK_1$, $m \geq 1$.*

Proof. Let $G \cong K_2 + mK_1$. Then a set containing a vertex in K_2 will form a minimum inverse ctd set. Hence, $\gamma'_{ctd}(G) = 1$. Conversely, assume $\gamma'_{ctd}(G) = 1$. Since $\gamma_{ctd}(G) \leq \gamma'_{ctd}(G)$, $\gamma_{ctd}(G) = 1$. Let $D = \{v\}$ be a minimum ctd set. Then $V - D$ is a tree and each vertex in $V - D$ is adjacent to v. Let $D' = \{u\}$ be an inverse ctd set of G w.r.t D. Then, $V - D'$ is a tree. Then u is adjacent to each vertex in $V - D'$. Let $u_1, u_2 \in V - \{u, v\}$. If u_1, u_2 are adjacent, then $< \{u_1, u_2, v\} >$ forms a triangle. Therefore, u_1 and u_2 are not adjacent. That is, any two vertices in $V - (u, v)$ are not adjacent. Hence, both u and v are adjacent to all the vertices in $V - \{u, v\}$ and $V - \{u\}$ and $V - \{v\}$ are stars. That is, $K_2 + mK_1$, $m \geq 1$. □

Theorem 2.13. *If G is a co-connected graph and $\delta(G) \geq 1$, then*

(1). $4 \leq \gamma'_{ctd}(G) + \gamma'_{ctd}(\overline{G}) \leq 2(p - 4)$.

(2). $4 \leq \gamma'_{ctd}(G).\gamma'_{ctd}(\overline{G}) \leq (p - 4)^2$.

Proof. If $\gamma'_{ctd}(G) = 1$, then $G \cong K_2 + mK_1$, $m \geq 1$. In this case, $\overline{G}$ is disconnected. Hence, $\gamma'_{ctd}(G) \geq 2$. Therefore, $\gamma'_{ctd}(G) + \gamma'_{ctd}(\overline{G}) \geq 4$. Upper bound follows, since $\gamma'_{ctd}(G) \leq p - 2$. □

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