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A Fixed Point Theorem for Four Self Maps on a Multiplicative Metric Space and Consequences

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Abstract: In this paper a fixed point theorem for a four self maps on a complete multiplicative metric space is proved and a fixed point theorem for six self maps is obtained as a corollary. Incidentally we obtain the result of Nisha Sharma et al. [8] as a corollary.

MSC: 54H25, 47H10.

Keywords: Multiplicative metric space, Multiplicative contraction, Commuting mappings, Weakly compatible maps and Common fixed point.

1. Introduction and Preliminaries

The study of fixed point theory become a subject of great interest due to its applications in Mathematics as well as in other areas of research. There are many researchers who have worked in fixed point theory of contractive mapping see [4, 11]. In [4], Banach presented a most out standing result concerning to contraction mapping. This famous result is known as Banach contraction principle. In [6], L. G.Huang, X. Zhang proved the contraction mapping principle in cone metric space. In 2012, Ozavsar and Cevikel [7] introduced the concept of multiplicative contraction mapping and proved some fixed point theorems of such mappings on a complete multiplicative metric space. They also gave some topological properties of the relevant multiplicative metric space. Recently Nisha Sharma et al. [8] studied related fixed point theorems for commuting and weakly compatible maps in a complete multiplicative metric spaces.

In this paper we proved the existence and uniqueness of fixed points of weakly compatible and commuting maps in a complete multiplicative metric space, which are improvements of the result of Nisha Sharma et al. [8].

Definition 1.1 ([7])**.** *Let X be a nonempty set. A multiplicative metric is a mapping $d : X \times X \to \mathbb{R}^+$ satisfying the following conditions:*

(1). $d(x, y) \geq 1$ for all $x, y \in X$ and $d(x, y) = 1$, if and only if $x = y$.

(2). $d(x, y) = d(y, x)$ for all $x, y \in X$.

(3). $d(x, y) \leq d(x, z).d(z, y)$ for all $x, y, z \in X$. (Multiplicative triangle inequality)

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Also (X,d) is called a multiplicative metric space.

Example 1.2 ([7]). *Let $d^* : (\mathbb{R}^+)^n \times (\mathbb{R}^+)^n \to \mathbb{R}^+$ be defined as follows $d^*(x,y) = \mid \frac{x_1}{y_1} \mid^* . \mid \frac{x_2}{y_2} \mid^* \cdots \mid \frac{x_n}{y_n} \mid^*$, where $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in \mathbb{R}^+$ and $\mid . \mid^* : \mathbb{R}^+ \to \mathbb{R}^+$ is defined $\mid a \mid^* = \begin{cases} a & if\ a \geq 1 \\ \frac{1}{a} & if\ a \leq 1 \end{cases}$. Then $((\mathbb{R}^+)^n, d^*)$ is a multiplicative metric space.*

Definition 1.3 (Multiplicative convergence [7]). *Let (X,d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. If for every multiplicative open ball $B_\epsilon(x) = \{y/d(x,y) < \epsilon\}$, $\epsilon > 1$ there exists a natural number N such that for $n \geq N$, $x_n \in B_\epsilon(x)$, the sequence $\{x_n\}$ is said to be multiplicative converging to x, denoted by $x_n \to x \quad (n \to \infty)$.*

Definition 1.4 ([7]). *Let (X,d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. the sequence $\{x_n\}$ is called a multiplicative Cauchy sequence if, for each $\epsilon > 1$, there exists $N \in \mathbf{N}$ such that $d(x_n, x_m) < \epsilon$, for all $m, n \geq N$.*

Definition 1.5 ([7]). *Let (X,d) be a multiplicative metric space. A mapping $f : X \to X$ is called a multiplicative contraction if there exists a real constant $\lambda \in [0,1)$ such that $d(fx, fy) \leq d(x,y)^\lambda$ for all $x, y \in X$.*

Definition 1.6 (Multiplicative continuity [7]). *Let (X, d_X) and (Y, d_Y) be two multiplicative metric spaces and $f : X \to Y$ be a function. If for every $\epsilon > 1$, there exists $\delta > 1$ such that $f(B_\delta(x)) \subset B_\epsilon(f(x))$, then we call f multiplicative continuous at $x \in X$.*

Definition 1.7 ([7]). *Let (X,d) be a multiplicative metric space. we call (X,d) is complete if every multiplicative Cauchy sequence in X is multiplicative convergent to some $x \in X$.*

Definition 1.8 ([7]). *Let S, T be self maps of a multiplicative metric space (X,d), then S, T are said to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = lim_{n\to\infty} Tx_n = z$ for some $z \in X$.*

Definition 1.9 ([3]). *Two self maps of multiplicative metric space S, T of a non empty set X are said to be weakly compatible is $STx = TSx$ whenever $Sx = Tx$.*

Recently Nisha Sharma et al. [8] proved the following fixed point theorem for commuting and weakly compatible maps in a complete multiplicative metric space.

Theorem 1.10 ([8]). *Let (X,d) be a complete multiplicative metric space and P, Q, R, S, T and U be self maps of X satisfying the following conditions*

(1). $TU(X) \subseteq P(X)$ and $RS(X) \subseteq Q(X)$ and

(2). $d(RSx, TUy) \leq (d(Px, Qy).d(Px, RSx).d(Qy, TUy).d(Px, TUy).d(Qy, RSx).d(TUy, RSx))^{\frac{\lambda}{3}}$ for all $x, y \in X$, $\lambda \in [0, \frac{1}{4})$ is a constant.

Assume that the pairs (TU, Q), (RS, P) are weakly compatible. Pairs (T, U), (T, Q), (U, S), (R, S), (R, P) and (S, P) are commuting pairs of maps. Then P, Q, R, S, T and U have a unique common fixed point in X.

2. Main Result

In this section we improve Theorem 1.10 (Nisha Sharma et al. [8]) by allowing λ in $[0, \frac{1}{2})$. Now we state and prove our first result.

Theorem 2.1. *Let (X,d) be a complete multiplicative metric space and P, Q, A, and B be self maps of X satisfying the following conditions*

(1). $A(X) \subseteq P(X)$ and $B(X) \subseteq Q(X)$ and

(2). $d(Bx,Ay) \leq \left\{ \begin{array}{c} d(Px,Qy).d(Px,Bx).d(Qy,Ay) \\ d(Px,Ay).d(Qy,Bx).d(Ay,Bx) \end{array} \right\}^{\frac{\lambda}{3}}$ for all $x,y \in X$, $\lambda \in [0,\frac{1}{2})$ is a constant.

Assume that the pairs (A,Q) and (B,P) are weakly compatible. Suppose either $P(X)$ or $Q(X)$ is closed. Then P,Q,A and B have a unique common fixed point in X.

Proof. Let $x_0 \in X$, by (i) we can define inductively a sequence $y_n \in X$ such that $y_{2n} = Bx_{2n} = Qx_{2n+1}$ and $y_{2n+1} = Ax_{2n+1} = Px_{2n+2}$ for all $n = 1,2,3,....$ Then

$$
\begin{aligned}
d(y_{2n},y_{2n+1}) &= d(Bx_{2n},Ax_{2n+1}) \\
&\leq \left\{ \begin{array}{c} d(Px_{2n},Qx_{2n+1}).d(Px_{2n},Bx_{2n}).d(Qx_{2n},Ax_{2n}) \\ d(Px_{2n},Ax_{2n+1}).d(Qx_{2n+1}\text{y},Bx_{2n}).d(Ax_{2n+1},Bx_{2n}) \end{array} \right\}^{\frac{\lambda}{3}} \\
&= \left\{ \begin{array}{c} d(Ax_{2n-1},Bx_{2n}).d(Ax_{2n-1},Bx_{2n}).d(Bx_{2n},Bx_{2n}) \\ d(Ax_{2n-1},Ax_{2n+1}).d(Bx_{2n}\text{y},Bx_{2n}).d(Ax_{2n+1},Bx_{2n}) \end{array} \right\}^{\frac{\lambda}{3}} \\
&\leq \left\{ \begin{array}{c} d(y_{2n-1},y_{2n}).d(y_{2n-1},y_{2n}).d(y_{2n},y_{2n}) \\ d(y_{2n-1},y_{2n+1}).d(y_{2n}\text{y},y_{2n}).d(y_{2n+1},y_{2n}) \end{array} \right\}^{\frac{\lambda}{3}} \\
&= \left\{ d^2(y_{2n-1},y_{2n}).d(y_{2n-1},y_{2n+1}).d^2(y_{2n+1},y_{2n}) \right\}^{\frac{\lambda}{3}} \\
&\leq \left\{ d^3(y_{2n-1},y_{2n}).d^3(y_{2n},y_{2n+1}) \right\}^{\frac{\lambda}{3}} \\
\therefore d(y_{2n},y_{2n+1}) &\leq d(y_{2n-1},y_{2n})^{\lambda}.d(y_{2n},y_{2n+1})^{\lambda} \\
\therefore d(y_{2n},y_{2n+1}) &\leq d(y_{2n-1},y_{2n})^{\frac{\lambda}{1-\lambda}} = d(y_{2n-1},y_{2n})^{h}. \ \ (\text{write } \frac{\lambda}{1-\lambda} = h) \\
d(y_{2n+1},y_{2n+2}) &= d(y_{2n+2},y_{2n+1}) = d(Bx_{2n+2},Ax_{2n+1}) \\
&\leq \left\{ \begin{array}{c} d(Px_{2n+2},Qx_{2n-1}).d(Px_{2n+2},Bx_{2n+2}).d(Qx_{2n+1},Ax_{2n+1}) \\ d(Px_{2n+2},Ax_{2n+1}).d(Qx_{2n+1}\text{y},Bx_{2n+2}).d(Ax_{2n+1},Bx_{2n+2}) \end{array} \right\}^{\frac{\lambda}{3}} \\
&\leq \left\{ \begin{array}{c} d(y_{2n+1},y_{2n}).d(y_{2n+1},y_{2n+2}).d(y_{2n},y_{2n+1}) \\ d(y_{2n+1},y_{2n+1}).d(y_{2n}\text{y},y_{2n+2}).d(y_{2n+1},y_{2n+2}) \end{array} \right\}^{\frac{\lambda}{3}} \\
&\leq \left\{ d^3(y_{2n},y_{2n+1}).d^3(y_{2n+1},y_{2n+2}) \right\}^{\frac{\lambda}{3}} \\
\therefore d(y_{2n+1},y_{2n+2}) &\leq d(y_{2n},y_{2n+1})^{\frac{\lambda}{1-\lambda}} = d(y_{2n-1},y_{2n})^{h}. \ \ (\text{write } \frac{\lambda}{1-\lambda} = h) \\
\therefore d(y_{n+1},y_n) &\leq d(y_n,y_{n-1})^{h} \\
&\leq [d(y_{n-2},y_{n-1})^{h}]^{h} \\
&= d(y_{n-1},y_{n-2})^{h^2} \\
&\leq \ldots \\
&= d(y_1,y_0)^{h^n}
\end{aligned}
$$

$$\therefore d(y_{n+1}, y_n) \leq d(y_1, y_0)^{h^n} \rightarrow \quad 1 \text{ as } n \rightarrow \infty \ (\because \ h < 1)$$

$$\therefore d(y_{n+1}, y_n) \rightarrow 1.$$

We show that $\{y_n\}$ is a multiplicative Cauchy sequence in X.

$$\begin{aligned} d(y_n, y_{n+k}) &\leq d(y_n, y_{n+1}).d(y_{n+1}, y_{n+2}) \dots d(y_{n+k-1}, y_{n+k}) \\ &\leq d(y_0, y_1)^{h^n}.d(y_0, y_1)^{h^{n+1}} \dots d(y_0, y_1)^{h^{n+k-1}} \\ &= [d(y_0, y_1)]^{h^n + h^{n+1} + h^{n+2} + \dots + h^{n+k-1}} \\ &= d(y_1, y_0)^{\frac{h^n}{1-h}} \rightarrow \quad 1 \text{ as } n \rightarrow \infty, \ k \rightarrow \infty \quad (\because \ h < 1) \end{aligned}$$

Therefore $d(y_n, y_{n+k}) \rightarrow 1$. Therefore $\{y_n\}$ is a multiplicative Cauchy sequence in X. Since X is Complete multiplicative metric space, so there exists $r \in X$, such that $y_n \rightarrow r$. i.e., $d(y_n, r) \rightarrow 1$. Therefore $d(y_{2n}, r) \rightarrow 1$ and $d(y_{n+1}, r) \rightarrow 1$ i.e., $d(Bx_{2n}, r) \rightarrow 1$ and $d(Ax_{2n+1}, r) \rightarrow 1$ i.e., $d(Qx_{2n+1}, r) \rightarrow 1$ and $d(Px_{2n+2}, r) \rightarrow 1$. Without loss of generality, suppose $Q(X)$ is closed, and $B(X) \subseteq Q(X)$, so there exists some $u \in X$ such that $Qu = r$. Now

$$\begin{aligned} d(Bx_{2n}, Au) &\leq \left\{ \begin{matrix} d(Px_{2n},Qu).d(Px_{2n},Bx_{2n}).d(Qu,Au) \\ d(Px_{2n},Au).d(Quy,Bx_{2n}).d(Au,Bx_{2n}) \end{matrix} \right\}^{\frac{\lambda}{3}} \\ &= \left\{ \begin{matrix} d(Px_{2n},r).d(Px_{2n},Bx_{2n}).d(r,Au) \\ d(Px_{2n},Au).d(ry,Bx_{2n}).d(Au,Bx_{2n}) \end{matrix} \right\}^{\frac{\lambda}{3}} \end{aligned}$$

On letting $n \rightarrow \infty$

$$d(r, Au) \leq \left\{ \begin{matrix} d(r, r).d(r, r).d(r, Au) \\ d(r, Au).d(r, r).d(Au, r) \end{matrix} \right\}^{\frac{\lambda}{3}}$$

Therefore $d(r, Au) \leq (d^3(r, Au))^{\frac{\lambda}{3}} = d(r, Au)^{\lambda} < d(r, Au)$, a contradiction if $r \neq Au$. Therefore $Au = r$. Therefore $Au = Qu = r$. Since (A, Q) are weakly Compatible, $QAu = AQu \Rightarrow Qr = Ar$. Therefore r is a coincident point of A and Q. Now

$$d(Bx_{2n}, Ar) \leq \left\{ \begin{matrix} d(Px_{2n},Qr).d(Px_{2n},Bx_{2n}).d(Qr,Ar) \\ d(Px_{2n},Ar).d(Qry,Bx_{2n}).d(Ar,Bx_{2n}) \end{matrix} \right\}^{\frac{\lambda}{3}}$$

On letting $n \rightarrow \infty$

$$\begin{aligned} d(r, Ar) &\leq \left\{ \begin{matrix} d(r, Qr).d(r, r).d(Qr, Ar) \\ d(r, Ar).d(Qr, r).d(Ar, r) \end{matrix} \right\}^{\frac{\lambda}{3}} \\ &\leq \left\{ d^2 \ (r, Qr).d^2 \ (Ar, r) \ .d(Qr, Ar) \right\}^{\frac{\lambda}{3}} \\ \therefore d(r, Ar) &\leq \left\{ d^2 \ (r, Ar).d^2 \ (r, Ar) \right\}^{\frac{\lambda}{3}} \end{aligned}$$

Therefore $d(r, Ar) \leq d(r, Ar)^{\frac{4\lambda}{3}} < d(r, Ar)$, a contradiction if $r \neq Ar$. Therefore $Ar = r$. Therefore $Ar = Qr = r$. Therefore r is a fixed point of A and Q. Now $Au = r \Rightarrow r \in A(X) \Rightarrow r \in P(X)$. So there exists $v \in X$ such that $r = Pv$,

$$d(Bv, Au) \leq \left\{ \begin{matrix} d(Pv, Qu).d(Pv, Bv).d(Qu, Au) \\ d(Pv, Au).d(Qu, Bv).d(Au, Bv) \end{matrix} \right\}^{\frac{\lambda}{3}}$$

$$= \left\{ \begin{array}{c} d(r,r).d(r,Bv).d(r,r) \\ d(r,r).d(r,Bv).d(r,Bv) \end{array} \right\}^{\frac{\lambda}{3}}$$

$$[d^3(r,Bv)]^{\frac{\lambda}{3}} = d(r,Bv)^{\lambda}$$

Therefore $d(Bv,r) = d(Bv,Au) \leq d(r,Bv)^{\lambda} < d(r,Bv)$, a contradiction if $r \neq Bv$. Therefore $Bv = r$. Since (P,B) are weakly Compatible,

$PBv = BPv \Rightarrow Pr = Br$,

$$\begin{aligned} d(Br,r) &= d(Br,Ar) \\ &\leq \left\{ \begin{array}{c} d(Pr,Qr).d(Pr,Br).d(Qr,Ar) \\ d(Pr,Ar).d(Qr,Br).d(Ar,Br) \end{array} \right\}^{\frac{\lambda}{3}} \\ &= \left\{ \begin{array}{c} d(Br,r).d(Pr,Bvr).d(r,r) \\ d(Br,r).d(r,Br).d(r,Br) \end{array} \right\}^{\frac{\lambda}{3}} \end{aligned}$$

Therefore $d(Br,r) \leq [d(Br,r)]^{\frac{4\lambda}{3}} < d(Br,r)$, a contradiction if $r \neq Br$. Therefore $Br = r$. Therefore $Br = Pr = r$. Therefore r is a fixed point of B and P.

and hence $Ar = Qr = Br = Pr = r$. Therefore r is a fixed point of A,B,P and Q.

Uniqueness: Let s be another common fixed point of A, B, P and Q. Then by (1)

$$\begin{aligned} d(r,s) = d(Br,As) &\leq \left\{ \begin{array}{c} d(Pr,Qs).d(Pr,Br).d(Qs,As) \\ d(Pr,As).d(Qs,Br).d(As,Br) \end{array} \right\}^{\frac{\lambda}{3}} \\ &= \left\{ \begin{array}{c} d(r,s).d(r,r).d(s,s) \\ d(r,s).d(s,r).d(s,r) \end{array} \right\}^{\frac{\lambda}{3}} \\ &= [d(r,s)^4]^{\frac{\lambda}{3}} = d(r,s)^{\frac{4\lambda}{3}} \end{aligned}$$

Therefore $d(r,s) < d(r,s)]^{\frac{4\lambda}{3}} < d(r,s)$, a contradiction if $r \neq s$. Therefore $r = s$. Therefore r is a unique common fixed point of A, B, P and Q. □

The following theorem is a corollary of Theorem 2.1 for six self maps.

Theorem 2.2. *Let (X,d) be a complete multiplicative metric space and P, Q, R, S, T and U be self maps of X satisfying the following conditions.*

(1). $TU(X) \subseteq P(X)$ and $RS(X) \subseteq Q(X)$ and

(2). $d(RSx,TUy) \leq \left\{ \begin{array}{c} d(Px,Qy).d(Px,Bx).d(Qy,Ay) \\ d(Px,Ay).d(Qy,Bx).d(Ay,Bx) \end{array} \right\}^{\frac{\lambda}{3}}$ *for all $x,y \in X$, $\lambda \in [0,\frac{1}{2})$ is a constant.*

Pairs (T,U), (T,Q), (U,Q), (R,S), (R,P) and (S,P) are commuting pairs. Then P, Q, R, S, T and U have a unique common fixed point in X.

Proof. Put $TU = A$, and $RS = B$ then (i) and (ii) of theorem 2.1 are satisfied. Hence P, Q, A, B have unique common fixed point say r, then $Ar = Qr = Br = Pr = r$ i.e., $TUr = Qr = RSr = Pr = r \Rightarrow T(TUr) = T(Qr) = Q(Tr)$

($\because (T,Q)$ commute) $\Rightarrow T(r) = Q(Tr)$. Therefore Tr is a fixed point of Q. Now $RS(r) = Pr \Rightarrow R(RSr) = R(Pr) = P(Rr)$ ($\because (R,P)$ commute) $\Rightarrow R(r) = P(Rr)$. Therefore Rr is a fixed point of P. Again $Ar = TU(r) \Rightarrow T(Ar) = T(TUr) = T(UTr) = TU(Tr)$ ($\because (T,U)$ commute) $\Rightarrow T(r) = TU(Tr) = A(Tr)$. Therefore Tr is a fixed point of A and $Br = RS(r) \Rightarrow P(Br) = R(RSr) = R(SRr) = RS(Rr)$ ($\because (R,S)$ commute). Therefore Rr is a fixed point of B. $Rr = P(Rr) = B(Rr)$ and $Tr = Q(Tr) = A(Tr)$. Now

$$\begin{aligned} UT(r) &= Qr \\ U(U(Tr)) &= U(Qr) \\ U(TU(r)) &= U(Qr) \\ U(r) &= Q(Ur) \\ Ur &= Q(Ur) \end{aligned}$$

Therefore Ur is a fixed point of Q and $Ar = TUr \Rightarrow U(Ar) = U(TUr)$ ($\because (U,A)$ commute) $\Rightarrow U(Ar) = U(r) \Rightarrow AUr = Ur$. Therefore Ur is a fixed point of A. Therefore Tr and Ur are fixed points of A and Q. Similarly Rr and Sr are fixed points of B and P.

$$\begin{aligned} d(Rr,Tr) &= d(BRr,ATr) \\ &\leq \left\{ \begin{array}{l} d(PRr,QTr).d(PRr,BRr).d(QTr,ATr) \\ d(PRr,ATr).d(QTr,BRr).d(ATr,BRr) \end{array} \right\}^{\frac{\lambda}{3}} \\ &= \left\{ \begin{array}{l} d(Rr,Tr).d(Rr,Rr).d(Tr,Tr) \\ d(Rr,Tr).d(Tr,Rr).d(Tr,Rr) \end{array} \right\}^{\frac{\lambda}{3}} \\ &= d(Rr,Tr)^{\frac{4\lambda}{3}} \end{aligned}$$

Therefore $d(Rr,Tr) \leq d(Rr,Tr)^{\frac{4\lambda}{3}} < d(Rr,Tr)$, a contradiction if $Rr \neq Tr$. Therefore $Rr = Tr$. Therefore Rr is a common fixed point of A, Q, P and B. i.e., TU, Q, P and RS. But r is a unique common fixed point of A, Q, P and B. i.e., TU, Q, P and RS. Therefore $Rr = r$ is a unique common fixed point of P, Q, R, S, T and U. Similarly we find $d(Sr,Ur) = d(BSr,AUr) \leq d(Sr,Ur)^{\frac{4\lambda}{3}} < d(Sr,Ur)$, a contradiction if $Sr \neq Ur$. Therefore r is a unique common fixed point of P, Q, R, S, T and U. □

Corollary 2.3. *In addition to Theorem 2.1, If x is a fixed point of P and B, and y is a fixed point of A and Q. Then $x = y$.*

Proof. By Theorem 2.1, A, B, P and Q have unique fixed point. Suppose x is fixed point of B and P, and y is fixed point of A and Q. Now

$$\begin{aligned} d(x,y) &= d(Bx,Ay) \\ &\leq \left\{ \begin{array}{l} d(Px,Qy).d(Px,Bx).d(Qy,Ay) \\ d(Px,Ay).d(Qy,Bx).d(Ay,Bx) \end{array} \right\}^{\frac{\lambda}{3}} \\ &= \left\{ \begin{array}{l} d(x,y).d(x,x).d(y,y) \\ d(x,y).d(y,x).d(y,x) \end{array} \right\}^{\frac{\lambda}{3}} \end{aligned}$$

$$= d(x,y)^{\frac{4\lambda}{3}}$$

$d(x,y) \leq d(x,y)^{\frac{4\lambda}{3}} < d(x,y)$, a contradiction if $x \neq y$. Therefore $x = y$. □

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