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Interval Oscillation Criteria for Second-order Forced Delay Differential Equation with Riemann-Stieltjes Integral

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Abstract: In this paper, we study the oscillatory behavior of second-order forced delay differential equation with Riemann-Stieltjes integral. By using the Riccati transformation technique, interval oscillation criteria of both El-Sayed type and Kong type are established, which generalize and extend some of the existing results. Finally, two examples are presented to illustrate the theoretical results.

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1. Introduction

In this paper, we are concerned with the interval oscillation criteria for the second order forced delay differential equation with Riemann-Stieltjes integral of the form

$$(p(t)\Phi_\alpha(x'(t)))' + q(t)\Phi_\alpha(x(\tau(t))) + \int_0^h r(t,s)\Phi_{\gamma(s)}(x(\psi(t,s)))d\xi(s) = e(t),\ t \geq t_0, \tag{1}$$

where $\Phi_*(z) = |z|^* \operatorname{sgn} z, 0 < h < \infty, \int_0^h f(s)d\xi(s)$ denotes the Riemann-Stieltjes integral of the function f on $[0,h]$ with respect to ξ and $\xi : [0,h] \to \mathbb{R}$ is non- decreasing, $\gamma(s)$ is strictly increasing continuous function on $[0,h]$ satisfying $0 \leq \gamma(0) < \alpha < \gamma(h)$; $p \in C^1[t_0,\infty)$ with $p(t) > 0, q, e \in C[t_0,\infty)$ and $r \in C([t_0,\infty)\times[0,h])$; $\tau : [t_0,\infty) \to [\mu,\infty), \psi : [t_0,\infty)\times[0,h] \to [\mu,\infty)$ with $\mu \leq t_0$ are continuous functions satisfying $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\psi(t,s) = \infty$ for $s \in [0,h]$ and $\tau(t), \psi(t,s) \leq t$. By a *solution* of equation (1), we mean a function $x(t) \in C^1([t_x,\infty),\mathbb{R}), t_x \geq t_0$, which has the property $(p(t)\Phi_\alpha(x'(t))) \in C^1([t_x,\infty),\mathbb{R})$ and satisfies equation (1) for $t \in [t_x,\infty)$. As usual, a nontrivial solution $x(t)$ of equation (1) is called oscillatory if it has arbitrarily large zeros on the interval $[t_x,\infty)$; otherwise, it is termed nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory. The theory of oscillation is an important branch of the qualitative theory of differential equations. In the past few decades, a great deal of effort has been spent in obtaining the sufficient conditions for the oscillation/nonoscillation of solutions of different classes of differential equations such as linear and nonlinear ordinary and functional differential equations; we refer the reader to the monographs [2, 5] and the references quoted therein. In recent years, people have been increasing interest in establishing interval oscillation criteria for

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the combined effects of linear, superlinear, sublinear terms and forcing terms. For instance, Sun and Wong [13] investigated the following forced differential equation with mixed nonlinearities

$$(p(t)x')' + q(t)x + \sum_{i=1}^{n} q_i(t)|x|^{\alpha_i}\mathrm{sgn}x = e(t), \tag{2}$$

where $p, q, q_i, e \in C[0, \infty)$ and $\alpha_1 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots > \alpha_n$. Sun and Meng [12] have studied the same equation by making use of some of the arguments developed by Kong [9]. In [11], Sun and Kong studied the oscillation of the second-order forced differential equation with the nonlinearities given by Riemann-Stieltjes integral of the form

$$(p(t)x')' + q(t)x + \int_0^b r(t,s)|x(t)|^{\alpha(s)}\mathrm{sgn}x(t)d\xi(s) = e(t), \tag{3}$$

where $p, q, e \in C[0, \infty)$ with $p(t) > 0, r \in C([0,\infty) \times [0, b)), \alpha \in C[0, b)$ is strictly increasing such that $0 \le \alpha(0) < 1 < \alpha(b-)$. In [11], Sun and Kong studied the oscillation of the second- order forced differential equation with the nonlinearities given by Riemann-Stieltjes integral of the form

$$(p(t)x')' + q(t)x + \int_0^b r(t,s)|x(t)|^{\alpha(s)}\mathrm{sgn}x(t)d\xi(s) = e(t), \tag{4}$$

where $p, q, e \in C[0, \infty)$ with $p(t) > 0, r \in C([0,\infty) \times [0, b)), \alpha \in C[0, b)$ is strictly increasing such that $0 \le \alpha(0) < 1 < \alpha(b-)$. In [8], Hassan and Kong extended the results in [11] to

$$(p(t)\phi_\gamma(x'(t)))' + q_0(t)\phi_\gamma(x(t)) + \int_0^b q(t,s)\phi_{\alpha(s)}(x(t))d\xi(s) = e(t), \tag{5}$$

where $p, q_0, e \in C[0, \infty)$ with $p(t) > 0, q \in C([0,\infty)\times[0, b)), \alpha \in C[0, b)$ is strictly increasing such that $0 \le \alpha(0) < \gamma < \alpha(b-)$. It is obvious that (2), (4) and (5) are special cases of (1). Motivated by the ideas in [1, 4, 7–11], we establish interval oscillation criteria for equation (1).

The organization of this paper is as follows. After this introduction, in Section 2, some important lemmas are given and we establish interval oscillation criteria of both the El-Sayed type and the Kong type for equation (1) and its special case. In Section 3, we give two examples to illustrate our main results.

2. Main Results

We denote by $L_\xi(0, h)$ the set of Riemann-Stieltjes integrable functions on $[0, h]$ with respect to ξ. Let $a \in (0, h)$ such that $\gamma(a) = \alpha$ and let γ^{-1} be the reciprocal of γ. We further assume that,

$$\gamma^{-1} \in L_\xi(0,h) \text{ such that } \int_0^a d\xi(s) > 0 \text{ and } \int_a^h d\xi(s) > 0.$$

We see that the condition $\gamma^{-1} \in L_\xi(0, h)$ is satisfied if either $\gamma(0) > 0$ or $\gamma(s) \to 0$ "slowly"as $s \to 0^+$, or $\xi(s)$ is constant in a right neighborhood of 0.

Lemma 2.1 ([6])**.** *Suppose X and Y are non-negative real numbers. Then*

$$\lambda XY^{\lambda-1} - X^\lambda \le (\lambda - 1)Y^\lambda, \ \lambda > 1, \tag{6}$$

where equality holds if and only if $X = Y$.

Lemma 2.2 ([8]). *Let*

$$m = \alpha \left(\int_a^h \gamma^{-1}(s) d\xi(s) \right) \left(\int_a^h d\xi(s) \right)^{-1}$$
$$n = \alpha \left(\int_0^a \gamma^{-1}(s) d\xi(s) \right) \left(\int_0^a d\xi(s) \right)^{-1}.$$

Then for any $\delta \in (m, n)$, there exists $\eta \in L_\xi(0,h)$ such that $\eta(s) > 0$ on $[0,h]$,

$$\int_0^h \gamma(s)\eta(s) d\xi(s) = \alpha, \tag{7}$$
$$\int_0^h \eta(s) d\xi(s) = \delta. \tag{8}$$

The following lemma is a generalized arithmetic-geometric mean inequality established in [11].

Lemma 2.3. *Let $u \in C[0,h]$ and $\eta \in L_\xi(0,h)$ satisfying $u \geq 0, \eta > 0$ on $[0,h]$ and $\int_0^h \eta(s) d\xi(s) = 1$. Then*

$$\int_0^h \eta(s) u(s) d\xi(s) \geq \exp\left(\int_0^h \eta(s) \ln[u(s)] d\xi(s) \right), \tag{9}$$

where we use the convention that $\ln 0 = -\infty$ and $e^{-\infty} = 0$.

The following two lemmas are generalization of Lemma 4.1 in [11].

Lemma 2.4. *Let $\tau \in C([t_0,\infty),[\mu,\infty))$ with $\mu \leq t_0$ be such that $0 \leq \tau(t) \leq t$ and $lim_{t\to\infty}\tau(t) = \infty, c, d \in [t_0,\infty)$ with $c < d$ and $\tau_* = min\{\tau(t) : t \in [c,d]\}$. Assume that $x \in C^1([\tau_*, d], \mathbb{R})$ is a positive function such that $p(t)\Phi_\alpha(x'(t))$ is nonincreasing on $[\tau_*, d]$. Then*

$$\frac{x(\tau(t))}{x(t)} \geq \frac{\mathcal{P}(\tau(t), \tau_*)}{\mathcal{P}(t, \tau_*)},\ t \in [c,d], \tag{10}$$

where $\mathcal{P}(t,a) = \int_a^t p^{-1/\alpha}(s) ds$.

Proof. Set $\upsilon(t) = p^{1/\alpha}(t)x'(t)$. It is easy to prove that $\upsilon(t)$ is nonincreasing on $[\tau_*, d]$ because $p(t)\Phi_\alpha(x'(t))$ is nonincreasing on $[\tau_*, d]$. Then we have

$$\begin{aligned} x(t) &= x(\tau_*) + \int_{\tau_*}^t x'(s) ds \\ &= x(\tau_*) + \int_{\tau_*}^t p^{-1/\alpha}(s)\upsilon(s) ds \\ &\geq \upsilon(t) \int_{\tau_*}^t p^{-1/\alpha}(s) ds \\ &= p^{1/\alpha}(t)\mathcal{P}(t,\tau_*)x'(t),\ t \in [\tau_*, d]. \end{aligned} \tag{11}$$

Next, for $s \in [\tau(t), t]$ and $t \in [c,d]$ we define

$$\kappa(s) := x(s) - p^{1/\alpha}(s)\mathcal{P}(s,\tau_*)x'(s). \tag{12}$$

Then (11) yields that $\kappa(s) \geq 0$ for $s \in [\tau(t), t]$ with $t \in [c, d]$. Consequently, for $t \in [c, d]$, we have

$$0 \leq \int_{\tau(t)}^{t} \frac{p^{-1/\alpha}\kappa(s)}{x^2(s)} ds = \int_{\tau(t)}^{t} \left[\frac{\mathcal{P}(s, \tau_*)}{x(s)}\right]' ds.$$

This implies that,

$$\frac{x(\tau(t))}{x(t)} \geq \frac{\mathcal{P}(\tau(t), \tau_*)}{\mathcal{P}(t, \tau_*)}, \, t \in [c, d]. \tag{13}$$

This completes the proof of Lemma 2.4. □

Similar to the proof of Lemma 2.4, we can get the following result.

Lemma 2.5. *Let $\psi(t, s) \in C([t_0, \infty) \times [0, h], [\mu, \infty))$ with $\mu \leq t_0$ be such that $0 \leq \psi(t, s) \leq t$ and $\lim_{t\to\infty} \psi(t, s) = \infty, c, d \in [t_0, \infty)$ with $c < d$ and $\psi_* = \min\{\psi(t, s) : (t, s) \in [c, d] \times [0, h]\}$. Assume that $x \in C^1([\psi_*, d], \mathbb{R})$ is a positive function for which $p(t)\Phi_\alpha(x'(t))$ is nonincreasing on $[\psi_*, d]$. Then*

$$\frac{x(\psi(t, s))}{x(t)} \geq \frac{\mathcal{P}(\psi(t, s), \psi_*)}{\mathcal{P}(t, \psi_*)}, \, t \in [c, d], \tag{14}$$

where $\mathcal{P}(a, t)$ is defined as in Lemma 2.4.

We note from the definition of m and n that $0 < m < 1 < n$. In the following, we will use the values of δ in the interval $(m, 1]$ to establish interval criteria for oscillation of equation (1). Following El-Sayed [4], for $c, d \in [t_0, \infty)$ with $c < d$, we define the function class

$$\mathcal{W}(c, d) := \{w \in C^1[c, d] \;:\; w(c) = 0 = w(d), w \not\equiv 0\}. \tag{15}$$

Our first result provides an oscillation criterion for equation (1) of the El-Sayed type.

Theorem 2.6. *Assume that $\tau(t), \psi(t, s) \leq t$ for $t \in [t_0, \infty)$ and $s \in [0, h]$. Suppose also that for any $T \geq t_0$, there exists subintervals $[c_i, d_i]$ of $[T, \infty), i = 1, 2$ such that $T < c_1 - \Psi_1 < c_1 < d_1 \leq c_2 < d_2$ and*

$$\begin{aligned} r(t, s) \geq 0, \;& (t, s) \in [\Psi_i, d_i] \times [0, h], \\ (-1)^i e(t) \geq 0, \;& t \in [\Psi_i, d_i], \end{aligned} \tag{16}$$

where $\Psi_i = min\{\tau_*^i, \psi_*^i\}$. For each $\delta \in (m, 1]$, let $\eta \in L_\xi(0, h)$ be defined as in Lemma 2.2. Further assume that for $i = 1, 2$, there exists a function $w_i \in \mathcal{W}(c_i, d_i)$ such that

$$\sup_{\delta\in(m,1]} \int_{c_i}^{d_i} \left[Q_i(t)|w_i(t)|^{\alpha+1} - p(t)|w_i'(t)|^{\alpha+1}\right] dt > 0, \tag{17}$$

where

$$Q_i(t) = q(t) \left[\frac{\mathcal{P}(\tau(t), \tau_*^i)}{\mathcal{P}(t, \tau_*^i)}\right]^\alpha + \left[\frac{|e(t)|}{1 - \delta}\right]^{1-\delta} \exp\left(\int_0^h \eta(s) \ln\left(\frac{r(t, s)}{\eta(s)} \left[\frac{\mathcal{P}(\psi(t, s), \psi_*^i)}{\mathcal{P}(t, \psi_*^i)}\right]^{\gamma(s)}\right) d\xi(s)\right). \tag{18}$$

Here we use the convention that $0^{1-\delta} = 0$ and $(1 - \delta)^{1-\delta} = 1$ for $\delta = 1$ due to the fact that $\lim_{t\to0+} t^t = 1$. Then equation (1) is oscillatory.

Proof. Assume, for the sake of contradiction, that equation (1) has a nonoscillatory solution $x(t)$ on an interval $[t, \infty)$ for $t \geq T \geq t_0$, which is eventually positive or negative. Without loss of generality, we may assume that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\psi(t,s)) > 0$ for all $t \geq T \geq t_0$ for some $t_0 \geq 0$. In this case the interval of t selected for the following discussion is $[c_1, d_1]$. When $x(t)$ is eventually negative, then the proof follows the same argument using the interval $[c_2, d_2]$ instead of $[c_1, d_1]$. Define

$$u(t) = \frac{p(t)\Phi_\alpha(x'(t))}{\Phi_\alpha(x(t))},\ t \geq T. \tag{19}$$

It follows from (1) that for $t \geq T$, we have

$$u'(t) = \frac{e(t)}{x^\alpha(t)} - \left(q(t) \left[\frac{x(\tau(t))}{x(t)} \right]^\alpha + \int_0^h r(t,s) \left[\frac{x(\psi(t,s))}{x(t)} \right]^{\gamma(s)} [x(t)]^{\gamma(s)-\alpha} d\xi(s) \right) - \frac{\alpha}{p^{1/\alpha}(t)} [u(t)]^{\frac{\alpha+1}{\alpha}}. \tag{20}$$

From the assumption, there exists a nontrivial interval $[c_1, d_1] \subset [T, \infty)$ such that (16) hold with $i = 1$. Then by Lemmas 2.4 and 2.5, we have that for $t \in [c_1, d_1]$

$$u'(t) \leq -\left(q(t) \left[\frac{\mathcal{P}(\tau(t), \tau_*^1)}{\mathcal{P}(t, \tau_*^1)} \right]^\alpha + \int_0^h r(t,s) \left[\frac{\mathcal{P}(\psi(t,s), \psi_*^1)}{\mathcal{P}(t, \psi_*^1)} \right]^{\gamma(s)} [x(t)]^{\gamma(s)-\alpha} d\xi(s) \right) + \frac{e(t)}{x^\alpha(t)} - \frac{\alpha}{p^{1/\alpha}(t)} [u(t)]^{\frac{\alpha+1}{\alpha}},\ t \in [c_1, d_1]. \tag{21}$$

There are two cases with respect to δ as follows:

Case 1: $\delta = 1$.

We first consider the case where the supremum in (17) is assumed at $\delta = 1$. From (21), we have that for $t \in [c_1, d_1]$

$$u'(t) \leq -\left(q(t) \left[\frac{\mathcal{P}(\tau(t), \tau_*^1)}{\mathcal{P}(t, \tau_*^1)} \right]^\alpha + \int_0^h r(t,s) \left[\frac{\mathcal{P}(\psi(t,s), \psi_*^1)}{\mathcal{P}(t, \psi_*^1)} \right]^{\gamma(s)} [x(t)]^{\gamma(s)-\alpha} d\xi(s) \right) - \frac{\alpha}{p^{1/\alpha}(t)} [u(t)]^{\frac{\alpha+1}{\alpha}}. \tag{22}$$

Let $\eta \in L_\xi(0,h)$ be defined as in Lemma 2.2 with $\delta = 1$. Then η satisfies (7) and (8) with $\delta = 1$. It follows that,

$$\int_0^h \eta(s)[\gamma(s) - \alpha] d\xi(s) = 0. \tag{23}$$

Therefore, by (23) and Lemma 2.3, we get

$$\begin{aligned} \int_0^h r(t,s) \left[\frac{\mathcal{P}(\psi(t,s), \psi_*^1)}{\mathcal{P}(t, \psi_*^1)} \right]^{\gamma(s)} [x(t)]^{\gamma(s)-\alpha} d\xi(s) &\geq \exp\left(\int_0^h \eta(s) \ln\left(\frac{r(t,s)}{\eta(s)} \left[\frac{\mathcal{P}(\psi(t,s), \psi_*^1)}{\mathcal{P}(t, \psi_*^1)} \right]^{\gamma(s)} [x(t)]^{\gamma(s)-\alpha} \right) d\xi(s) \right) \\ &= \exp\left(\int_0^h \eta(s) \ln\left(\frac{r(t,s)}{\eta(s)} \left[\frac{\mathcal{P}(\psi(t,s), \psi_*^1)}{\mathcal{P}(t, \psi_*^1)} \right]^{\gamma(s)} \right) d\xi(s) \right). \end{aligned} \tag{24}$$

Now, substituting (24) into (22), we obtain

$$u'(t) \leq -Q_1(t) - \frac{\alpha}{p^{1/\alpha}(t)} [u(t)]^{\frac{\alpha+1}{\alpha}} \text{ for } t \in [c_1, d_1], \tag{25}$$

where $Q_1(t)$ is defined by (18) with $\delta = 1$.

Case 2: $\delta \neq 1$ and $\delta \in (m, 1)$.

Now we consider the case where the supremum in (17) is assumed at $\delta \in (m, 1)$. Let $\tilde{\eta}(s) = \delta^{-1}\eta(s)$. Then from (7) and (8), we have

$$\int_0^h \tilde{\eta}(s) d\xi(s) = 1 \text{ and } \int_0^h \tilde{\eta}(s)[\delta\gamma(s) - \alpha] d\xi(s) = 0. \tag{26}$$

Hence for $t \in [c_1, d_1]$,

$$\begin{aligned}\int_0^h r(t,s)\left[\frac{\mathcal{P}(\psi(t,s),\psi_*^1)}{\mathcal{P}(t,\psi_*^1)}\right]^{\gamma(s)} [x(t)]^{(\gamma(s)-\alpha)} d\xi(s) - e(t)x^{-\alpha}(t) \\ = \int_0^h \tilde{\eta}(s)\left(\delta\eta^{-1}(s)r(t,s)\left[\frac{\mathcal{P}(\psi(t,s),\psi_*^1)}{\mathcal{P}(t,\psi_*^1)}\right]^{\gamma(s)} [x(t)]^{(\gamma(s)-\alpha)} + |e(t)|\, x^{-\alpha}(t)\right) d\xi(s). \end{aligned} \tag{27}$$

If we let

$$a = \eta^{-1}(s)r(t,s)\left[\frac{\mathcal{P}(\psi(t,s),\psi_*^1)}{\mathcal{P}(t,\psi_*^1)}\right]^{\gamma(s)} [x(t)]^{\gamma(s)-\alpha}, \quad b = \frac{1}{1-\delta}\left(|e(t)|\, x^{-\alpha}(t)\right), \ j = \delta \text{ and } k = 1-\delta,$$

then from the Young inequality ($aj + bk \geq a^j b^k$, where $j + k = 1$, $j, k > 0$, $a, b \geq 0$), we get

$$\begin{aligned}\delta\eta^{-1}(s)r(t,s)\left[\frac{\mathcal{P}(\psi(t,s),\psi_*^1)}{\mathcal{P}(t,\psi_*^1)}\right]^{\gamma(s)} [x(t)]^{\gamma(s)-\alpha} + \frac{1}{1-\delta}\left(|e(t)|\, x^{-\alpha}(t)\right)(1-\delta) \\ \geq \left(\frac{r(t,s)}{\eta(s)}\right)^{\delta}\left[\frac{\mathcal{P}(\psi(t,s),\psi_*^1)}{\mathcal{P}(t,\psi_*^1)}\right]^{\delta\gamma(s)} [x(t)]^{\delta\gamma(s)-\alpha}\left[\frac{|e(t)|}{(1-\delta)}\right]^{1-\delta}.\end{aligned} \tag{28}$$

Substituting (28) into (27) and using Lemma 2.3, we have

$$\begin{aligned}&\int_0^h r(t,s)\left[\frac{\mathcal{P}(\psi(t,s),\psi_*^1)}{\mathcal{P}(t,\psi_*^1)}\right]^{\gamma(s)} [x(t)]^{(\gamma(s)-\alpha)} d\xi(s) + |e(t)|\, x^{-\alpha}(t) \\ &\geq \int_0^h \tilde{\eta}(s)\left[\left(\frac{r(t,s)}{\eta(s)}\right)^{\delta}\left[\frac{\mathcal{P}(\psi(t,s),\psi_*^1)}{\mathcal{P}(t,\psi_*^1)}\right]^{\delta\gamma(s)} [x(t)]^{\delta\gamma(s)-\alpha}\left[\frac{|e(t)|}{(1-\delta)}\right]^{1-\delta}\right] d\xi(s) \\ &\geq \exp\left(\int_0^h \tilde{\eta}(s)\ln\left[\left(\frac{r(t,s)}{\eta(s)}\right)^{\delta}\left[\frac{\mathcal{P}(\psi(t,s),\psi_*^1)}{\mathcal{P}(t,\psi_*^1)}\right]^{\delta\gamma(s)} [x(t)]^{\delta\gamma(s)-\alpha}\left[\frac{|e(t)|}{(1-\delta)}\right]^{1-\delta}\right] d\xi(s)\right) \\ &= \left[\frac{|e(t)|}{(1-\delta)}\right]^{1-\delta}\exp\left(\int_0^h \eta(s)\left(\ln\left[\left(\frac{r(t,s)}{\eta(s)}\right)\left[\frac{\mathcal{P}(\psi(t,s),\psi_*^1)}{\mathcal{P}(t,\psi_*^1)}\right]^{\gamma(s)}\right]\right) d\xi(s)\right).\end{aligned} \tag{29}$$

Using the above inequality in (21), we have

$$u'(t) \leq -Q_1(t) - \frac{\alpha}{p^{1/\alpha}(t)}[u(t)]^{\frac{\alpha+1}{\alpha}} \text{ for } t \in [c_1, d_1], \tag{30}$$

where $Q_1(t)$ is defined by (18) with $\delta \in (m, 1)$. Thus from the above two cases, we have

$$u'(t) \leq -Q_1(t) - \frac{\alpha}{p^{1/\alpha}(t)}[u(t)]^{\frac{\alpha+1}{\alpha}} \text{ for } t \in [c_1, d_1], \tag{31}$$

where $Q_1(t)$ is defined by (18) with $\delta \in (m, 1]$. Multiplying both sides of the above inequality by $|w_1(t)|^{\alpha+1}$, integrating every term from c_1 to d_1 and using integration by parts, we get

$$\int_{c_1}^{d_1} Q_1(t)|w_1(t)|^{\alpha+1}dt \leq \int_{c_1}^{d_1}\left[(\alpha+1)|u(t)||w_1^{\alpha}(t)w_1'(t)| - \frac{\alpha}{p^{1/\alpha}(t)}|u(t)|^{\frac{\alpha+1}{\alpha}}|w_1(t)|^{\alpha+1}\right] dt. \tag{32}$$

Letting

$$\lambda = 1 + \frac{1}{\alpha},\ X = \left[\frac{\alpha}{p^{1/\alpha}(t)}\right]^{\alpha/\alpha+1}|\omega_1^{\alpha}(t)|\,|u(t)| \text{ and } Y = [\alpha p(t)]^{\alpha/\alpha+1}\left|\omega_1'(t)\right|^{\alpha},$$

then by using Lemma 2.1, we get

$$\left[(\alpha+1)|u(t)||w_1^{\alpha}(t)w_1'(t)| - \frac{\alpha}{p^{1/\alpha}(t)}|u(t)|^{\frac{\alpha+1}{\alpha}}|w_1(t)|^{\alpha+1}\right] \le p(t)\left|w_1'(t)\right|^{\alpha+1}. \tag{33}$$

From (32) and (33), we get

$$\int_{c_1}^{d_1}[Q_1(t)|w_1(t)|^{\alpha+1} - p(t)\left|w_1'(t)\right|^{\alpha+1}]dt \le 0. \tag{34}$$

This contradicts (17) with $i = 1$. Thus the proof is complete. □

Next, following the ideas of Kong [9] and Philos [10], we establish a Kong's type interval oscillation criterion for equation (1). First, we introduce the class of functions $\mathcal{H}$ which will be used in the sequel. Denote $D = \{(t,s) : t_0 \le s \le t\}$, we say that a function $H \in C(D,\mathbb{R})$ belong to the function class $\mathcal{H}$, denoted by $H \in \mathcal{H}$, if it satisfies the following conditions

(H_1) $H(t,t) = 0$ for $t \ge t_0$ and $H(t,s) > 0$ for $t > s \ge t_0$,

(H_2) H has a continuous partial derivatives $\partial H/\partial t$ and $\partial H/\partial s$ on D such that

$$\frac{\partial H(t,s)}{\partial t} = h_1(t,s)H^{\alpha/\alpha+1}(t,s) \text{ and } \frac{\partial H(t,s)}{\partial s} = -h_2(t,s)H^{\alpha/\alpha+1}(t,s),$$

where $h_1, h_2 \in L_{\text{loc}}(D,\mathbb{R})$.

Theorem 2.7. *Assume that $\tau(t), \psi(t,s) \le t$ for $t \in [t_0,\infty)$ and $s \in [0,h]$. Suppose also that for any $T \ge t_0$, there exists subintervals $[c_i,d_i]$ of $[T,\infty)$ such that (16) holds for $i = 1,2$. For each $\delta \in (m,1]$, let $\eta \in L_\xi(0,h)$ be defined as in Lemma 2.2. Further assume that for $i = 1,2$, there exist a constant $\rho_i \in (c_i,d_i)$ and a function $H \in \mathcal{H}$ such that*

$$\begin{aligned}\sup_{\delta\in(m,1]}\Bigg\{&\frac{1}{H(\rho_i,c_i)}\int_{c_i}^{\rho_i}\left[Q_i(t)H(t,c_i) - \frac{1}{(\alpha+1)^{\alpha+1}}p(t)|h_1(t,c_i)|^{\alpha+1}\right]dt \\ &+ \frac{1}{H(d_i,\rho_i)}\int_{\rho_i}^{d_i}\left[Q_i(t)H(d_i,t) - \frac{1}{(\alpha+1)^{\alpha+1}}p(t)|h_2(d_i,t)|^{\alpha+1}\right]dt\Bigg\} > 0,\end{aligned} \tag{35}$$

where $Q_i(t)$ is defined by (18). Then equation (1) is oscillatory.

Proof. To arrive at a contradiction, let us suppose that $x(t)$ is a non-oscillatory solution of equation (1). Without loss of generality, we assume that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\psi(t,s)) > 0$ for $t \ge T \ge t_0$. In this case the interval of t selected for the following discussion is $[c_1,d_1]$. Continuing as in Theorem 2.6, we can get (31). Multiplying both sides of (31) by $H(t,c_1)$, integrating it from c_1 to ρ_1 and using integration by parts we have

$$\int_{c_1}^{\rho_1}Q_1(t)H(t,c_1)dt \le -H(\rho_1,c_1)u(\rho_1) + \int_{c_1}^{\rho_1}\left[|u(t)||h_1(t,c_1)|H^{\frac{\alpha}{\alpha+1}}(t,c_1) - \frac{\alpha}{p^{1/\alpha}(t)}|u(t)|^{\frac{\alpha+1}{\alpha}}H(t,c_1)\right]dt. \tag{36}$$

Letting

$$\lambda = 1+\frac{1}{\alpha},\ X = \left[\frac{\alpha}{p^{1/\alpha}(t)}\right]^{\alpha/\alpha+1}H^{\alpha/\alpha+1}(t,c_1)\,|u(t)| \text{ and } Y = \left[\frac{\alpha}{(\alpha+1)^{\alpha+1}}p(t)\right]^{\alpha/\alpha+1}|h_1(t,c_1)|^{\alpha},$$

then by Lemma 2.1, we get

$$\left[|u(t)||h_1(t,c_1)|H^{\frac{\alpha}{\alpha+1}}(t,c_1) - \frac{\alpha}{p^{1/\alpha}(t)}|u(t)|^{\frac{\alpha+1}{\alpha}}H(t,c_1)\right] \le \frac{1}{(\alpha+1)^{\alpha+1}}p(t)|h_1(t,c_1)|^{\alpha+1}. \tag{37}$$

It follows from (36) and (37), that

$$\int_{c_1}^{\rho_1}\left[Q_1(t)H(t,c_1)-\frac{1}{(\alpha+1)^{\alpha+1}}p(t)|h_1(t,c_1)|^{\alpha+1}\right]dt\leq -H(\rho_1,c_1)u(\rho_1). \tag{38}$$

Similarly, multiplying both sides of (31) by $H(d_1,t)$ and using similar analysis as above, we can obtain

$$\int_{\rho_1}^{d_1}\left[Q_1(t)H(d_1,t)-\frac{1}{(\alpha+1)^{\alpha+1}}p(t)|h_2(d_1,t)|^{\alpha+1}\right]\leq H(d_1,\rho_1)u(\rho_1). \tag{39}$$

By dividing (38) and (39) by $H(\rho_1,c_1)$ and $H(d_1,\rho_1)$, respectively and then adding them together, we have

$$\begin{aligned}&\frac{1}{H(\rho_1,c_1)}\int_{c_1}^{\rho_1}\left[Q_1(t)H(t,c_1)-\frac{1}{(\alpha+1)^{\alpha+1}}p(t)|h_1(t,c_1)|^{\alpha+1}\right]dt\\&+\frac{1}{H(d_1,\rho_1)}\int_{\rho_1}^{d_1}\left[Q_1(t)H(d_1,t)-\frac{1}{(\alpha+1)^{\alpha+1}}p(t)|h_2(d_1,t)|^{\alpha+1}\right]dt\leq 0.\end{aligned} \tag{40}$$

This leads to contradiction to (35) with $i=1$. When $x(t)$ is eventually negative, we can consider $[c_2,d_2]$ and reach a similar contradiction. Hence the proof is complete. □

Now we interpret the results for equation (1) for the special case as in [11]. That is, for $N\in\mathbb{N}$ and $s\in[0,N+1)$, we let

$$\xi(s)=\sum_{j=1}^{N}\chi(s-j)\text{ with }\chi(s)=\begin{cases}1, & s\geq 0\\ 0, & s<0;\end{cases}$$

$\gamma\in C[0,N+1)$ such that $\gamma(j)=\beta_j$ for $j=1,2,3,...,N$, satisfying $\beta_1>\cdots>\beta_l>\alpha>\beta_{l+1}>\cdots>\beta_N>0$, $q(t)=q_0(t), r(t,j)=q_j(t)\in C[t_0,\infty), \tau(t)=\tau_0(t)$ and $\psi(t,j)=\tau_j(t)$ for $j=1,2,3,...,N$. Then equation (1) reduces to

$$(p(t)\Phi_\alpha(x'(t)))'+q_0(t)\Phi_\alpha(x(\tau_0(t)))+\sum_{j=1}^{N}q_j(t)\Phi_{\beta_j}(x(\tau_j(t)))=e(t), \tag{41}$$

where $p\in C^1[t_0,\infty)$ with $p(t)>0, q_0,q_j,e\in C[t_0,\infty)$; $\beta_1>\cdots>\beta_l>\alpha>\beta_{l+1}>\cdots>\beta_N>0$; $\tau_0(t)$ and $\tau_j(t)$ are continuous functions satisfying $\lim_{t\to\infty}\tau_0(t)=\lim_{t\to\infty}\tau_j(t)=\infty$ and we obtain the following results for equation (41).

Lemma 2.8. *Let*

$$m=\frac{\alpha}{l}\left(\sum_{j=1}^{l}\beta_j^{-1}\right)\ \textit{and}\ \ n=\frac{\alpha}{N-l}\left(\sum_{j=l+1}^{N}\beta_j^{-1}\right).$$

Then for any $\delta\in(m,n)$, there exists an N- tuple $(\eta_1,\eta_2,...,\eta_N)$ with $\eta_j>0$ satisfying

$$\sum_{j=1}^{N}\beta_j\eta_j=\alpha\ \textit{and}\ \sum_{j=1}^{N}\eta_j=\delta. \tag{42}$$

Proof. Define

$$\eta_j^1=\begin{cases}\alpha\beta_j^{-1}/l, & j=1,2,...,l\\ 0, & j=l+1,...,N\end{cases}$$

and

$$\eta_j^2=\begin{cases}0, & j=1,2,...,l\\ \alpha\beta_j^{-1}/N-l, & j=l+1,...,N.\end{cases}$$

Clearly, for $i = 1, 2$, we get

$$\sum_{j=1}^{N} \beta_j \eta_j^i = \alpha.$$

Moreover,

$$\sum_{j=1}^{N} \eta_j^1 = m \text{ and } \sum_{j=1}^{N} \eta_j^2 = n.$$

For $p^* \in [0, 1]$, let

$$\eta_j(p^*) = (1 - p^*)\eta_j^1 + p^*\eta_j^2,\ j = 1, 2, ..., N.$$

Then it is easy to see that

$$\sum_{j=1}^{N} \beta_j \eta_j(p^*) = \alpha.$$

Moreover, since $\eta_j(0) = \eta_j^1$ and $\eta_j(1) = \eta_j^2$, we have

$$\sum_{j=1}^{N} \eta_j(0) = m < 1 \text{ and } \sum_{j=1}^{N} \eta_j(1) = n > 1.$$

By the continuous dependence of $\eta_j(p^*)$ on p^*, there exists $p \in (0, 1)$ such that $\eta_j := \eta_j(p)$ satisfies that

$$\sum_{j=1}^{N} \eta_j = \delta.$$

Note that $\eta_j > 0$ for $j = 1, 2, ..., N$ and $\sum_{j=1}^{N} \beta_j \eta_j = \alpha$. This completes the proof of Lemma 2.8. □

Theorem 2.9. *Assume that $\tau_0(t), \tau_j(t) \leq t$ for $t \in [t_0, \infty)$ and $j = 1, 2, ..., N$. Suppose also that for any $T \geq t_0$, there exists subintervals $[c_i, d_i]$ of $[T, \infty), i = 1, 2$ such that $T < c_1 - \Theta_1 < c_1 < d_1 \leq c_2 < d_2$ and*

$$q_j(t) \geq 0,\ (-1)^i e(t) \geq 0,\ t \in [\Theta_i, d_i], \tag{43}$$

where $\Theta_i = min\{\pi_*^i, {\tau_j}_*^i : j = 1, 2, ..., N\}, \pi_*^i = min\{\tau_0(t) : t \in [c_i, d_i]\}$ and ${\tau_j}_*^i = min\{\tau_j(t) : t \in [c_i, d_i]\}$. For each $\delta \in (m, 1]$, let $(\eta_1, \eta_2, ..., \eta_N)$ be defined as in Lemma 2.8. We further assume that, there exists a function $w_i \in \mathcal{W}(c_i, d_i)$ for $i = 1, 2$ satisfying (17) with

$$Q_i(t) = q_0(t) \left[\frac{P(\tau_0(t), \pi_*^i)}{P(t, \pi_*^i)}\right]^{\alpha} + \left[\frac{|e(t)|}{1 - \delta}\right]^{1-\delta} \prod_{j=1}^{N} \left[\frac{q_j(t)}{\eta_j}\right]^{\eta_j} \left[\frac{P(\tau_j(t), {\tau_j}_*^i)}{P(t, {\tau_j}_*^i)}\right]^{\beta_j \eta_j}. \tag{44}$$

Here we use the convention that $0^{1-\delta} = 0$ and $(1 - \delta)^{1-\delta} = 1$ for $\delta = 1$. Then equation (41) is oscillatory.

Proof. To arrive at a contradiction, let us suppose that $x(t)$ is a non-oscillatory solution of equation (41). Without loss of generality, we assume that $x(t) > 0$, $x(\tau_0(t)) > 0$ and $x(\tau_j(t)) > 0$ for $t \geq T \geq t_0$. In this case the interval of t selected for the following discussion is $[c_1, d_1]$. When $x(t)$ is eventually negative, the proof follows the same way using the interval $[c_2, d_2]$, instead of $[c_1, d_1]$. Define $u(t)$ by (19). It follows from (41) that for $t \geq T$, we have

$$u'(t) = -\left(\frac{q_0(t)(x(\tau_0(t)))^{\alpha}}{x^{\alpha}(t)} + \frac{\sum\limits_{j=1}^{N} q_j(t)(x(\tau_j(t)))^{\beta_j}}{x^{\alpha}(t)} + \frac{|e(t)|}{x^{\alpha}(t)}\right) - \frac{\alpha}{p^{1/\alpha}(t)}[u(t)]^{\frac{\alpha+1}{\alpha}}. \tag{45}$$

From the assumption, there exists a nontrivial interval $[c_1, d_1] \subset [T, \infty)$ such that (43) hold with $i = 1$. There are two cases with respect to δ as follows:

Case 1: $\delta \neq 1$, ie, $\delta \in (m, 1)$

We first consider the case where the supremum in (17) is assumed at $\delta \neq 1$. From (45), we have that for $t \in [c_1, d_1]$

$$u'(t) = -\left(\frac{q_0(t)(x(\tau_0(t))^{\alpha}}{x^{\alpha}(t)} + \frac{\sum_{j=1}^{N} q_j(t)(x(\tau_j(t)))^{\beta_j}}{x^{\alpha}(t)} + \frac{|e(t)|}{x^{\alpha}(t)} \right) - \frac{\alpha}{p^{1/\alpha}(t)}[u(t)]^{\frac{\alpha+1}{\alpha}},\ t \in [c_1, d_1]. \tag{46}$$

By Lemma 2.8, there exist $\eta_j > 0$, $j = 1, \ldots, N$, such that $\sum_{j=1}^{N} \beta_j \eta_j = \alpha$ and $\sum_{j=1}^{N} \eta_j = \delta$.

Define $\eta_0 := 1 - \sum_{j=1}^{N} \eta_j$ and let

$$\begin{aligned} u_0 &:= \eta_0^{-1} \left| \frac{e(t) x(\tau_0(t))}{x^{\alpha}(t)} \right| x^{-1}(\tau_0(t)), \\ u_j &:= \eta_j^{-1} q_j(t) \frac{x(\tau_j(t))}{x^{\alpha}(t)}\, x^{\beta_j - 1}(\tau_j(t)), \quad j = 1, 2, \ldots, N\,. \end{aligned}$$

Then by the arithmetic-geometric mean inequality (see Beckenbach and Bellman [3]),

$$\sum_{j=0}^{N} \eta_j u_j \geq \prod_{j=0}^{N} u_j^{\eta_j},\ u_j \geq 0 \text{ and } \eta_j > 0, \tag{47}$$

we have

$$u'(t) \leq -q_0(t) \left(\frac{x(\tau_0(t))}{x(t)} \right)^{\alpha} - \eta_0^{-\eta_0} \left[\frac{|e(t)|^{\eta_0}}{x^{\alpha\eta_0}(t)} \right] \prod_{j=1}^{N} \eta_j^{-\eta_j} q_j^{\eta_j}(t) \left(\frac{x^{\beta_j}(\tau_j(t))}{x^{\alpha}(t)} \right)^{\eta_j(t)} - \frac{\alpha}{p^{1/\alpha}(t)}[u(t)]^{\frac{\alpha+1}{\alpha}}. \tag{48}$$

By using Lemma 2.8, we have

$$\frac{1}{x^{\alpha\eta_0}(t)} \prod_{j=1}^{N} \frac{x^{\beta_j \eta_j}(\tau_j(t))}{(x^{\eta_j}(t))^{\alpha}} = \frac{x^{\beta_1\eta_1 + \beta_2\eta_2 + \cdots + \beta_N\eta_N}(\tau_j(t))}{x^{\alpha\eta_0}(x^{\eta_1+\eta_2+\cdots+\eta_n}(t))^{\alpha}} = \frac{(x(\tau_j(t)))^{\sum_{j=1}^{N}\beta_j\eta_j}}{(x(t))^{\alpha\eta_o + \alpha\sum_{j=1}^{N}\eta_j}} = \left(\frac{x(\tau_j(t))}{x(t)} \right)^{\alpha}.$$

Therefore, (48) becomes

$$\begin{aligned} u'(t) &\leq -q_0(t) \left(\frac{x(\tau_0(t))}{x(t)} \right)^{\alpha} - \left[\frac{|e(t)|}{\eta_0} \right]^{\eta_0} \left(\prod_{j=1}^{N} \left[\frac{q_j(t)}{\eta_j} \right]^{\eta_j} \right) \left(\frac{x(\tau_j(t))}{x(t)} \right)^{\alpha} - \frac{\alpha}{p^{1/\alpha}(t)}[u(t)]^{\frac{\alpha+1}{\alpha}} \\ &= -\left[q_0(t) \left(\frac{x(\tau_0(t))}{x(t)} \right)^{\alpha} + \left[\frac{|e(t)|}{1-\delta} \right]^{1-\delta} \left(\prod_{j=1}^{N} \left[\frac{q_j(t)}{\eta_j} \right]^{\eta_j} \left[\frac{x(\tau_j(t))}{x(t)} \right]^{\beta_j\eta_j} \right) \right] - \frac{\alpha}{p^{1/\alpha}(t)}[u(t)]^{\frac{\alpha+1}{\alpha}}. \end{aligned}$$

Then by Lemmas 2.4 and 2.5, we have that for $t \in [c_1, d_1]$

$$\begin{aligned} u'(t) \ &\leq\ -\left(q_0(t) \left[\frac{\mathcal{P}(\tau_0(t), \pi_*^1)}{\mathcal{P}(t, \pi_*^1)} \right]^{\alpha} + \left[\frac{|e(t)|}{1-\delta} \right]^{1-\delta} \prod_{j=1}^{N} \left[\frac{q_j(t)}{\eta_j} \right]^{\eta_j} \left[\frac{\mathcal{P}(\tau_j(t), \tau_{j*}^{1})}{\mathcal{P}(t, \tau_{j*}^{1})} \right]^{\beta_j\eta_j} \right) - \frac{\alpha}{p^{1/\alpha}(t)}[u(t)]^{\frac{\alpha+1}{\alpha}} \\ &=\ -Q_1(t) - \frac{\alpha}{p^{1/\alpha}(t)}[u(t)]^{\frac{\alpha+1}{\alpha}} \text{ for } t \in [c_1, d_1], \end{aligned} \tag{49}$$

where $Q_1(t)$ is defined by (44) with $\delta \in (m, 1)$.

Case 2: $\delta = 1$

We consider the case where the supremum in (17) is assumed at $\delta = 1$. From (45), we have that for $t \in [c_1, d_1]$

$$u'(t) \leq -\left(\frac{q_0(t)(x(\tau_0(t)))^\alpha}{x^\alpha(t)} + \frac{\sum_{j=1}^{N} q_j(t)(x(\tau_j(t)))^{\beta_j}}{x^\alpha(t)} \right) - \frac{\alpha}{p^{1/\alpha}(t)}[u(t)]^{\frac{\alpha+1}{\alpha}}, \ t \in [c_1, d_1]. \tag{50}$$

Let η_j, $j = 1, 2, \ldots, N$, be defined as in Lemma 2.8 with $\delta = 1$. From (42) we have

$$\sum_{j=1}^{N} \beta_j \eta_j = \alpha \text{ and } \sum_{j=1}^{N} \eta_j = 1.$$

Then by using the arithmetic-geometric mean inequality (47), we have

$$\sum_{j=1}^{N} \eta_j \frac{q_j(t)}{\eta_j} \frac{(x(\tau_j(t)))^{\beta_j}}{x^\alpha(t)} \geq \prod_{j=1}^{N} \left[\frac{q_j(t)}{\eta_j}\right]^{\eta_j} \left[\frac{(x(\tau_j(t)))^{\beta_j}}{x^\alpha(t)}\right]^{\eta_j} = \prod_{j=1}^{N} \left[\frac{q_j(t)}{\eta_j}\right]^{\eta_j} \left[\frac{x(\tau_j(t))}{x(t)}\right]^{\beta_j \eta_j}. \tag{51}$$

Substituting the above inequality in (50), we get

$$u'(t) \leq -\left(q_0(t) \left[\frac{x(\tau_0(t))}{x(t)}\right]^\alpha + \prod_{j=1}^{N} \left[\frac{q_j(t)}{\eta_j}\right]^{\eta_j} \left[\frac{x(\tau_j(t))}{x(t)}\right]^{\beta_j \eta_j} \right) - \frac{\alpha}{p^{1/\alpha}(t)}[u(t)]^{\frac{\alpha+1}{\alpha}}, \ t \in [c_1, d_1]. \tag{52}$$

Then by Lemmas 2.4 and 2.5, we have that for $t \in [c_1, d_1]$

$$u'(t) \leq -Q_1(t) - \frac{\alpha}{p^{1/\alpha}(t)}[u(t)]^{\frac{\alpha+1}{\alpha}}, \ t \in [c_1, d_1], \tag{53}$$

where $Q_1(t)$ is defined by (44) with $\delta = 1$. The rest of the proof is similar to that of Theorem 2.6 and hence is omitted. This completes the proof of Theorem 2.9. □

Theorem 2.10. *Assume that $\tau_0(t), \tau_j(t) \leq t$ for $t \in [t_0, \infty)$ and $j = 1, 2, ..., N$. Suppose also that for any $T \geq t_0$, there exists subintervals $[c_i, d_i]$ of $[T, \infty)$ such that (43) holds for $i = 1, 2$. For each $\delta \in (m, 1]$, let $(\eta_1, \eta_2, ..., \eta_N)$ be defined as in Lemma 2.8. We further assume that for $i = 1, 2$, there exist constants $\rho_i \in (c_i, d_i)$ and a function $H \in \mathcal{H}$ such that (35) holds, where $Q_i(t)$ is defined by (44). Then equation (41) is oscillatory.*

Proof. Proceeding as in the proof of Theorem 2.9, we get inequalities (49) and (53) for the cases when $\delta \in (m, 1)$ and $\delta = 1$, respectively. The rest of the proof is similar to that of Theorem 2.7 and hence is omitted. This completes the proof of Theorem 2.10. □

Remark 2.11. *We observe that in Theorem 2.6 and Theorem 2.9 , if the supremum in (17) is assumed at $\delta = 1$, the effect of $e(t)$ is neglected in some extent. This implies that the magnitude of $e(t) \in [c_i, d_i]$ cannot be large. For otherwise, the supremum would have been taken at some $\delta \in (m, 1)$. Similar remark holds for Theorem 2.7 and Theorem 2.10.*

Remark 2.12. *When $\alpha = 1, \gamma(s) = \alpha(s)$ and $\tau_0(t) = \tau_j(t) = 1$, Theorem 2.6 and Theorem 2.7 reduces to Theorem* 2.1 *and Theorem* 2.2 *of [11].*

Remark 2.13. *When $\alpha = 1, \gamma(s) = \alpha(s)$ and $\tau_0(t) = \tau_j(t) = 1$, Theorem 2.9 and Theorem 2.10 reduces to Theorem* 2.3 *and Theorem* 2.4 *of [11].*

3. Examples

In this section we give two examples to illustrate our main results.

Example 3.1. Consider the second order forced delay differential equation of the form

$$(e^{2t}\Phi_\alpha(x'(t)))' + m_1 \sin t\Phi_\alpha\left(x\left(t-\frac{\pi}{4}\right)\right) + \int_0^1 m_2 \sin t\Phi_{(3s)}\left(x\left(t-\frac{\pi}{8}\right)\right)ds = e(t),\ t \geq 0. \tag{54}$$

Here,

$$p(t) = e^{2t},\ q(t) = m_1 \sin t,\ h = 1,\ r(t,s) = m_2 \sin t,\ \gamma(s) = 3s,\ \xi(s) = s,\ \tau(t) = t - \frac{\pi}{4},\ \psi(t,s) = t - \frac{\pi}{8},$$

where m_1 and m_2 are positive constants.

For any $T \geq 0$, we can choose k large enough such that $T < c_1 = 2k\pi + \frac{\pi}{4}$, $d_1 = c_2 = 2k\pi + \frac{\pi}{2}$ and $d_2 = 2k\pi + \pi$, then it is easy to see that

$$\tau_*^1 = 2k\pi,\ \tau_*^2 = 2k\pi + \pi/4,\ \psi_*^1 = 2k\pi + \pi/8 \text{ and } \psi_*^2 = 2k\pi + 3\pi/8.$$

Therefore, $\Psi_1 = 2k\pi$ and $\Psi_2 = 2k\pi + \pi/4$. Let,

$$e(t) = \begin{cases} -e^{2t}\sin 2t, & t \in [2k\pi, 2k\pi + \pi/2], \\ e^t \cos^2 t, & t \in [2k\pi + \pi/4, \pi]. \end{cases}$$

Taking $\alpha = 1$ and $a = 1/3$, we have

$$m = \left(\int_{1/3}^1 \frac{1}{3s}ds\right)\left(\int_{1/3}^1 ds\right)^{-1} = \ln(\sqrt{3}).$$

For any $\delta \in (\ln(\sqrt{3}), 1]$, set

$$\eta(s) = \frac{\delta}{3\delta - 1}s^{\frac{2-3\delta}{3\delta-1}}.$$

It is easy to verify that (7) and (8) valid. If we take $\delta = 2/3$ then $\eta(s) = 2/3$. Therefore for $t \in [c_1, d_1]$, we have

$$\begin{aligned} Q_1(t) &= m_1 \sin t\left[\frac{1-e^{-2(t-\pi/4)}}{1-e^{-2t}}\right] + |3e^{-2t}\sin 2t|^{(1/3)} \exp\left\{\int_0^1 (2/3)\ln\left(\frac{m_2 \sin t}{(2/3)}\left[\frac{e^{-\pi/4} - e^{-2(t-\pi/8)}}{e^{-\pi/4} - e^{-2t}}\right]^{(3s)}\right)ds\right\} \\ &= m_1 \sin t\left[\frac{1-e^{-2(t-\pi/4)}}{1-e^{-2t}}\right] + |3e^{-2t}\sin 2t|^{(1/3)}\left[\frac{m_2 \sin t}{(2/3)}\right]^{(2/3)}\left[\frac{e^{-\pi/4} - e^{-2(t-\pi/8)}}{e^{-\pi/4} - e^{-2t}}\right]. \end{aligned}$$

Taking $w_1(t) = \sin 4t$, we have

$$\begin{aligned} \int_{c_1}^{d_1} Q_1(t)|w_1(t)|^{\alpha+1}dt &= \int_{\pi/4}^{\pi/2} m_1 \sin t\left[\frac{1-e^{-2(t-\pi/4)}}{1-e^{-2t}}\right]|\sin 4t|^2 dt \\ &\quad + \int_{\pi/4}^{\pi/2} |3e^{-2t}\sin 2t|^{(1/3)}\left[\frac{m_2 \sin t}{(2/3)}\right]^{(2/3)}\left[\frac{e^{-\pi/4} - e^{-2(t-\pi/8)}}{e^{-\pi/4} - e^{-2t}}\right]|\sin 4t|^2 dt \\ &= m_1(0.2108) + m_2(0.8798) \end{aligned} \tag{55}$$

and

$$\int_{c_1}^{d_1} p(t)|w_1'(t)|^{\alpha+1}dt = \int_{\pi/4}^{\pi/2} e^{2t}|4\cos 4t|^2 dt = 77.6339. \tag{56}$$

Therefore, by (55) and (56), if we choose m_1 and m_2 large enough so that

$$m_1(0.2108) + m_2(0.8798) - 77.6339 > 0, \tag{57}$$

then (17) holds for $t \in [c_1, d_1]$.

Similarly for $t \in [c_2, d_2]$, if we choose $w_2(t) = \sin 2t$ then we can get the following condition

$$m_1(0.4054) + m_2(3.8937) - 38.4263 > 0. \tag{58}$$

Hence, by Theorem 2.6, equation (54) is oscillatory if (57) and (58) hold.

Example 3.2. Consider the second order forced delay differential equation of the form

$$\left(\left(\frac{1}{1+t^2}\right)\Phi_\alpha(x'(t))\right)' + m_1(1+\sin t)\Phi_\alpha\left(x\left(t-\frac{\pi}{12}\right)\right) + \int_0^1 (m_2 e^{3t})\Phi_{\left(\frac{3}{2}\sqrt{s}\right)}\left(x\left(t-\frac{\pi}{8}\right)\right) ds = e(t),\ t \geq 0, \tag{59}$$

where m_1 and m_2 are positive constants.

Here we have, $p(t) = \frac{1}{1+t^2}$, $q(t) = m_1(1+\sin t)$, $h = 1$, $r(t,s) = m_2 e^{3t}$, $\gamma(s) = \frac{3}{2}\sqrt{s}$, $\xi(s) = s$, $\tau(t) = t - \frac{\pi}{12}$, $\psi(t,s) = t - \frac{\pi}{8}$. For any $T \geq 0$, we can choose k large enough such that $T < c_1 = 2k\pi + \frac{\pi}{6} < \rho_1 = 2k\pi + \frac{\pi}{4} < d_1 = 2k\pi + \frac{\pi}{3} < c_2 = 2k\pi + \frac{\pi}{2} < \rho_2 = 2k\pi + \frac{2\pi}{3}$ and $d_2 = 2k\pi + \pi$, then it is easy to see that

$$\tau_*^1 = 2k\pi + \pi/12,\ \tau_*^2 = 2k\pi + 5\pi/12,\ \psi_*^1 = 2k\pi + \pi/24 \text{ and } \psi_*^2 = 2k\pi + 3\pi/8.$$

Therefore, $\Psi_1 = 2k\pi + \pi/24$ and $\Psi_2 = 2k\pi + 3\pi/8$. Take $\alpha = 1$ and $H(t,s) = (t-s)^2$. Then $h_1(t,s) = -h_2(t,s) = 2$. Let $\eta(s) = 1$. It is easy to verify that (7) and (8) are valid for $\delta = 1$. Assume that $e(t) \in C[0,\infty)$ is any function satisfying $(-1)^i e(t) \geq 0$ on $[\Psi_i, d_i]$ for $i = 1, 2$. Then for $t \in [c_1, d_1]$, we have

$$\begin{aligned}
Q_1(t) &= m_1(1+\sin t)\left[\frac{[(t-\pi/12) + (t-\pi/12)^3/3] - [(\pi/12) + (\pi/12)^3/3]}{(t+t^3/3) - [\pi/12 + (\pi/12)^3/3]}\right] \\
&\quad + \exp\left\{\int_0^1 \ln\left(m_2 e^{3t}\left[\frac{[(t-\pi/8) + (t-\pi/8)^3/3] - [(\pi/24) + (\pi/24)^3/3]}{(t+t^3/3) - [\pi/24 + (\pi/24)^3/3]}\right]^{\frac{3}{2}\sqrt{s}}\right) ds\right\} \\
&= m_1(1+\sin t)\left[\frac{[(t-\pi/12) + (t-\pi/12)^3/3] - [(\pi/12) + (\pi/12)^3/3]}{(t+t^3/3) - [\pi/12 + (\pi/12)^3/3]}\right] \\
&\quad + \left(m_2 e^{3t}\right)\left[\frac{[(t-\pi/8) + (t-\pi/8)^3/3] - [(\pi/24) + (\pi/24)^3/3]}{(t+t^3/3) - [\pi/24 + (\pi/24)^3/3]}\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{c_1}^{\rho_1} Q_1(t)H(t,c_1)dt &= \int_{\pi/6}^{\pi/4} m_1(1+\sin t)\left[\frac{[(t-\pi/12) + (t-\pi/12)^3/3] - [(\pi/12) + (\pi/12)^3/3]}{(t+t^3/3) - [\pi/12 + (\pi/12)^3/3]}\right](t-(\pi/6))^2 dt \\
&\quad + \int_{\pi/6}^{\pi/4} m_2(e^{3t})\left[\frac{[(t-\pi/8) + (t-\pi/8)^3/3] - [(\pi/24) + (\pi/24)^3/3]}{(t+t^3/3) - [\pi/24 + (\pi/24)^3/3]}\right](t-(\pi/6))^2 dt \\
&= m_1(0.0037673) + m_2(0.0153683)
\end{aligned} \tag{60}$$

and

$$\int_{c_1}^{\rho_1} \frac{1}{(\alpha+1)^{\alpha+1}} p(t)|h_1(t,c_1)|^{\alpha+1} dt = \int_{\pi/6}^{\pi/4}\left(\frac{1}{1+t^2}\right) dt = 0.183426. \tag{61}$$

From (60) and (61), we have

$$\begin{aligned}\frac{1}{H(\rho_1,c_1)}\int_{c_1}^{\rho_1}\left[Q_1(t)H(t,c_1)-\frac{1}{(\alpha+1)^{\alpha+1}}p(t)|h_1(t,c_1)|^{\alpha+1}\right]dt\\=\left(\frac{1}{(\pi/12)^2}\right)[m_1(0.0037673)+m_2(0.0153683)-0.183426]\,.\end{aligned}\tag{62}$$

Also,

$$\int_{\rho_1}^{d_1}Q_1(t)H(d_1,t)dt=m_1(0.00514878)+m_2(0.030316)\tag{63}$$

and

$$\int_{\rho_1}^{d_1}\frac{1}{(\alpha+1)^{\alpha+1}}p(t)|h_2(d_1,t)|^{\alpha+1}dt=0.142675.\tag{64}$$

From (63) and (64), we have

$$\begin{aligned}\frac{1}{H(d_1,\rho_1)}\int_{\rho_1}^{d_1}\left[Q_1(t)H(d_1,t)-\frac{1}{(\alpha+1)^{\alpha+1}}p(t)|h_2(d_1,t)|^{\alpha+1}\right]dt\\=\left(\frac{1}{(\pi/12)^2}\right)[m_1(0.00514878)+m_2(0.030316)-0.142675]\,.\end{aligned}\tag{65}$$

Therefore, (62) and (65) for $t\in[c_1,d_1]$, if we choose m_1 and m_2 large enough so that

$$\left(\frac{1}{(\pi/12)^2}\right)[(m_1(0.0089)+m_2(0.1839))-1.3261]>0\tag{66}$$

then (35) will be satisfied.

Similarly (35) is satisfied for $t\in[c_2,d_2]$, if we choose m_1 and m_2 large enough so that

$$\begin{aligned}\left(\frac{1}{(\pi/6)^2}\right)[m_1(0.0476)+m_2(7.5933)-0.121454]\\+\left(\frac{1}{(\pi/3)^2}\right)[m_1(0.4241)+m_2(323.098)-0.1373]>0.\end{aligned}\tag{67}$$

Hence, by Theorem 2.7, equation (59) is oscillatory if (66) and (67) hold.

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