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On Certain Sequence Space

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Abstract: In this paper, we introduce a new sequence space ℓ_s^p, $p \geq 1$, which turns out to be an infinite dimensional separable Banach space in which Hölder inequality does not hold. It is shown that it is a proper subspace of ℓ^p to which ℓ^p is topologically equivalent. Apart from studying various algebraic and topological properties of ℓ_s^p, its Köthe- Toeplitz duals have also been computed.

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1. Introduction and Preliminaries

By ω we shall denote the space of all complex sequences; ℓ^∞, c and c_0 denote the spaces of all bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively normed by $\|x\| = \sup_k |x_k|$. ℓ^p, $p \geq 1$ denotes the linear space of all absolutely $p-summable$ scalar sequences, normed by $\|x\|_p = (\sum_k |x_k|^p)^{\frac{1}{p}}$. The following concepts are of long standing [1, 3, 4, 6]. A complete metric linear space is called a Frèchet space. Let X be a linear subspace of ω such that X is a Frèchet space with continuous coordinate projections. Then we say that X is an FK space. If the metric of an FK space is given by a complete norm then we say that X is a BK space. We say that an FK space X has AK, or has the AK property, if (e_k), the sequence of unit vectors, is a Schauder basis for X.

A sequence space X is called

(1). normal (or solid) if $y = (y_k) \in X$ whenever $|y_k| \leq |x_k|$, $k \geq 1$, for some $x = (x_k) \in X$,

(2). monotone if it contains the canonical preimages of all its stepspaces,

(3). sequence algebra if $xy = (x_k y_k) \in X$ whenever $x = (x_k), y = (y_k) \in X$,

(4). convergence free when, if $x = (x_k)$ is in X and if $y_k = 0$ whenever $x_k = 0$, then $y = (y_k)$ is in X.

The idea of dual sequence spaces was introduced by Köthe and Toeplitz [5] whose main results concerned α-duals; the α-dual of $X \subset \omega$ being defined as

$$X^\alpha = \{a = (a_k) \in \omega : \sum_k |a_k x_k| < \infty \quad for\ all \quad x = (x_k) \in X\}.$$

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In the same paper [5], they also introduced another kind of dual, namely, the β-dual (see [2] also, where it is called the g-dual by Chillingworth) defined as

$$X^{\beta} = \{a = (a_k) \in \omega : \sum_k a_k x_k \quad converges\ for\ all \quad x = (x_k) \in X\}.$$

Obviously $\phi \subset X^{\alpha} \subset X^{\beta}$, where ϕ is the well-known sequence space of finitely non-zero scalar sequences. Also if $X \subset Y$, then $Y^{\eta} \subset X^{\eta}$ for $\eta = \alpha$, or β. For any sequence space X, we denote $(X^{\delta})^{\eta}$ by $X^{\delta\eta}$ where $\delta, \eta = \alpha$, or β. It is clear that $X \subset X^{\eta\eta}$ where $\eta = \alpha$, or β. For a sequence space X, if $X = X^{\alpha\alpha}$ then X is called a Köthe space or a perfect sequence space. We now introduce a new sequence space ℓ_s^p as follows:

Definition 1.1. *For* $1 \leq p < \infty$, *we define*

$$\ell_s^p = \{x = (x_k) \in \omega : \left(\sum_{i=1}^{k} x_i\right) \in \ell^p.$$

For $p = 1$, *sequence space* ℓ_s^p *reduces to* ℓ_s*—the space introduced by Mishra et al. [7].*

The main purpose of this paper is to determine the Köthe- Toeplitz duals of the newly introduced sequence space ℓ_s^p and to study some of its algebraic and topological properties.

2. Main Results

Our first result gives a linear topological structure of the space ℓ_s^p .

Theorem 2.1. ℓ_s^p *is a BK space with respect to norm* $\|x\|_{s(p)} = \left(\sum_{k=1}^{\infty} |\sum_{i=1}^{k} x_i|^p\right)^{\frac{1}{p}}$.

Proposition 2.2. ℓ_s^p *has Schauder basis namely* $\{\delta^1, \delta^2, \delta^3 \dots\}$ *where* $\delta^{(k)} = (0, 0, 0, \dots, 1, -1, 0, 0, \dots)$, *1 is in the* k^{th} *place and -1 in the* $(k+1)^{th}$ *place for* $k = 1, 2, \dots$.

Proof. Let $x = (x_1, x_2, x_3, \dots) \in \ell_s^p$. Then $\sum_{k=1}^{\infty} \left(|\sum_{i=1}^{k} x_i|\right)^p < \infty$. Now

$$\begin{aligned}\|x - \sum_{k=1}^{n}(\sum_{i=1}^{k} x_i)\delta^{(k)}\|_{s(p)} &= \|(0, 0, 0, \dots, 0, x_1 + x_2 + \dots + x_{n+1}, x_{n+2}, \dots)\|_{s(p)} \\ &= \left(\sum_{k=n+1}^{\infty} |\sum_{i=1}^{k} x_i|^p\right)^{\frac{1}{p}} \\ &\to 0 \,\text{as}\;\; n \to \infty\end{aligned}$$

so that $x = \sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} x_i\right) \delta^{(k)}$. If also we had $x = \sum_k a_k \delta^{(k)}$, then it is easy to see that $a_k = \sum_{i=1}^{k} x_i$, $k \in \mathbb{N}$. Thus every $x = (x_1, x_2, x_3, \dots) \in \ell_s^p$ has a unique representation as $x = \sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} x_i\right) \delta^{(k)}$. □

Remark 2.3. *It was shown in [7] that* ℓ_s *has the AK property. Unfortunately, this is not true in view of the following.*

Proposition 2.4. ℓ_s^p *does not have the AK property.*

Proof. The sequence (e_k) of unit vectors is not a Schauder basis for ℓ_s^p. Infact $e_k \notin \ell_s^p$, $k \geq 1$. □

Proposition 2.5. ℓ_s^p *is separable.*

The proof follows from the fact that if a normed space has Schauder basis, then it is separable.

Proposition 2.6. *The continuous dual of ℓ_s^p is ℓ_q; here $1 < p < \infty$ and q is the conjugate of p, that is, $\frac{1}{p} + \frac{1}{q} = 1$.*

Proposition 2.7 ([7]). *The space ℓ_s, (i.e., ℓ_s^p when $p = 1$) is not reflexive.*

The next result takes care of the case when $p > 1$.

Proposition 2.8. *ℓ_s^p $(1 < p < \infty)$ is reflexive.*

Proof. ℓ_s^p is a Banach space and $(\ell_s^p)' = \ell^q$. The proof follows from the fact that a Banach space is reflexive if and only if its dual is reflexive. □

Remark 2.9. *It was shown in [7] that ℓ_s is solid, which is incorrect as is clear from the following.*

Proposition 2.10. *ℓ_s^p is not monotone and hence neither normal (solid) nor convergence free.*

Proof. Consider $x = (x_k) = (1, -1, 0, 0, \ldots) \in \ell_s^p$. Take $y = (y_k) = (1, 0, 0, \ldots)$,then $(y_k) \notin \ell_s^p$ and so ℓ_s^p is not monotone. It is well known ([1, 4]) that every convergence free space is normal and normal space is monotone. Consequently, ℓ_s^p is neither monotone nor convergence free. □

Corollary 2.11. *ℓ_s^p is not perfect.*

The proof follows from the Proposition 2.10 and the fact ([1, 4]) that perfectness $\Rightarrow$ normality.

Lemma 2.12. $(\ell_s^p)^\alpha \subset (\ell_s^p)^\beta \subset \ell^\infty$.

Proof. Let, if possible, $y = (y_k) \in (\ell_s^p)^\beta$ such that $(y_k) \notin \ell^\infty$. Hence there exists a strictly increasing sequence $(n(i)) \subset \mathbb{N}$, $n(1) > 1$ with $|y_{n(i)}| > 2^{i+1}$. If

$$x_k = \begin{cases} \frac{-1}{2}, & \text{if} \quad k = 1; \\ \frac{1}{2^{i+1}}, & k = n(i); \\ 0, & k \neq n(i); i \geq 1 \end{cases}$$

then $(x_k) \in \ell_s^p$ but $\sum\limits_k x_k y_k = \frac{-y_1}{2} + \sum\limits_i \frac{y_{n(i)}}{2^{i+1}}$ is not convergent since $|\frac{y_{n(i)}}{2^{i+1}}| > 1$ for all $i \geq 1$, a contradiction to the fact $(y_k) \in (\ell_s^p)^\beta$. Hence $y = (y_k) \in \ell^\infty$. □

Proposition 2.13. $(\ell_s^p)^\alpha = (\ell_s^p)^\beta = \{(y_k) : (y_k - y_{k+1}) \in \ell^q\} \cap \ell^\infty = D$ *where* $(1 \leq p < \infty)$.

Proof. Let $y = (y_k) \in (\ell_s^p)^\beta$ and $z = (z_k) \in \ell^p$. Then the sequence (w_k) defined by $w_k = z_k - z_{k+1}$, $k \geq 1$ where $z_0 = 0$, belongs to ℓ_s^p, since $(\sum\limits_{i=1}^{k} w_i) = (z_k) \in \ell^p$. As $(y_k) \in (\ell_s^p)^\beta$ so $\sum\limits_k y_k w_k$ converges. But

$$\begin{aligned} \sum_{i=1}^{n} (z_i - z_{i-1}) y_i &= \sum_{i=1}^{n-1} (z_i y_i - z_{i-1} y_i) + (z_n - z_{n-1}) y_n \\ &= \sum_{i=1}^{n-1} z_i (y_i - y_{i+1}) + \sum_{i=1}^{n-1} (z_i y_{i+1} - z_{i-1} y_i) + (z_n - z_{n-1}) y_n \\ &= \sum_{i=1}^{n-1} z_i (y_i - y_{i+1}) + z_n y_n. \end{aligned}$$

Using Lemma 2.12, $(y_k) \in \ell^\infty$. As $(z_n) \in \ell^p \subset c_0$, so $\sum\limits_i (z_i - z_{i-1}) y_i = \sum\limits_i (y_i - y_{i+1}) z_i$. As $\sum\limits_i w_i y_i < \infty$, so $\sum\limits_i z_i (y_i - y_{i+1}) < \infty$ for all $z = (z_i) \in \ell^p$. Consequently,$(y_i - y_{i+1}) \in (\ell^p)^\beta = \ell^q$, i.e., $(y_i) \in D$. To prove the reverse inclusion, let $y = (y_k) \in D$.

Then $\sum_k |y_k - y_{k+1}|^q < \infty$ and $(y_k) \in \ell^\infty$. Let $x = (x_k) \in \ell^p_s$, then the sequence (w_k), $w_k = \sum_{i=1}^k x_i$, $k \geq 1$ is an element of ℓ^p. As $(\ell^p)^\alpha = \ell^q$, so the series $\sum_k w_k(y_k - y_{k+1})$ is absolutely convergent. Also for integers $n, m \in \mathbb{N}$ with $n > m$ we have

$$|\sum_{k=m}^{n}(w_k - w_{k-1})y_k| \leq |\sum_{k=m}^{n-1}(y_k - y_{k+1})w_k| + |w_n y_n - w_{m-1}y_m|.$$

As $(w_k) \in \ell^p \subset c_0$ and $(y_k) \in \ell^\infty$, the right-hand side of the above inequality converges to 0 as $m, n \to \infty$. Hence the series $\sum_k(w_k - w_{k-1})y_k$ or $\sum_k x_k y_k$ converges and so $D \subset (\ell^p_s)^\beta$. □

Corollary 2.14 ([7])**.** $(\ell_s)^\alpha = (\ell_s)^\beta = \ell^\infty$.

Taking $p = 1$ in Proposition 2.13, we have

$$(\ell_s)^\alpha = (\ell_s)^\beta = \{(y_k) : (y_k - y_{k+1}) \in \ell^\infty\} \cap \ell^\infty = \ell^\infty(\Delta) \cap \ell^\infty = \ell^\infty.$$

Proposition 2.15.

(a). $\ell^p_s \subset \ell^r_s$ for $p < r$.

(b). ℓ^p_s are distinct for distinct p.

Proof.

(a). The proof follows from the fact that $\ell^p \subset \ell^r$ for $p < r$.

(b). Let $1 \leq p < r < \infty$. Choose $q \in \mathbb{R}$ such that $p < q < r$. Take

$$x_k = \begin{cases} 1, \text{for } k = 1; \\ (\frac{1}{k})^{\frac{1}{q}} - (\frac{1}{k-1})^{\frac{1}{q}},\ k > 1,\ k \in \mathbb{N}. \end{cases}$$

Now $\sum_k |\sum_{i=1}^k x_i|^r = \sum_k (\frac{1}{k})^{\frac{r}{q}} < \infty$ and so $x = (x_k) \in \ell^r_s$. But $\sum_k |\sum_{i=1}^k x_i|^p = \sum_k (\frac{1}{k})^{\frac{p}{q}}$ is divergent and so $x = (x_k) \notin \ell^p_s$. □

Proposition 2.16.

(a). $\ell^p_s \subset \ell^p$; inclusion is strict.

(b). ℓ^p_s is homeomorphic to ℓ^p.

Proof.

(a). Let $x = (x_k) \in \ell^p_s$. Then $\sum_k |\sum_{i=1}^k x_i|^p < \infty$, i.e., $y = (y_k) \in \ell^p$ where $y_k = \sum_{i=1}^k x_i$. Clearly $x_k = y_k - y_{k-1}$, for $k \geq 1$ with $y_0 = 0$. As $(y_k), (y_{k-1}) \in \ell^p$ so $(y_k - y_{k-1}) = (x_k) \in \ell^p$. For strict inclusion, we have $e_1 = (1, 0, 0, \ldots) \in \ell^p$ but $e_1 \notin \ell^p_s$.

(b). $\sigma : \ell^p_s \to \ell^p$ defined by $\sigma((x_k)) = (\sum_{i=1}^k x_i)$ is a linear map and $\|\sigma((x_k))\|_p = \|(x_k)\|_{s(p)}$. That is σ is an isometry and hence a homeomorphism. Thus ℓ^p is topologically equivalent to a proper subspace of it.

Clearly the space ℓ^2_s (i.e., ℓ^p_s for $p = 2$) is a Hilbert space with the inner product $\langle x, y\rangle = \left\langle \sum_{i=1}^k x_i, \sum_{i=1}^k y_i \right\rangle$. However for the case $p \neq 2$ we have the following □

Proposition 2.17. *The space ℓ^p_s, $(p \neq 2)$ is not an inner product space and, hence, not a Hilbert space.*

Proof. Take $x = (1, -1, 0, 0, \dots) \in \ell_s^p$ and $y = (0, 0, 1, -1, \dots) \in \ell_s^p$. Then $\|x\|_{s(p)} = 1$, $\|y\|_{s(p)} = 1$, $\|x+y\|_{s(p)} = 2^{\frac{1}{p}}$ and $\|x-y\|_{s(p)} = 2^{\frac{1}{p}}$. If $p \neq 2$ then $\|x+y\|_{s(p)}^2 + \|x-y\|_{s(p)}^2 \neq 2\|x\|_{s(p)}^2 + 2\|y\|_{s(p)}^2$, i.e.,parallelogram law does not hold if $p \neq 2$, which means that the norm can not be obtained from the inner product. Hence, the space ℓ_s^p, with $(p \neq 2)$ is not a Banach space. □

Remark 2.18. *Taking $p = 1$ in Proposition 2.17, it follows that the first part of the Theorem 6 of [7], i.e., " ℓ_s is an inner product space" is incorrect.*

Proposition 2.19. *Hölder inequality does not hold in ℓ_s^p.*

Proof. Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Consider $x = (1, -1, 0, 0, \dots)$ and $y = (1, -1, 0, \dots)$. Now $\sum_k |\sum_{i=1}^k x_i|^p = 1^p + 0 + 0 + \dots = 1 < \infty$ so $x \in \ell_s^p$, $\sum_k |\sum_{i=1}^k y_i|^q = 1^q + 0 + 0 + \dots = 1 < \infty$, so $y \in \ell_s^q$ and $\sum_k |\sum_{i=1}^k x_i y_i| = 1 + 2 + 2 + \dots \to \infty$. This implies that $\sum_k |\sum_{i=1}^k x_i y_i| > (\sum_k |\sum_{i=1}^k x_i|^p)^{\frac{1}{p}} . (\sum_k |\sum_{i=1}^k y_i|^q)^{\frac{1}{q}}$. □

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