Int. J. Math. And Appl., 6(1–B)(2018), 229–242
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

On the Interval Oscillation of Impulsive Partial Differential Equations with Damping Term

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Abstract: In this paper, we present some sufficient conditions for the oscillations of all solutions of impulsive partial differential equations. The results obtained here are based on the effect of impulses, delay and damping term in the sequence of subintervals of $\mathbb{R}_+$, which develops some well-known results for the equations without impulses, delay and the equations without damping term. Moreover, an example is presented to illustrate our main results.

MSC: 35B05, 35L70, 35R10, 35R12.

Keywords: Oscillation, Impulse, Partial differential equations, Forcing term.

1. Introduction

In recent years, the theory of impulsive differential equations emerge as an important area of research, since such equations have many applications in the control theory, physics, biology, population dynamics, economics, etc. Because of difficulties caused by impulsive perturbations there is a less consideration regarding the oscillation problem for impulsive differential equation [4, 5, 9, 10, 14, 23]. In [11], the problem of interval oscillation criteria of impulsive differential equation with damping of the form

$$(r(t)\phi_\alpha(x'))' + p(t)\phi_\alpha(x') + q(t)\phi_\beta(x') = e(t), \quad t \neq \tau_k,$$
$$\Delta(r(t)\phi_\alpha(x')) + q_i\phi_\beta(x) = e_i, \quad t = \tau_k, k = 1, 2, \cdots$$

was studied by Ozbekler in the year 2009. Using the same approach in [5], Huang et.al. considered the oscillation of second order forced FDE with impulses

$$x''(t) + p(t)f(x(t-\tau)) = e(t), \quad t \neq t_k,$$
$$x(t_k^+) = a_k x(t_k), \quad x'(t_k^+) = b_k x'(t_k), \quad k = 1, 2, \cdots$$

and established some interval oscillation criteria which generalized some known results for the equations without delay or impulses [2, 6, 12, 16, 21]. In the last decades, interval oscillation of impulsive differential equations was stimulating the interest of many researchers, see for examples [3, 8, 11, 13, 15, 17, 19, 20]. For more details, one can refer the monographs

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[1, 7, 18, 22] and reference cited therein. Most of the existing literature determined on the interval oscillation criteria for the case of without delay and with out damping and only a very few papers appeared for the case of with delays. As far as authors knowledge, it seems that there has been no paper dealing with interval oscillation criteria for impulsive partial differential equations. Motivated by this gap, we consider the following impulsive partial differential equations with damping term of the form

$$\left.\begin{aligned}&\frac{\partial}{\partial t}\left[r(t)g\left(\frac{\partial}{\partial t}u(x,t)\right)\right]+p(t)g\left(\frac{\partial}{\partial t}u(x,t)\right)+q(x,t)f(u(x,t-\tau))+\sum_{i=1}^{n}q_i(x,t)f_i(u(x,t-\tau))\\&=a(t)\Delta u(x,t)+\sum_{j=1}^{m}a_j(t)\Delta u(x,t-\rho_j)+F(x,t),\quad t\neq t_k,\\&u(x,t_k^+)=(1+\alpha_k)u(x,t_k),\\&u_t(x,t_k^+)=(1+\beta_k)u_t(x,t_k),\quad k=1,2,\cdots,\qquad (x,t)\in\Omega\times\mathbb{R}_+\equiv G,\end{aligned}\right\}\tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$, Δ is the Laplacian in the Euclidean space $\mathbb{R}^N$ and $\mathbb{R}_+=[0,+\infty)$. Equation (1) is enhancement with the boundary condition,

$$\frac{\partial u(x,t)}{\partial\gamma}+\mu(x,t)u(x,t)=0,\qquad (x,t)\in\partial\Omega\times\mathbb{R}_+,\tag{2}$$

where γ is the outer surface normal vector to $\partial\Omega$ and $\mu(x,t)\in C(\partial\Omega\times[0,+\infty),[0,+\infty))$.

In the sequel, we assume that the following hypotheses (A) hold:

(A_1) $r(t)\in C^1(\mathbb{R}_+,(0,+\infty)), p(t)\in C(\mathbb{R}_+,\mathbb{R}),\ q(x,t),q_i(x,t)\in C(\bar{G},\mathbb{R}_+),\ q(t)=\min_{x\in\bar{\Omega}}q(x,t),\ q_i(t)=\min_{x\in\bar{\Omega}}q_i(x,t),\ i=1,2,\cdots,n,\ f,f_i\in C(\mathbb{R},\mathbb{R})$ are convex in $\mathbb{R}_+$ with $uf(u)>0,\ uf_i(u)>0$ and $\frac{f(u)}{u}\geq\epsilon>0,\ \frac{f_i(u)}{u}\geq\epsilon_i>0$ for $u\neq0$, $i=1,2,\cdots,n,\ t-\tau<t,\ t-\rho_j<t,\ \lim_{t\to+\infty}t-\tau=\lim_{t\to+\infty}t-\rho_j=+\infty,\ j=1,2,\cdots,m.$

(A_2) $F\in C(\bar{G},\mathbb{R}),g\in C(\mathbb{R},\mathbb{R})$ is convex in $\mathbb{R}_+$ with $ug(u)>0,\ g(u)\leq\theta u$ for $u\neq0,\ g^{-1}\in C(\mathbb{R},\mathbb{R})$ is continuous function with $ug^{-1}(u)>0$ for $u\neq0$ and there exist positive constant η such that $g^{-1}(uv)\leq\eta g^{-1}(u)g^{-1}(v)$ for $uv\neq0$.

(A_3) $a(t),a_j(t)\in PC(\mathbb{R}_+,\mathbb{R}_+),\ j=1,2,...,m$, where PC represents the class of functions which are piecewise continuous in t with discontinuities of first kind only at $t=t_k,\ k=1,2,\cdots$, and left continuous at $t=t_k,\ k=1,2,\cdots$.

(A_4) $u(x,t)$ and its derivative $u_t(x,t)$ are piecewise continuous in t with discontinuities of first kind only at $t=t_k$, $k=1,2,\cdots$, and left continuous at $t=t_k,\ u(x,t_k)=u(x,t_k^-),\ u_t(x,t_k)=u_t(x,t_k^-),\ k=1,2,\cdots$.

(A_5) α_k,β_k are real constants satisfying $\alpha_k>-1,\alpha_k\leq\beta_k,\ 0<t_1<\cdots<t_k<\cdots$ and $\lim_{t\to+\infty}t_k=+\infty,k=1,2,\cdots$.

Definition 1.1. *A solution u of the problem (1)-(2) is a function $u\in C^2(\bar{\Omega}\times[t_{-1},+\infty),\mathbb{R})\cap C(\bar{\Omega}\times[\hat{t}_{-1},+\infty),\mathbb{R})$ that satisfies (1), where*

$$t_{-1}:=\min\left\{0,\ \min_{1\leq j\leq m}\left\{\inf_{t\geq0}t-\rho_j\right\}\right\},\quad \hat{t}_{-1}:=\min\left\{0,\ \inf_{t\geq0}t-\tau\right\}.$$

Definition 1.2. *The solution u of the problem (1)-(2) is said to be oscillatory in the domain G, if it has arbitrary large zeros. Otherwise it is non-oscillatory.*

For convenience, we introduce the following notations:

$$U(t)=\frac{1}{|\Omega|}\int_\Omega u(x,t)dx,\quad Q(t)=\epsilon q(t)+\sum_{i=1}^{n}\epsilon_i q_i(t),\quad\text{where}\quad |\Omega|=\int_\Omega dx.$$

2. Main Results

In this section, the intervals $[c_1, d_1]$ and $[c_2, d_2]$ are considered to establish oscillation criteria. So we also assume that

(A_6) $c_s, d_s \notin \{t_k\}$, $s = 1, 2$, $k = 1, 2, \cdots$, with $c_1 < d_1$, $c_2 < d_2$ and $r(t) > 0$, $q(t) \geq 0$, $q_i(t) \geq 0$, $i = 1, 2, \cdots, n$ for $t \in [c_1 - \tau, d_1] \cup [c_2 - \tau, d_2]$ and $F(t)$ has different signs in $[c_1 - \tau, d_1]$ and $[c_2 - \tau, d_2]$, for instance, let

$$F(t) \leq 0 \quad \text{for} \quad t \in [c_1 - \tau, d_1], \quad \text{and} \quad F(t) \geq 0 \quad \text{for} \quad t \in [c_2 - \tau, d_2].$$

Denote

$$I(s) := \max\{i : \ t_0 < t_i < s\}, \quad r_s := \max\{r(t) : \ t \in [c_s, d_s]\}, \quad s = 1, 2.$$
$$D_u(c_s, d_s) = \left\{u \in C^1([c_s, d_s], \mathbb{R}) \ \ u(t) \not\equiv 0, \ u(c_s) = u(d_s) = 0, \ s = 1, 2\right\}.$$

Lemma 2.1. *If the impulsive differential inequality*

$$\left.\begin{array}{l} [r(t)g(U'(t))]' + p(t)g(U'(t)) + \epsilon q(t)U(t-\tau) + \sum\limits_{i=1}^{n} \epsilon_i q_i(t)U(t-\tau) \leq F(t), \quad t \neq t_k \\ U(t_k^+) = (1 + \alpha_k)U(t_k) \\ U'(t_k^+) = (1 + \beta_k)U'(t_k), \quad k = 1, 2, \cdots \end{array}\right\} \tag{3}$$

has no eventually positive solution, then every solution of the boundary value problem defined by (1)-(2) is oscillatory in G.

Proof. Suppose to the contrary that there is a non-oscillatory solution $u(x,t)$ of the boundary value problem $(1) - (2)$. Without loss of generality, we may assume that $u(x,t) > 0$ in $\Omega \times [t_0, +\infty)$ for some $t_0 > 0$, $u(x, t-\tau) > 0$ and $u(x, t-\rho_j) > 0$, $j = 1, 2, \cdots, m$. For $t \neq t_k, t \geq t_0, k = 1, 2, \cdots$, we multiplying both sides of equation (1) by $\frac{1}{|\Omega|}$ and integrating with respect to x over the domain Ω, we obtain

$$\left.\begin{array}{l} \frac{d}{dt}\left[r(t)g\left(\frac{d}{dt}\left(\frac{1}{|\Omega|}\int_\Omega u(x,t)dx\right)\right)\right] + p(t)g\left(\frac{d}{dt}\left(\frac{1}{|\Omega|}\int_\Omega u(x,t)dx\right)\right) + \frac{1}{|\Omega|}\int_\Omega q(x,t)f(u(x,t-\tau))dx \\ + \frac{1}{|\Omega|}\sum\limits_{i=1}^{n}\int_\Omega q_i(x,t)f_i(u(x,t-\tau))dx = a(t)\frac{1}{|\Omega|}\int_\Omega \Delta u(x,t)dx \\ + \sum\limits_{j=1}^{m} a_j(t)\frac{1}{|\Omega|}\int_\Omega \Delta u(x,t-\rho_j)dx + \frac{1}{|\Omega|}\int_\Omega F(x,t)dx. \end{array}\right\} \tag{4}$$

From Green's formula and the boundary condition (2), we see that

$$\int_\Omega \Delta u(x,t)dx = \int_{\partial\Omega} \frac{\partial u(x,t)}{\partial \gamma} dS = -\int_{\partial\Omega} \mu(x,t)u(x,t)dS \leq 0 \tag{5}$$

and for $j = 1, 2, \cdots, m$, we have

$$\int_\Omega \Delta u(x,t-\rho_j)dx = \int_{\partial\Omega} \frac{\partial u(x,t-\rho_j)}{\partial \gamma} dS = -\int_{\partial\Omega} \mu(x,t)u(x,t-\rho_j)dS \leq 0, \tag{6}$$

where dS is surface component on $\partial\Omega$. Furthermore applying Jensen's inequality for convex functions and using the assumptions on (A_1), we get that

$$\int_\Omega q(x,t)f(u(x,t-\tau))dx \geq q(t)\int_\Omega f(u(x,t-\tau))dx$$

$$\geq \epsilon q(t)\int_{\Omega} u(x,t-\tau)dx, \tag{7}$$

and for $i=1,2,\cdots,n$

$$\begin{aligned}\int_{\Omega} q_i(x,t)f_i(u(x,t-\tau))dx &\geq q_i(t)\int_{\Omega} f_i(u(x,t-\tau))dx\\ &\geq \epsilon_i q_i(t)\int_{\Omega} u(x,t-\tau)dx.\end{aligned} \tag{8}$$

Take

$$F(t)=\frac{1}{|\Omega|}\int_{\Omega} F(x,t)dx. \tag{9}$$

Combining (4)-(9), we get that

$$\left[r(t)g(U'(t))\right]'+p(t)g(U'(t))+\epsilon q(t)U(t-\tau)+\sum_{i=1}^{n}\epsilon_i q_i(t)U(t-\tau)\leq F(t).$$

For $t=t_k, k=1,2,\cdots$, multiplying both sides of the equation (1) by $\frac{1}{|\Omega|}$, integrating with respect to x over the domain Ω, and from (A_5), we obtain

$$\begin{aligned}\frac{1}{|\Omega|}\int_{\Omega} u(x,t_k^+)dx &= (1+\alpha_k)\frac{1}{|\Omega|}\int_{\Omega} u(x,t_k)dx\\ \frac{1}{|\Omega|}\int_{\Omega} u_t(x,t_k^+)dx &= (1+\beta_k)\frac{1}{|\Omega|}\int_{\Omega} u_t(x,t_k)dx,\end{aligned}$$

since $U(t)=\frac{1}{|\Omega|}\int_{\Omega} u(x,t)dx$, we have

$$\begin{aligned}U(t_k^+) &= (1+\alpha_k)U(t_k)\\ U'(t_k^+) &= (1+\beta_k)U'(t_k).\end{aligned}$$

Therefore $U(t)$ is an eventually positive solution of (3), which contradicts the hypothesis and completes the proof. □

Theorem 2.2. *Assume that conditions $(A_1)-(A_5)$ hold, furthermore for any $T\geq 0$ there exist c_s,d_s satisfying (A_6) with $T\leq c_1<d_1$, $T\leq c_2<d_2$ and*
(i) $\int_{t_0}^{\infty} g^{-1}\left(\frac{1}{r(s)}\right)ds=\infty$, (ii) $u(t)\in D_u(c_s,d_s)$ such that

$$\begin{aligned}&\int_{c_s}^{d_s} r(t)\left[\theta(u'(t))^2-p(t)u^2(t)\frac{w(t)}{r(t)}\right]dt-\int_{c_s}^{t_{I(c_s)+1}} Q(t)u^2(t)R_{I(c_s)}^s(t)dt-\sum_{k=I(c_s)+1}^{I(d_s)-1}\int_{t_k}^{t_{k+1}} Q(t)u^2(t)R_k^s(t)dt\\ &-\int_{t_{I(d_s)}}^{d_s} Q(t)u^2(t)R_{I(d_s)}^s(t)dt<r_s\Pi_{c_s}^{d_s}[u^2(t)],\end{aligned} \tag{10}$$

for $I(c_s)<I(d_s)$, $s=1,2$, where $Q(t)=\epsilon q(t)+\sum_{i=1}^{n}\epsilon_i q_i(t)$,

$$R_k^s(t)=\begin{cases}\dfrac{t-t_k}{(1+\alpha_k)\tau+(1+\beta_k)(t-t_k)}, & t\in(t_k,t_k+\tau)\\ \dfrac{t-t_k-\tau}{t-t_k}, & t\in[t_k+\tau,t_{k+1}),\end{cases}$$

then every solution of the boundary value problem (1) − (2) is oscillatory in G.

Proof. Assume to the contrary that $u(t)$ is a non-oscillatory solution of (2) and (1.2). Without loss of generality we may assume that $U(t)$ is an eventually positive solution of (3). Then there exists $t_1 \geq t_0$ such that $U(t) > 0$ for $t \geq t_1$. Therefore it follows from (3) that

$$\left[r(t)g(U'(t))\right]' \leq F(t) - p(t)g(U'(t)) - \epsilon q(t)U(t-\tau) - \sum_{i=1}^{n} \epsilon_i q_i(t)U(t-\tau) \leq 0 \quad \text{for} \quad t \in [t_1, +\infty). \tag{11}$$

Thus $U'(t) \geq 0$ or $U'(t) < 0$, $t \geq t_1$ for some $t_1 \geq t_0$. We now claim that

$$U'(t) \geq 0 \quad \text{for} \quad t \geq t_1. \tag{12}$$

Suppose not, then $U'(t) < 0$ and there exists $t_2 \in [t_1, +\infty)$ such that $U'(t_2) < 0$. Since $r(t)g(U'(t))$ is strictly decreasing on $[t_1, +\infty)$. It is clear that

$$r(t)g(U'(t)) < r(t_2)g(U'(t_2)) := -c,$$

where $c > 0$ is a constant for $t \in [t_2, +\infty)$, we have

$$\begin{aligned} r(t)g(U'(t)) &< -c \\ U'(t) &< g^{-1}\left(\frac{-c}{r(t)}\right) \\ U'(t) &\leq -c_0 g^{-1}\left(\frac{1}{r(t)}\right), \quad \text{where} \quad c_0 = \eta g^{-1}(c) \text{ for } t \in [t_2, +\infty). \end{aligned}$$

Integrating the above inequality from t_2 to t, we have

$$U(t) \leq U(t_2) - c_0 \int_{t_2}^{t} g^{-1}\left(\frac{1}{r(s)}\right) ds.$$

Letting $t \to +\infty$, we get

$$\lim_{t \to +\infty} U(t) = -\infty$$

which contradiction proves that (12) holds. Define the Riccati Transformation

$$w(t) := \frac{r(t)g(U'(t))}{U(t)}. \tag{13}$$

It follows from (3) that $w(t)$ satisfies

$$w'(t) \leq \frac{F(t)}{U(t)} - \left[\epsilon q(t) + \sum_{i=1}^{n} \epsilon_i q_i(t)\right] \frac{U(t-\tau)}{U(t)} - \frac{w^2(t)}{\theta r(t)} - \frac{p(t)w(t)}{r(t)}.$$

By the assumption, we can choose $c_1, d_1 \geq t_0$ such that $r(t) \geq 0$, $q(t) \geq 0$ and $q_i(t) \geq 0$ for $t \in [c_1 - \tau, d_1]$, $i = 1, 2, \cdots, n$ and $F(t) \leq 0$ for $t \in [c_1 - \tau, d_1]$ from (3) we can easily to see that

$$w'(t) \leq -Q(t)\frac{U(t-\tau)}{U(t)} - \frac{w^2(t)}{\theta r(t)} - \frac{p(t)w(t)}{r(t)}. \tag{14}$$

For $t = t_k,\ k = 1, 2, \cdots$, one has

$$w(t_k^+) = \frac{r(t_k^+)g(U'(t_k^+))}{U(t_k^+)} \leq \frac{(1+\beta_k)}{(1+\alpha_k)} w(t_k). \tag{15}$$

At first, we consider the case in which $I(c_1) < I(d_1)$. In this case, all the impulsive moments in $[c_1, d_1]$ are $t_{I(c_1)+1},\ t_{I(c_1)+2}, \cdots,\ t_{I(d_1)}$. Choose $u(t) \in D_u(c_1, d_1)$ and multiplying by $u^2(t)$ on both sides of (14), integrating it from c_1 to d_1, we obtain

$$\begin{aligned}
&\int_{c_1}^{t_{I(c_1)+1}} u^2(t)w'(t)dt + \int_{t_{I(c_1)+1}}^{t_{I(c_1)+2}} u^2(t)w'(t)dt + \cdots + \int_{t_{I(d_1)}}^{d_1} u^2(t)w'(t)dt \\
&\leq -\int_{c_1}^{t_{I(c_1)+1}} u^2(t)\frac{w^2(t)}{\theta r(t)}dt - \int_{t_{I(c_1)+1}}^{t_{I(c_1)+2}} u^2(t)\frac{w^2(t)}{\theta r(t)}dt - \cdots - \int_{t_{I(d_1)}}^{d_1} u^2(t)\frac{w^2(t)}{\theta r(t)}dt \\
&- \int_{c_1}^{t_{I(c_1)+1}} u^2(t)Q(t)\frac{U(t-\tau)}{U(t)}dt - \int_{t_{I(c_1)+1}}^{t_{I(c_1)+1}+\tau} u^2(t)Q(t)\frac{U(t-\tau)}{U(t)}dt - \int_{t_{I(c_1)+1}+\tau}^{t_{I(c_1)+2}} u^2(t)Q(t)\frac{U(t-\tau)}{U(t)}dt \\
&- \cdots - \int_{t_{I(d_1)-1}+\tau}^{t_{I(d_1)}} u^2(t)Q(t)\frac{U(t-\tau)}{U(t)}dt - \int_{t_{I(d_1)}}^{d_1} u^2(t)Q(t)\frac{U(t-\tau)}{U(t)}dt - \int_{c_1}^{d_1} u^2(t)p(t)\frac{w(t)}{r(t)}dt.
\end{aligned}$$

Using the integration by parts on the left-hand side, and noting that the condition $u(c_1) = u(d_1) = 0$, we get

$$\begin{aligned}
&\sum_{k=I(c_1)+1}^{I(d_1)} u^2(t_k)\left[w(t_k) - w(t_k^+)\right] \leq -\int_{c_1}^{t_{I(c_1)+1}} \theta r(t)\left[u'(t) - \frac{u(t)w(t)}{\theta r(t)}\right]^2 dt \\
&- \sum_{k=I(c_1)+1}^{I(d_1)-1} \int_{t_k}^{t_{k+1}} \theta r(t)\left[u'(t) - \frac{u(t)w(t)}{\theta r(t)}\right]^2 dt - \int_{t_{I(d_1)}}^{d_1} \theta r(t)\left[u'(t) - \frac{u(t)w(t)}{\theta r(t)}\right]^2 \\
&- \int_{c_1}^{t_{I(c_1)+1}} u^2(t)Q(t)\frac{U(t-\tau)}{U(t)}dt - \sum_{k=I(c_1)+1}^{I(d_1)-1}\left[\int_{t_k}^{t_k+\tau} u^2(t)Q(t)\frac{U(t-\tau)}{U(t)}dt + \int_{t_k+\tau}^{t_{k+1}} u^2(t)Q(t)\frac{U(t-\tau)}{U(t)}dt\right] \\
&- \int_{t_{I(d_1)}}^{d_1} u^2(t)Q(t)\frac{U(t-\tau)}{U(t)}dt - \int_{c_1}^{d_1} p(t)u^2(t)\frac{w(t)}{r(t)}dt.
\end{aligned} \tag{16}$$

Now for $t \in [c_1, d_1]\backslash\tau_k, k \in \mathbb{N}$ from (3), it is clear that

$$\left[r(t)g(U'(t))\right]' + p(t)g(U'(t)) \leq F(t) - \epsilon q(t)U(t-\tau) - \sum_{i=1}^{n} \epsilon_i q_i(t)U(t-\tau) \leq 0.$$

That is,

$$\left(U'(t)\right)' + \left[\frac{r'(t)+p(t)}{r(t)}\right]U'(t) \leq 0.$$

This implies that

$$U'(t)exp\int_{c_1}^{t}\frac{r'(s)+p(s)}{r(s)}ds$$

is non-decreasing on $[c_1, d_1]\backslash t_k$. There are several case, we will estimate $\frac{U(t-\tau)}{U(t)}$ in each interval of t.

Case 1: For $t \in (t_k, t_{k+1}] \subset [c_1, d_1]$. We consider two sub cases:

Case 1.1: If $t \in [t_k + \tau, t_{k+1}]$, then $t - \tau \in [t_k, t_{k+1} - \tau]$ and there are no impulsive moments in $(t - \tau, t)$, then for any $t \in [t_k + \tau, t_{k+1}]$ one has

$$U(s) - U(t_k^+) = U'(\xi_1)(s - t_k), \quad \xi_1 \in (t_k, s),$$

$$U(s) \geq U'(\xi_1)(s-t_k).$$

Since $U'(t)exp\int_{c_1}^{t}\dfrac{r'(v)+p(v)}{r(v)}dv$ is non-increasing in $[c_1,t]$, we have

$$U'(\xi)exp\int_{c_1}^{\xi}\frac{r'(v)+p(v)}{r(v)}dv \geq U'(s)exp\int_{c_1}^{s}\frac{r'(v)+p(v)}{r(v)}dv.$$

From the fact that $r(t)$ is non-decreasing, we get

$$U(s) \geq \frac{U'(s)exp\int_{c_1}^{s}\frac{r'(v)+p(v)}{r(v)}dv}{exp\int_{c_1}^{\xi}\frac{r'(v)+p(v)}{r(v)}dv}(s-t_k)$$

$$U(s) \geq U'(s)\left(s-t_k\right).$$

We obtain

$$\frac{U'(s)}{U(s)} < \frac{1}{s-t_k}.$$

Integrating it from $t-\tau$ to t, we have

$$\frac{U(t-\tau)}{U(t)} > \frac{t-\tau-t_k}{t-t_k} > 0.$$

Case 1.2: If $t \in (t_k, t_k+\tau)$ then $t-\tau \in (t_k-\tau, t_k)$ and there is an impulsive moment t_k in $(t-\tau, t)$. Similar to Case 1.1, we obtain

$$\frac{U'(s)}{U(s)} < \frac{1}{s-t_k+\tau}, \quad for \;\; any \;\; s \in (t_k-\tau, t_k).$$

Integrating it from $t-\tau$ to t, we get

$$\frac{U(t-\tau)}{U(t_k)} > \frac{t-t_k}{\tau} \geq 0, \quad t \in (t_k, t_k+\tau). \tag{17}$$

For any $t \in (t_k, t_k+\tau)$, we have

$$U(t)-U(t_k^+) < U'(t_k^+)(t-t_k).$$

Using the impulsive conditions in equation (3), we get

$$U(t)-(1+\alpha_k)U(t_k) < (1+\beta_k)U'(t_k)(t-t_k)$$
$$\frac{U(t)}{U(t_k)} < (1+\beta_k)\frac{U'(t_k)}{U(t_k)}(t-t_k)+(1+\alpha_k).$$

Using $\frac{U'(t_k)}{U(t_k)} < \frac{1}{\tau}$, we obtain

$$\frac{U(t)}{U(t_k)} < (1+\alpha_k)+\frac{1}{\tau}(1+\beta_k)(t-t_k).$$

That is,

$$\frac{U(t_k)}{U(t)} > \frac{\tau}{(1+\alpha_k)\tau + (1+\beta_k)(t-t_k)}. \tag{18}$$

From (17) and (18), we get

$$\frac{U(t-\tau)}{U(t)} > \frac{t-t_k}{(1+\alpha_k)\tau + (1+\beta_k)(t-t_k)} \geq 0.$$

Case 2: If $t \in [c_1, t_{I(c_1)+1})$, we consider three sub cases:

Case 2.1: If $t_{I(c_1)} > c_1 - \tau$ and $t \in [t_{I(c_1)} + \tau, t_{I(c_1)+1}]$ then $t - \tau \in [t_{I(c_1)}, t_{I(c_1)+1} - \tau]$ and there are no impulsive moments in $(t-\tau, t)$. Making a similar analysis of the Case 1.1 and using Mean-value Theorem on $(t_{I(c_1)}, t_{I(c_1)+1}]$, we get

$$\frac{U(t-\tau)}{U(t)} > \frac{t-\tau-t_{I(c_1)}}{t-t_{I(c_1)}} \geq 0.$$

Case 2.2: If $t_{I(c_1)} > c_1 - \tau$ and $t \in [c_1, t_{I(c_1)} + \tau)$, then $t - \tau \in [c_1 - \tau, t_{I(c_1)})$ and there is an impulsive moments $t_{I(c_1)}$ in $(t-\tau, t)$. Making a similar analysis of the Case 1.2, we have

$$\frac{U(t-\tau)}{U(t)} > \frac{t-t_{I(c_1)}}{(1+\alpha_{I(c_1)})\tau + (1+\beta_{I(c_1)})(t-t_{I(c_1)})} \geq 0.$$

Case 2.3: If $t_{I(c_1)} < c_1 - \tau$ then for any $t \in [c_1, t_{I(c_1)+1}]$, $t - \tau \in [c_1 - \tau, t_{I(c_1)+1} - \tau]$ and there are no impulsive moments in $(t-\tau, t)$. Making a similar analysis of the Case 1.1, we obtain

$$\frac{U(t-\tau)}{U(t)} > \frac{t-\tau-t_{I(c_1)}}{t-t_{I(c_1)}} \geq 0.$$

Case 3: For $t \in (t_{I(d_1)}, d_1]$, there are three sub cases:

Case 3.1: If $t_{I(d_1)} + \tau < d_1$ and $t \in [t_{I(d_1)} + \tau, d_1]$ then $t - \tau \in [t_{I(d_1)}, d_1 - \tau]$ and there are no impulsive moments in $(t-\tau, t)$. Making a similar analysis of the Case 2.1, we have

$$\frac{U(t-\tau)}{U(t)} > \frac{t-\tau-t_{I(d_1)}}{t-t_{I(d_1)}} \geq 0.$$

Case 3.2: If $t_{I(d_1)} + \tau < d_1$ and $t \in [t_{I(d_1)}, t_{I(d_1)} + \tau)$, then $t - \tau \in [t_{I(d_1)} - \tau, t_{I(d_1)})$ and there is an impulsive moments $t_{I(d_1)}$ in $(t-\tau, t)$. Making a similar analysis of the Case 2.2, we obtain

$$\frac{U(t-\tau)}{U(t)} > \frac{t-t_{I(d_1)}}{(1+\alpha_{I(d_1)})\tau + (1+\beta_{I(d_1)})(t-t_{I(d_1)})} \geq 0.$$

Case 3.3: If $t_{I(d_1)} + \tau \geq d_1$ then for any $t \in (t_{I(d_1)}, d_1]$, we get $t - \tau \in (t_{I(d_1)} - \tau, d_1 - \tau]$ and there is an impulsive moments $t_{I(d_1)}$ in $(t-\tau, t)$. Making a similar analysis of the Case 3.2, we get

$$\frac{U(t-\tau)}{U(t)} > \frac{t-t_{I(d_1)}}{(1+\alpha_{I(d_1)})\tau + (1+\beta_{I(d_1)})(t-t_{I(d_1)})} \geq 0.$$

Combining all these cases, we have

$$\frac{U(t-\tau)}{U(t)} > \begin{cases} R^1_{I(c_1)}(t) & \text{for} \quad t \in [c_1, t_{I(c_1)+1}] \\ R^1_k(t) & \text{for} \quad t \in (t_k, t_{k+1}], \quad k = I(c_1)+1, \cdots, I(d_1)-1 \\ R^1_{I(d_1)}(t) & \text{for} \quad t \in (t_{I(d_1)+1}, d_1]. \end{cases}$$

Hence by (16), we have

$$\sum_{k=I(c_1)+1}^{I(d_1)} u^2(t_k)\left[w(t_k)-w(t_k^+)\right] \le \int_{c_1}^{d_1} r(t)\left[\theta(u'(t))^2 - p(t)\frac{u^2(t)w(t)}{r^2(t)}\right]dt - \int_{c_1}^{t_{I(c_1)+1}} u^2(t)Q(t)R^1_{I(c_1)}(t)dt$$
$$- \sum_{k=I(c_1)+1}^{I(d_1)-1}\int_{t_k}^{t_{k+1}} u^2(t)Q(t)R^1_k(t)dt - \int_{t_{I(d_1)}}^{d_1} u^2(t)Q(t)R^1_{I(d_1)}(t)dt. \tag{19}$$

For all $t \in (c_1, t_{I(c_1)+1}]$, where

$$U(t) \ge U'(\xi)\,(t-c_1)\,, \quad \xi \in (c_1, t) \tag{20}$$

By the monotonicity of $U'(t)exp\int_{c_1}^{t}\frac{r'(t)+p(t)}{r(t)}ds$ and (20) we have

$$U(t) \ge \frac{U'(t)exp\int_{c_1}^{t}\frac{r'(s)+p(s)}{r(s)}ds}{exp\int_{c_1}^{\xi}\frac{r'(s)+p(s)}{r(s)}ds}(t-c_1)$$
$$U(t) \ge U'(t)\,(t-c_1)$$

for some $\xi \in (c_1, t)$. It follows

$$\frac{U'(t)}{U(t)} \le \frac{1}{t-c_1}.$$

Taking $t \to t_{I(c_1)+1}$, it follows that

$$\frac{w(t)}{r(t)} \le \frac{\theta}{t-c_1}.$$

Then we get,

$$w(t_{I(c_1)+1}) \le \frac{\theta r(t)}{t_{I(c_1)+1}-c_1},$$
$$w(t_{I(c_1)+1}) \le \frac{r_1}{t_{I(c_1)+1}-c_1}. \tag{21}$$

Similarly we can prove that on $(t_{k-1}, t_k]$, $k = I(c_1)+2, \cdots, I(d_1)$,

$$w(t_k) \le \frac{r_1}{t_k - t_{k-1}}. \tag{22}$$

Hence (21) and (22), we have

$$\sum_{k=I(c_1)+1}^{I(d_1)}\left[\frac{\alpha_k-\beta_k}{1+\alpha_k}\right]u^2(t_k)w(t_k) \ge r_1\left[\frac{\alpha_{I(c_1)+1}-\beta_{I(c_1)+1}}{(1+\alpha_{I(c_1)+1})(t_{I(c_1)+1}-c_1)}u^2(t_{I(c_1)+1}) + \sum_{k=I(c_1)+2}^{I(d_1)}\frac{\alpha_k-\beta_k}{(1+\alpha_k)(t_k-t_{k-1})}u^2(t_k)\right]$$
$$\ge r_1\Pi_{c_1}^{d_1}[u^2(t)]. \tag{23}$$

Thus we have

$$\sum_{k=I(c_1)+1}^{I(d_1)}\left[\frac{\alpha_k-\beta_k}{1+\alpha_k}\right]u^2(t_k)w(t_k)\geq r_1\Pi_{c_1}^{d_1}[u^2(t)].$$

Therefore (19), we get

$$\begin{aligned}&\int_{c_1}^{d_1}r(t)\left[\theta(u'(t))^2-p(t)\frac{u^2(t)w(t)}{r^2(t)}\right]dt-\int_{c_1}^{t_{I(c_1)+1}}u^2(t)Q(t)R^1_{I(c_1)}(t)dt\\&-\sum_{k=I(c_1)+1}^{I(d_1)-1}\int_{t_k}^{t_{k+1}}u^2(t)Q(t)R^1_k(t)dt-\int_{t_{I(d_1)}}^{d_1}u^2(t)Q(t)R^1_{I(d_1)}(t)dt>r_1\Pi_{c_1}^{d_1}[u^2(t)]\end{aligned}$$

which contradicts (10).

If $I(c_1)=I(d_1)$ then $\Pi_{c_1}^{d_1}[u^2(t)]=0$ and there are no impulsive moments in $[c_1,d_1]$. Similar to the proof of (19), we obtain

$$\int_{c_1}^{d_1}r(t)\left[\theta(u'(t))^2-p(t)\frac{u^2(t)w(t)}{r^2(t)}\right]dt-\int_{c_1}^{t_{I(c_1)+1}}u^2(t)Q(t)R^1_{I(c_1)}(t)dt>0.$$

This again contradicts our assumption. Finally if $U(t)$ is eventually negative, we can consider $[c_2,d_2]$ and reach similar contradiction. The proof of theorem is complete. □

Next we obtain some new oscillatory results for (1) − (2), by using integral average condition of Philos type. Let $D=\{(t,s):t_0\leq s\leq t\}$, $H_1,H_2\in C^1(D,\mathbb{R})$. If $H_1,H_2\in\mathbb{H}$, then $H_1(t,t)=H_2(t,t)=0$ and $H_1(t,s)>0,H_2(t,s)>0$ for $t>s$ and $h_1,h_2\in L_{loc}(D,\mathbb{R})$ such that

$$\frac{\partial H_1(t,s)}{\partial t}=h_1(t,s)H_1(t,s),\quad\frac{\partial H_2(t,s)}{\partial s}=-h_2(t,s)H_2(t,s).\tag{24}$$

For $\lambda\in(c_s,d_s)$, $s=1,2$,

$$\begin{aligned}\Gamma_{1,s}=&\int_{c_s}^{t_{I(c_s)+1}}H_1(t,c_s)Q(t)R^s_{I(c_s)}(t)dt+\sum_{k=I(c_s)+1}^{I(\lambda_s)-1}\int_{t_k}^{t_{k+1}}H_1(t,c_s)Q(t)R^s_k(t)dt+\int_{t_{I(\lambda_s)}}^{\lambda_s}H_1(t,c_s)Q(t)R^s_{I(d_s)}(t)dt\\&-\int_{c_s}^{d_s}\frac{w(t)}{r(t)}\left[h_1(t,c_s)r(t)-\frac{w(t)}{\theta}-p(t)\right]H_1(t,c_s)dt\end{aligned}$$

and

$$\begin{aligned}\Gamma_{2,s}=&\int_{\lambda_s}^{t_{I(\lambda_s)+1}}H_2(d_s,t)Q(t)R^s_{I(\lambda_s)}(t)dt+\sum_{k=I(\lambda_s)+1}^{I(d_s)-1}\int_{t_k}^{t_{k+1}}H_2(d_s,t)Q(t)R^s_k(t)dt+\int_{t_{I(d_s)}}^{d_s}H_2(d_s,t)Q(t)R^s_{I(d_s)}(t)dt\\&-\int_{c_s}^{d_s}\frac{w(t)}{r(t)}\left[h_2(d_s,t)r(t)-\frac{w(t)}{\theta}-p(t)\right]H_2(d_s,t)dt.\end{aligned}$$

Theorem 2.3. *Assume that conditions $(A_1)-(A_5)$ hold, furthermore for any $T\geq 0$ there exist c_s,d_s satisfying (H_6) with $c_1<\lambda_1<d_1\leq c_2<\lambda_2<d_2$. If there exists $H_1,H_2\in\mathcal{H}$ such that*

$$\frac{1}{H_1(\lambda_1,c_1)}\Gamma_{1,1}+\frac{1}{H_2(d_1,\lambda_1)}\Gamma_{2,1}>\Lambda(H_1,H_2;c_s,d_s)\tag{25}$$

where

$$\Lambda(H_1,H_2;c_s,d_s)=-\left\{\frac{r_s}{H_1(\lambda_s,c_s)}\Pi_{c_s}^{\lambda_s}[H_1(.,c_s)]+\frac{r_s}{H_2(d_s,\lambda_s)}\Pi_{\lambda_s}^{d_s}[H_2(d_s,.)]\right\},\tag{26}$$

then every solution of the boundary value problem (1) − (2) is oscillatory in G.

Proof. Suppose to the contrary that there is a non-oscillatory solution $u(x,t)$ of the boundary value problem (1) − (2). Notice whether or not there are impulsive moments in $[c_1,\lambda_1]$ and $[\lambda_1,d_1]$, we should consider the following cases $I(c_1) < I(\lambda_1) < I(d_1), I(c_1) = I(\lambda_1) < I(d_1), I(c_1) < I(\lambda_1) = I(d_1)$ and $I(c_1) = I(\lambda_1) = I(d_1)$.

Moreover, the impulsive moments of $U(t-\tau)$ having following two cases, $t_{I(\lambda_s)} + \tau > \lambda_s$ and $t_{I(\lambda_s)} + \tau \le \lambda_s$.

Consider the case $I(c_1) < I(\lambda_1) < I(d_1)$, with $t_{I(\lambda_s)} + \tau > \lambda_s$.

For this case, the impulsive moments are $t_{I(\lambda_1)+1},\ t_{I(\lambda_1)+2}, \cdots,\ t_{I(d_1)}$ in $[\lambda_1, d_1]$.

Multiplying by $H_1(t,c_1)$ on both sides on (14), integrating it from c_1 to λ_1, we obtain

$$\int_{c_1}^{\lambda_1} H_1(t,c_1)Q(t)\frac{U(t-\tau)}{U(t)}dt \le -\int_{c_1}^{\lambda_1} H_1(t,c_1)w'(t)dt - \int_{c_1}^{\lambda_1} H_1(t,c_1)\frac{w^2(t)}{\theta r(t)}dt - \int_{c_1}^{\lambda_1} H_1(t,c_1)p(t)\frac{w(t)}{r(t)}dt.$$

Applying integration by parts on the R.H.S of first integral we get,

$$\int_{c_1}^{\lambda_1} H_1(t,c_1)Q(t)\frac{U(t-\tau)}{U(t)}dt \le -\sum_{k=I(c_1)+1}^{I(\lambda_1)} H_1(t_k,c_1)\left[w(t_k) - w(t_k^+)\right] - H_1(\lambda_1,c_1)w(\lambda_1)$$
$$+\left(\int_{c_1}^{t_{I(c_1)+1}} + \sum_{k=I(c_1)+1}^{I\lambda_1)-1}\int_{t_k}^{t_{k+1}} + \int_{t_{I(\lambda_1)}}^{\lambda_1}\right)\left[h_1(t,c_1)w(t) - \frac{p(t)w(t)}{r(t)} - \frac{w^2(t)}{\theta r(t)}\right] H_1(t,c_1)dt$$

Then we get,

$$\int_{c_1}^{\lambda_1} H_1(t,c_1)Q(t)\frac{U(t-\tau)}{U(t)}dt - \int_{c_1}^{\lambda_1}\left[h_1(t,c_1)w(t) - \frac{p(t)w(t)}{r(t)} - \frac{w^2(t)}{\theta r(t)}\right] H_1(t,c_1)dt$$
$$\le -\sum_{k=I(c_1)+1}^{I(\lambda_1)} H_1(t_k,c_1)\left[\frac{\alpha_k-\beta_k}{1+\alpha_k}\right] w(t_k) - H_1(\lambda_1,c_1)w(\lambda_1). \tag{27}$$

By Theorem 2.2, we divide the interval $[c_1,\lambda_1]$ into several and calculating the function $\dfrac{U(t-\tau)}{U(t)}$, we obtain

$$\int_{c_1}^{\lambda_1} H_1(t,c_1)Q(t)\frac{U(t-\tau)}{U(t)}dt \ge \int_{c_1}^{t_{I(c_1)+1}} H_1(t,c_1)Q(t)R^1_{I(c_1)}(t)dt + \sum_{k=I(c_1)+1}^{I(d_1)-1}\int_{t_k}^{t_{k+1}} H_1(t,c_1)Q(t)R^1_{k(t)}dt$$
$$+\int_{t_{I(d_1)}}^{d_1} H_1(t,c_1)Q(t)R^1_{I(d_1)}(t)dt. \tag{28}$$

From (27) and (28), we obtain

$$\int_{c_1}^{t_{I(c_1)+1}} H_1(t,c_1)Q(t)R^1_{I(c_1)}(t)dt + \sum_{k=I(c_1)+1}^{I(d_1)-1}\int_{t_k}^{t_{k+1}} H_1(t_k,c_1)Q(t)R^1_{k(t)}dt$$
$$+\int_{t_{I(d_1)}}^{d_1} H_1(t,c_1)Q(t)R^1_{I(d_1)}(t)dt - \int_{c_1}^{\lambda_1}\left[h_1(t,c_1)w(t) - \frac{p(t)w(t)}{r(t)} - \frac{w^2(t)}{\theta r(t)}\right] H_1(t,c_1)dt$$
$$< -\sum_{k=I(c_1)+1}^{I(\lambda_1)} H_1(t_k,c_1)\left[\frac{\alpha_k-\beta_k}{1+\alpha_k}\right] w(t_k) - H_1(\lambda_1,c_1)w(\lambda_1). \tag{29}$$

On the other hand multiplying both sides of (14) by $H_2(d_1,t)$ and integrating from λ_1 to d_1 and using the similar of above, we get

$$\int_{\lambda_1}^{t_{I(\lambda_1)+1}} H_2(d_1,t)Q(t)R^1_{I(\lambda_1)}(t)dt + \sum_{k=I(\lambda_1)+1}^{I(d_1)-1}\int_{t_k}^{t_{k+1}} H_2(d_1,t_k)Q(t)R^1_{k(t)}dt + \int_{t_{I(d_1)}}^{d_1} H_2(d_1,t)Q(t)R^1_{I(d_1)}(t)dt$$
$$-\int_{\lambda_1}^{d_1}\left[h_2(d_1,t)w(t) - \frac{p(t)w(t)}{r(t)} - \frac{w^2(t)}{\theta r(t)}\right] H_2(d_1,t)dt$$

$$< -\sum_{k=I(\lambda_1)+1}^{I(d_1)} H_2(d_1,t_k)\left[\frac{\alpha_k-\beta_k}{1+\alpha_k}\right]w(t_k)+H_2(d_1,\lambda_1)w(\lambda_1). \tag{30}$$

Dividing (29) and (30) by $H_1(\lambda_1,c_1)$ and $H_2(d_1,\lambda_1)$ respectively and adding them, we get

$$\frac{1}{H_1(\lambda_1,c_1)}\Gamma_{1,1}+\frac{1}{H_2(d_1\lambda_1)}\Gamma_{2,1}\le -\left[\frac{1}{H_1(\lambda_1,c_1)}\sum_{k=I(c_1)+1}^{I(\lambda_1)} H_1(t_k,c_1)\left[\frac{\alpha_k-\beta_k}{1+\alpha_k}\right]w(t_k)\right.$$
$$\left.+\frac{1}{H_2(d_1,\lambda_1)}\sum_{k=I(\lambda_1)+1}^{I(d_1)} H_2(d_1,t_k)\left[\frac{\alpha_k-\beta_k}{1+\alpha_k}\right]w(t_k)\right]. \tag{31}$$

Using the method as in (23) , we obtain

$$\left.\begin{array}{l} -\sum\limits_{k=I(c_1)+1}^{I(\lambda_1)} H_1(t_k,c_1)\left[\dfrac{\alpha_k-\beta_k}{1+\alpha_k}\right]w(t_k)\le -r_1\Pi_{c_1}^{\lambda_1}[H_1(.,c_1)] \\ -\sum\limits_{k=I(\lambda_1)+1}^{I(d_1)} H_2(d_1,t_k)\left[\dfrac{\alpha_k-\beta_k}{1+\alpha_k}\right]w(t_k)\le -r_1\Pi_{\lambda_1}^{d_1}[H_2(d_1,.)]. \end{array}\right\} \tag{32}$$

From (31) and (32), we obtain

$$\frac{1}{H_1(\lambda_1,c_1)}\Gamma_{1,1}+\frac{1}{H_2(d_1,\lambda_1)}\Gamma_{2,1}\le -\left\{\frac{r_1}{H_1(\lambda_1,c_1)}\Pi_{c_1}^{\lambda_1}[H_1(.,c_1)]+\frac{r_1}{H_2(d_1,\lambda_1)}\Pi_{\lambda_1}^{d_1}[H_2(d_1,.)]\right\}\le \Lambda(H_1,H_2;c_1,d_1) \tag{33}$$

which is contradiction to the condition (25). Suppose $u(x,t)<0$, we take interval $[c_2,d_2]$ for equation (1).The proof is similar and hence omitted. □

3. Example

In this section, we present an example to illustrate our results established in Section 2.

Example 3.1. *Consider the following impulsive partial differential equation*

$$\left.\begin{array}{l} \dfrac{\partial}{\partial t}\left[\dfrac{1}{53}g\left(\dfrac{\partial}{\partial t}u(x,t)\right)\right]+tg\left(\dfrac{\partial}{\partial t}u(x,t)\right)+\dfrac{4}{3}u(x,t-\pi/8)+\dfrac{2}{3}u(x,t-\pi/8) \\ =\dfrac{4}{3}\Delta u(x,t)+\dfrac{1}{53}\Delta u(x,t-\pi)+F(x,t), \quad t\neq 2k\pi\pm\frac{\pi}{4} \\ u(x,t_k^+)=\dfrac{5}{2}u(x,t_k),\ u_t(x,t_k^+)=\dfrac{7}{2}u_t(x,t_k), \quad t_k=2k\pi\pm\frac{\pi}{4}, \quad k=1,2,\cdots. \end{array}\right\} \tag{34}$$

for $(x,t)\in(0,\pi)\times\mathbb{R}_+$, with the boundary condition

$$u_x(0,t)+u(0,t)=u_x(\pi,t)+u(\pi,t)=0, \quad t\neq t_k, \quad k=1,2,\cdots. \tag{35}$$

Here $r(t)=1/53, p(t)=t, a(t)=4/3, a_1(t)=1/53,\ q(t)=4/3,\ q_1(t)=2/3,\ \rho_1(t)=\pi,\ f(u)=f_1(u)=u,\ F(t)=2e^{-x}\left[t\cos t+\sin(t-\frac{\pi}{8})\right],\ \alpha_k=3/2, \beta_k=5/2$. Let $\tau=\frac{\pi}{8}, t_{k+1}-t_k=\frac{\pi}{2}>\frac{\pi}{8}$. Also for any $T>0$, we choose k large enough such that $T<c_1=4k\pi-\frac{\pi}{2}<d_1=4k\pi$ and $c_2=4k\pi+\frac{\pi}{8}<d_2=4k\pi+\frac{\pi}{2}, k=1,2,\cdots,$. Then there is an impulsive movement $t_k=4k\pi-\frac{\pi}{4}$ in $[c_1,d_1]$ and an impulsive moment $t_{k+1}=4k\pi+\frac{\pi}{4}$ in $[c_2,d_2]$.

For $\epsilon=\epsilon_1=1$, we have $Q(t)=2$, and we take $u(t)=\sin 4t, t_{I(c_1)}=4k\pi-\frac{7\pi}{4}, t_{I(d_1)}=4k\pi-\frac{\pi}{4}$, then by using simple calculation, the left side of Equation (10) is the following :

$$\int_{c_1}^{d_1} r(t)\left[\theta(u'(t))^2-p(t)u^2(t)\frac{w(t)}{r(t)}\right]dt-\int_{c_1}^{t_{I(c_1)+1}} Q(t)u^2(t)R^1_{I(c_1)}(t)dt$$

$$
\begin{aligned}
&-\sum_{k=I(c_1)+1}^{I(d_1)-1}\int_{t_{k-1}}^{t_k} Q(t)u^2(t)R_k^1(t)dt-\int_{t_{I(d_1)}}^{d_1} Q(t)u^2(t)R_{I(d_1)}^1(t)dt\\
\geq &1/53\int_{4k\pi-\frac{\pi}{2}}^{4k\pi}\left[2(16\cos^2 4t)-t\sin^2 4t(-2)\right]dt\\
&-2\int_{4k\pi-\frac{\pi}{2}}^{4k\pi-\frac{\pi}{4}}\sin^2 4t\left(\frac{t-\frac{\pi}{8}-4k\pi+\frac{7\pi}{4}}{t-4k\pi+\frac{7\pi}{4}}\right)dt\\
&-2\int_{4k\pi-\frac{\pi}{4}}^{4k\pi-\frac{\pi}{8}}\sin^2 4t\left(\frac{t-4k\pi+\frac{\pi}{4}}{(\frac{5}{2})\frac{\pi}{8}+(\frac{7}{2})(t-4k\pi+\frac{\pi}{4})}\right)dt\\
&-2\int_{4k\pi-\frac{\pi}{8}}^{4k\pi}\sin^2 4t\left(\frac{t-\frac{\pi}{8}-4k\pi+\frac{\pi}{4}}{t-4k\pi+\frac{\pi}{4}}\right)dt\\
\simeq &-0.0275.
\end{aligned}
$$

But $I(c_1)=k+1, I(d_1)=k, r_1=2$, *we have*

$$
\begin{aligned}
r_1\Pi_{c_1}^{d_1}[u^2(t)] =&2\left[\frac{\alpha_{I(c_1)+1}-\beta_{I(c_1)+1}}{(1+\alpha_{I(c_1)+1})(t_{I(c_1)+1}-c_1)}\sin^2(4t_{I(c_1)+1})\right],\\
=&0.
\end{aligned}
$$

Therefore the condition (10) *is satisfied in* $[c_1, d_1]$. *Similarly, we can prove that for* $t\in[c_2, d_2]$. *Hence by Theorem 2.2, every solution of* (34) $-$ (35) *is oscillatory. In fact* $u(x,t)=e^{-x}sint$ *is one such solution.*

Conclusion: In this article, we obtained some new sufficient conditions for all solutions of impulsive partial differential equations with damping term to be oscillatory, which extend and take a broad view of some known results in [3, 13, 16].

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