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Results on Intuitionistic Fuzzy k-ideals of Semiring

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Abstract: In this paper, the product of intuitionistic fuzzy k-ideals is introduced. Also some basic properties are derived. The relationship between intuitionistic fuzzy k-ideal A, B and $A \times B$ are proposed. Some theorems related to the above concepts are stated and proved.

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1. Introduction

The theory of fuzzy sets was introduced by L.A. Zadeh [10] in 1965. Rosenfeld [8] used the idea of fuzzy set to introduce the notions of fuzzy subgroups. Nobuaki Kuroki [5–7] is the pioneer of fuzzy ideal theory of semigroups. The idea of fuzzy subsemigroup was also introduced by Kuroki [5, 7]. In [6], Kuroki characterized several classes of semigroups in terms of fuzzy left, fuzzy right and fuzzy bi-ideals. Henriksen introduced the concept of k-ideals with the property that if the semiring S is a ring then a complex in S is a k-ideal iff it is a ring ideal. In 1986 Atanassov [1] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. In this paper, we prove that the Cartesian product of two intuitionistic fuzzy k-ideals of semi-ring S is also an intuitionistic fuzzy k-ideal. Conversely, we show that if $A \times B$ is an intuitionistic fuzzy left k-ideal of $S \times S$, then either A or B is an intuitionistic fuzzy left k-ideal of S.

2. Preliminaries

Definition 2.1. *A mathematical system $(S, *)$ is said to be a semi-group if $\forall a, b, c \in S, (a * b) * c = a * (b * c)$.*

Definition 2.2. *A semi-group $(S, *)$ is said to be commutative if for all $a, b \in S$, $a * b = b * a$:*

Definition 2.3. *A semi-ring S is a structure consisting of a non-empty set S together with two binary operations on S called addition and multiplication(denoted in the usual manner) such that*

- *together with addition is a semigroup,*
- *S together with multiplication is a semigroup, and*

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- $a(b+c) = ab + ac$ *and* $(a+b)c = ac + bc$ *for all* $a, b, c \in S$.

Definition 2.4. *A nonempty subset* I *of a semiring* S *is called an ideal if*

1. $a, b \in I$ *implies* $a + b \in I$,
2. $a \in I, s \in S$ *implies* $s.a \in I$ *and* $a.s \in I$.

Definition 2.5. *A left ideal* A *of* S *is called a left k-ideal of* S *if* $y, z \in A$, $x \in S$, *and* $x + y = z$ *implies* $x \in A$.

Definition 2.6. *An intuitionistic fuzzy sets defined on a non-empty set* X *as objects having the form* $A = \{< x, \mu_A(x), \gamma_A(x) > /x \in X\}$, *where the functions* $\mu_A : X \to [0, 1]$ *and* $\gamma_A : X \to [0, 1]$ *denote the degree of membership and the degree of non-membership of each element* $x \in X$ *to the set* A *respectively, and* $0 \leq \mu_A(x) + \gamma_A(x) \leq 1, \forall x \in X$.

Definition 2.7. *Let* A *and* B *be two intuitionistic fuzzy subsets of a set* X*: Then the following expressions hold:*

(1). $A \subseteq B$ *iff* $\mu_A(x) \leq \mu_B(x)$ *and* $\gamma_A(x) \geq \gamma_B(x)$,

(2). $A = B$ *iff* $A \subseteq B$ *and* $B \subseteq A$

(3). $A^C = \{< x, \mu_A(x), \gamma_A(x) > /x \in X\}$,

(4). $A \cap B = \{< x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\gamma_A(x), \gamma_B(x)\} > /x \in X\}\}$,

(5). $A \cup B = \{< x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\gamma_A(x), \gamma_B(x) > /x \in X\}$*:*

Definition 2.8. *An intuitionistic fuzzy set* $A = (\mu, \gamma)$ *in a semiring* S *is called an intuitionistic fuzzy left ideal of* S *if it satisfies* $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$
$\gamma(x + y) \leq \max\{\gamma(x), \gamma(y)\}$ $\forall x, y \in S$ *and* $\mu(x + y) \geq \mu(y); \gamma(x + y) \leq \gamma(y)$, $\forall x, y \in S$.

Definition 2.9. *If* $A = (\mu, \gamma)$ *is an intuitionistic fuzzy set in a set* S, *the strongest intuitionistic fuzzy relation on* S *that is an intuitionistic fuzzy relation on* A *is* $A_s = (\mu_{A_s}, \gamma_{A_s})$, *given by* $\mu_{A_s}(x, y) = \min\{\mu(x), \mu(y)\}$ *and* $\gamma_{A_s}(x, y) = \max\{\gamma(x), \gamma(y)\}, \forall x, y \in S$.

Definition 2.10. *A non-empty intuitionistic fuzzy subset* $A = (\mu_A, \gamma_A)$ *of a semigroup* S *is called an intuitionistic fuzzy left(right) ideal of* S *if*

(1). $\mu_A(xy) \geq \mu_A(y)(resp. \mu_A(xy) \geq \mu_A(x)), \forall x, y \in S$,

(2). $\gamma_A(xy) \leq \gamma_A(y)(resp. \gamma_A(xy) \leq \gamma_A(x)), \forall x, y \in S$

Definition 2.11. *A non-empty intuitionistic fuzzy subset* $A = (\mu_A, \gamma_A)$ *of a semigroup* S *is called an intuitionistic fuzzy two-sided ideal or an intuitionistic fuzzy ideal of* S *if it is both an intuitionistic fuzzy left and an intuitionistic fuzzy right ideal of* S.

3. Main Results

Theorem 3.1. *For a given intuitionistic fuzzy set* A *in a semiring* S *with the zero element,* A_s *can be the strongest intuitionistic fuzzy relation on* S. *If* A_s *is an intuitionistic fuzzy left k-ideal of* $S \times S$, *then* $\mu_A(a) \leq \mu_A(0); \gamma_A(a) \geq \gamma_A(0)$ *for all* $a \in S$.

Proof. If A_s is an intuitionistic fuzzy left $k-$ideal of $S\times S$, then $\mu_{A_s}(a,a)\leq\mu_{A_s}(0,0);\gamma_{A_s}(a,a)\geq\gamma_{A_s}(0,0)$ for all $a\in S\Rightarrow\min\{\mu_A(a),\mu_A(a)\}\leq\min\{\mu_A(0),\mu_A(0)\}$ and $\max\{\gamma_A(a),\gamma_A(a)\}\geq\max\{\gamma_A(0),\gamma_A(0)\}$, which implies that $\mu(a)\leq\mu(0);\gamma(a)\geq\gamma(0)$. □

Theorem 3.2. *Let $A=(\mu_A,\gamma_A)$ and $B=(\mu_B,\gamma_B)$ be intuitionistic fuzzy left k-ideals of a semiring S. Then $A\times B$ is an intuitionistic fuzzy left k-ideal of $S\times S$.*

Proof. Let $(x_1,x_2),(y_1,y_2)\in S\times S$. Then

$$\begin{aligned}(\mu_A\times\mu_B)((x_1,x_2)+(y_1,y_2))&=(\mu_A\times\mu_B)(x_1+y_1,x_2+y_2)\\&=\min\{\mu_A(x_1+y_1),\mu_B(x_2+y_2)\}\\&\geq\min\{\min\{\mu_A(x_1),\mu_A(y_1)\},\min\{\mu_B(x_2),\mu_B(y_2)\}\}\\&=\min\{\min\{\mu_A(x_1),\mu_B(x_2)\},\min\{\mu_A(y_1),\mu_B(y_2)\}\}\\&=\min\{(\mu_A\times\mu_B)(x_1,x_2),(\mu_A\times\mu_B)(y_1,y_2)\}\end{aligned}\tag{1}$$

Similarly,

$$\begin{aligned}(\gamma_A\times\gamma_B)((x_1,x_2)+(y_1,y_2))&=(\gamma_A\times\gamma_B)(x_1+y_1,x_2+y_2)\\&=\max\{\gamma_A(x_1+y_1),\gamma_B(x_2+y_2)\}\\&\leq\max\{\max\{\gamma_A(x_1),\gamma_A(y_1)\},\max\{\gamma_B(x_2),\gamma_B(y_2)\}\}\\&=\max\{\max\{\gamma_A(x_1),\gamma_B(x_2)\},\max\{\gamma_A(y_1),\gamma_B(y_2)\}\}\\&=\max\{(\gamma_A\times\gamma_B)(x_1,x_2),(\gamma_A\times\gamma_B)(y_1,y_2)\}\end{aligned}\tag{2}$$

$$\begin{aligned}(\mu_A\times\mu_B)((x_1,x_2)(y_1,y_2))&=(\mu_A\times\mu_B)(x_1y_1,x_2y_2)\\&=\min\{\mu_A(x_1y_1),\mu_B(x_2y_2)\}\\&\geq\min\mu_A(y_1),\mu_B(y_2)\\&=(\mu_A\times\mu_B)(y_1,y_2)\end{aligned}\tag{3}$$

Similarly,

$$\begin{aligned}(\gamma_A\times\gamma_B)((x_1,x_2)(y_1,y_2))&=(\gamma_A\times\gamma_B)(x_1y_1,x_2y_2)\\&=\max\{\gamma_A(x_1y_1),\gamma_B(x_2y_2)\}\\&\leq\max\{\gamma_A(y_1),\gamma_B(y_2)\}\\&=(\gamma_A\times\gamma_B)(y_1,y_2)\end{aligned}\tag{4}$$

Hence $A\times B$ is an intuitionistic fuzzy left ideal of $S\times S$. Now let $(a_1,a_2),(b_1,b_2),(x_1,x_2)\in S\times S$ be such that $(x_1,x_2)+(a_1,a_2)=(b_1,b_2)$ i.e., $(x_1+a_1,x_2+a_2)=(b_1,b_2)$. It follows that $x_1+a_1=b_1$ and $x_2+a_2=b_2$. Therefore,

$$\begin{aligned}(\mu_A\times\mu_B)(x_1,x_2)&=\min\{\mu_A(x_1),\mu_B(x_2)\}\\&\geq\min\{\min\{\mu_A(a_1),\mu_A(b_1)\},\min\{\mu_B(a_2),\mu_B(b_2)\}\}\\&=\min\{\min\{\mu_A(a_1),\mu_B(a_2)\},\min\{\mu_A(b_1),\mu_B(b_2)\}\}\\&=\min\{(\mu_A\times\mu_B)(a_1,a_2),(\mu_A\times\mu_B)(b_1,b_2)\}\end{aligned}\tag{5}$$

and

$$\begin{aligned}(\gamma_A \times \gamma_B)(x_1, x_2) &= \max\{\gamma_A(x_1), \gamma_B(x_2)\} \\ &\leq \max\{\max\{\gamma_A(a_1), \gamma_A(b_1)\}, \max\{\gamma_B(a_2), \gamma_B(b_2)\}\} \\ &= \max\{\max\{\gamma_A(a_1), \gamma_B(a_2)\}, \max\{\gamma_A(b_1), \gamma_B(b_2)\}\} \\ &= \max\{(\gamma_A \times \gamma_B)(a_1, a_2), (\gamma_A \times \gamma_B)(b_1, b_2)\}\end{aligned} \tag{6}$$

Hence $A \times B$ is an intuitionistic fuzzy left $k-$ideal of $S \times S$. □

Theorem 3.3. *Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy sets in a semiring S with the zero element such that $A \times B$ is an intuitionistic fuzzy left $k-$ideal of $S \times S$. Then*

(1). Either $\mu_A(x) \leq \mu_A(0)$ and $\gamma_A(x) \geq \gamma_A(0)$ or $\mu_B(x) \leq \mu_B(0)$ and $\gamma_B(x) \geq \gamma_B(0)$ for all $x \in S$.

(2). If $\mu_A(x) \leq \mu_A(0)$ and $\gamma_B(x) \geq \gamma_B(0)$ for all $x \in S$, then either $\mu_A(x) \leq \mu_B(0); \gamma_A(x) \geq \gamma_B(0)$ or $\mu_B(x) \leq \mu_B(0); \gamma_B(x) \geq \gamma_B(0)$.

(3). If $\mu_B(x) \leq \mu_B(0); \gamma_B(x) \geq \gamma_B(0)$ for all $x \in S$, then either $\mu_A(x) \leq \mu_A(0); \gamma_A(x) \geq \gamma_A(0)$ or $\mu_B(x) \leq \mu_A(0); \gamma_B(x) \geq \gamma_A(0)$

(4). If $\mu_B(x) \leq \mu_A(0); \gamma_B(x) \geq \gamma_A(0)$ for any $x \in S$, then B is an intuitionistic fuzzy left k-ideal of S.

(5). If $\mu_A(x) \leq \mu_A(0); \gamma_A(x) \geq \gamma_A(0)$ for all $x \in S$ and $\mu_B(y) > \mu_B(0)$ and $\gamma_B(y) \leq \gamma_B(0)$ for some y in S. Then A is an intuitionistic fuzzy left k-ideal of S.

Proof.

(1). Suppose that $\mu_A(x) > \mu_A(0); \gamma_A(x) < \gamma_A(0)$ and $\mu_B(x) > \mu_B(0); \gamma_B(x) < \gamma_B(0)$. Then

$$(\mu_A \times \mu_B)(x, y) > \min\{\mu_A(x), \mu_B(y)\} = (\mu_A \times \mu_B)(0, 0)$$
$$(\gamma_A \times \gamma_B)(x, y) < \max\{\gamma_A(x), \gamma_B(y)\} = (\gamma_A \times \gamma_B)(0, 0)$$

Which is a contradiction. Hence we obtain (1).

(2). Let us assume that there exist $x, y \in S$ such that $\mu_A(x) > \mu_B(0); \gamma_A(x) < \gamma_B(0)$ and $\mu_B(y) > \mu_B(0); \gamma_B(y) < \gamma_B(0)$. Then $(\mu_A \times \mu_B)(0, 0) = \min\{\mu_A(0), \mu_B(0)\} = \mu_B(0)$ and $(\gamma_A \times \gamma_B)(0, 0) = \max\{\gamma_A(0), \gamma_B(0)\} = \gamma_B(0)$. Hence,

$$(\mu_A \times \mu_B)(x, y) = \min\{\mu_A(x), \mu_B(x)\} > \mu_B(0) = (\mu_A \times \mu_B)(0, 0),$$
$$(\gamma_A \times \gamma_B)(x, y) = \max\{\gamma_A(x), \gamma_B(x)\} < \gamma_B(0) = (\gamma_A \times \gamma_B)(0, 0)$$

This is a contradiction. Hence (2) holds.

(3). Similarly we can prove (3).

(4). If $\mu_B(x) \leq \mu_A(0); \gamma_B(x) \geq \gamma_B(0)$ for any $x \in S$, then

$$\begin{aligned} \mu_B(x+y) &= \min\{\mu_A(0), \mu_B(x+y)\} \\ &= (\mu_A \times \mu_B)(0, x+y) \\ &= (\mu_A \times \mu_B)((0,x)+(0,y)) \\ &\geq \min\{(\mu_A \times \mu_B)(0,x), (\mu_A \times \mu_B)(0,y)\} \\ &= \min\{\min\{\mu_A(0), \mu_B(x)\}, \min\{\mu_A(0), \mu_B(y)\}\} \\ &= \min\{\mu_B(x), \mu_B(y)\} \end{aligned} \tag{7}$$

$$\begin{aligned} \gamma_B(x+y) &= \max\{\gamma_A(0), \gamma_B(x+y)\} \\ &= (\gamma_A \times \gamma_B)(0, x+y) \\ &= (\gamma_A \times \gamma_B)((0,x)+(0,y)) \\ &\leq \max\{(\gamma_A \times \gamma_B)(0,x), (\gamma_A \times \gamma_B)(0,y)\} \\ &= \max\{\max\{\gamma_A(0), \gamma_B(x)\}, \max\{\gamma_A(0), \gamma_B(y)\}\} \\ &= \max\{\gamma_B(x), \gamma_B(y)\} \end{aligned} \tag{8}$$

and

$$\begin{aligned} \mu_B(xy) &= \min\{\mu_A(0), \mu_B(xy)\} \\ &= (\mu_A \times \mu_B)(0, xy) \\ &= (\mu_A \times \mu_B)((0,x)(0,y)) \\ &\geq (\mu_A \times \mu_B)(0,y) \\ &= \min\{\mu_A(0), \mu_B(y)\} \\ &= \mu_B(y) \end{aligned} \tag{9}$$

$$\begin{aligned} \gamma_B(xy) &= \max\{\gamma_A(0), \gamma_B(xy)\} \\ &= (\gamma_A \times \gamma_B)(0, xy) \\ &= (\gamma_A \times \gamma_B)((0,x)(0,y)) \\ &\leq (\gamma_A \times \gamma_B)(0,y) \\ &= \max\{\gamma_A(0), \gamma_B(y)\} \\ &= \gamma_B(y) \end{aligned} \tag{10}$$

for all $x, y \in S$. Hence B is an intuitionistic fuzzy left ideal of S. Now let $a, b, x \in S$ be such that $x + a = b$. Then $(0, x) + (0, a) = (0, b)$ and so

$$\begin{aligned} \mu_B(x) &= \min\{\mu_A(0), \mu_B(x)\} \\ &= (\mu_A \times \mu_B)(0, x) \\ &\geq \min\{(\mu_A \times \mu_B)(0,a), (\mu_A \times \mu_B)(0,b)\} \\ &= \min\{\min\{\mu_A(0), \mu_B(a)\}, \min\{\mu_A(0), \mu_B(b)\}\} \\ &= \min\{\mu_B(a), \mu_B(a)\} \end{aligned} \tag{11}$$

$$\begin{aligned}\gamma_B(x) &= \max\{\gamma_A(0), \gamma_B(x)\}\\ &= (\gamma_A \times \gamma_B)(0, x)\\ &\leq \max\{(\gamma_A \times \gamma_B)(0, a), (\gamma_A \times \gamma_B)(0, b)\}\\ &= \max\{\max\{\gamma_A(0), \gamma_B(a)\}, \max\{\gamma_A(0), \gamma_B(b)\}\}\\ &= \max\{\gamma_B(a), \gamma_B(a)\}\end{aligned} \tag{12}$$

Hence μ_B is an intuitionistic fuzzy left k-ideal of S.

(5). Assume that $\mu_A(x) \leq \mu_A(0); \gamma_A(x) \geq \gamma_A(0)$ for all $x \in S$ and $\mu_B(y) > \mu_A(0); \gamma_B(y) < \gamma_A(0)$ for some $y \in S$. Then $\mu_B(0) \geq \mu_B(x) > \mu_A(0); \gamma_B(0) \leq \gamma_B(x) < \gamma_A(0)$. Since $\mu_A(0) \geq \mu_A(x); \mu_A(0) \geq \mu_A(x)$. Hence

$$(\mu_A \times \mu_B)(0, x) = \min\{\mu_A(x), \mu_B(0)\} = \mu_A(x)$$

$$(\gamma_A \times \gamma_B)(0, x) = \max\{\gamma_A(x), \gamma_B(0)\} = \gamma_A(x)$$

for all $x \in S$. Thus

$$\begin{aligned}\mu_A(x + y) &= (\mu_A \times \mu_B)(x + y, 0)\\ &= (\mu_A \times \mu_B)((x, 0) + (y, 0))\\ &\geq \min\{(\mu_A \times \mu_B)(x, 0), (\mu_A \times \mu_B)(y, 0)\}\\ &= \min\{\mu_A(x), \mu_A(y)\}\end{aligned} \tag{13}$$

$$\begin{aligned}\gamma_A(x + y) &= (\gamma_A \times \gamma_B)(x + y, 0)\\ &= (\gamma_A \times \gamma_B)((x, 0) + (y, 0))\\ &\leq \max\{(\gamma_A \times \gamma_B)(x, 0), (\gamma_A \times \gamma_B)(y, 0)\}\\ &= \max\{\gamma_A(x), \gamma_A(y)\}\end{aligned} \tag{14}$$

and

$$\begin{aligned}\mu_A(xy) &= (\mu_A \times \mu_B)(xy, 0)\\ &= (\mu_A \times \mu_B)((x, 0)(y, 0))\\ &\geq \min\{(\mu_A \times \mu_B)(y, 0) = \mu_B(y)\end{aligned} \tag{15}$$

$$\begin{aligned}\gamma_A(xy) &= (\gamma_A \times \gamma_B)(xy, 0)\\ &= (\gamma_A \times \gamma_B)((x, 0)(y, 0))\\ &\leq \max\{(\gamma_A \times \gamma_B)(y, 0) = \gamma_B(y)\end{aligned} \tag{16}$$

for all $x, y \in S$. Now let $a, b, x \in S$ be such that $x + a = b$ and so $(x, 0) + (a, 0) = (b, 0)$. Then

$$\begin{aligned}\mu_A(x) &= (\mu_A \times \mu_B)(x, 0)\\ &\geq \min\{(\mu_A \times \mu_B)(a, 0), (\mu_A \times \mu_B)(b, 0)\}\\ &= \min\{\mu_A(a), \mu_B(b)\}\end{aligned} \tag{17}$$

$$\begin{aligned}\gamma_A(x) &= (\gamma_A \times \gamma_B)(x, 0)\\ &\leq \max\{(\gamma_A \times \gamma_B)(a, 0), (\gamma_A \times \gamma_B)(b, 0)\}\\ &= \max\{\gamma_A(a), \gamma_B(b)\}\end{aligned} \tag{18}$$

Consequently, A is an intuitionistic fuzzy left $k-$ideal of S. Hence the proof. □

Theorem 3.4. *Let A be an intuitionistic fuzzy set in a semiring S and let A_s be the strongest intuitionistic fuzzy relation on S. Then A is an intuitionistic fuzzy left k-ideal of S if and only if A_s is an intuitionistic fuzzy left k-ideal of $S \times S$.*

Proof. Let $A = (\mu, \gamma)$ be an intuitionistic fuzzy left k-ideal of S. Let $(x_1, x_2, (y_1, y_2) \in S \times S$. Then

$$\begin{aligned} \mu_{A_s}((x_1,x_2)+(y_1,y_2)) &= \mu_{A_s}(x_1+y_1, x_2+y_2) \\ &= \min\{\mu(x_1+y_1), \mu(x_2+y_2)\} \\ &\geq \min\{\min\{\mu(x_1),\mu(y_1)\}, \min\{\mu(x_2),\mu(y_2)\}\} \\ &= \min\{\min\{\mu(x_1),\mu(x_2)\}, \min\{\mu(y_1),\mu(y_2)\}\} \\ &= \min\{\mu_{A_s}(x_1,x_2), \mu_{A_s}(y_1,y_2)\} \end{aligned} \tag{19}$$

$$\begin{aligned} \gamma_{A_s}((x_1,x_2)+(y_1,y_2)) &= \gamma_{A_s}(x_1+y_1, x_2+y_2) \\ &= \max\{\gamma(x_1+y_1), \gamma(x_2+y_2)\} \\ &\leq \max\{\max\{\gamma(x_1),\gamma(y_1)\}, \max\{\gamma(x_2),\gamma(y_2)\}\} \\ &= \max\{\max\{\gamma(x_1),\gamma(x_2)\}, \max\{\gamma(y_1),\gamma(y_2)\}\} \\ &= \max\{\gamma_{A_s}(x_1,x_2), \gamma_{A_s}(y_1,y_2)\} \end{aligned} \tag{20}$$

and

$$\begin{aligned} \mu_{A_s}((x_1,x_2)(y_1,y_2)) &= \mu_{A_s}(x_1y_1, x_2y_2) \\ &= \min\{\mu(x_1y_1), \mu(x_2y_2)\} \\ &\geq \min\{\mu(y_1), \mu(y_2)\} \\ &= \mu_{A_s}(y_1,y_2) \end{aligned} \tag{21}$$

$$\begin{aligned} \gamma_{A_s}((x_1,x_2)(y_1,y_2)) &= \gamma_{A_s}(x_1y_1, x_2y_2) \\ &= \max\{\gamma(x_1y_1), \gamma(x_2y_2)\} \\ &\leq \max\{\gamma(y_1), \gamma(y_2)\} \\ &= \gamma_{A_s}(y_1,y_2) \end{aligned} \tag{22}$$

Let $(a_1, a_2), (b_1, b_2), S \times S$ be such that $(x_1, x_2) + (a_1, a_2) = (b_1, b_2)$. Then $(x_1 + a_1, x_2 + a_2) = (b_1, b_2)$,it follows that $x_1 + a_1 = b_1$ and $x_2 + a_2 = b_2$. Thus

$$\begin{aligned} \mu_{A_s}(x_1,x_2) &= \min\{\mu(x_1), \mu(x_2)\} \\ &\geq \min\{\min\{\mu(a_1),\mu(b_1)\}, \min\{\mu(a_2),\mu(b_2)\}\} \\ &= \min\{\min\{\mu(a_1),\mu(a_2)\}, \min\{\mu(b_1),\mu(b_2)\}\} \\ &= \min\{\mu_{A_s}(a_1,a_2), \mu_{A_s}(b_1,b_2)\} \end{aligned} \tag{23}$$

$$\begin{aligned} \gamma_{A_s}(x_1,x_2) &= \max\{\gamma(x_1), \gamma(x_2)\} \\ &\leq \max\{\max\{\gamma(a_1),\gamma(b_1)\}, \max\{\gamma(a_2),\gamma(b_2)\}\} \\ &= \max\{\max\{\gamma(a_1),\gamma(a_2)\}, \max\{\gamma(b_1),\gamma(b_2)\}\} \\ &= \max\{\gamma_{A_s}(a_1,a_2), \gamma_{A_s}(b_1,b_2)\} \end{aligned} \tag{24}$$

HenceA_s is an intuitionistic fuzzy left $k-$ ideal of $S \times S$.

Conversely, suppose that A_s is an intuitionistic fuzzy left $k-$ ideal of $S \times S$. Let $x_1, x_2, y_1, y_2 \in S$. Then

$$\begin{aligned} \min\{\mu(x_1+y_1), \mu(x_2+y_2)\} &= \mu_{A_s}(x_1+y_1, x_2+y_2) \\ &\geq \min\{\mu_{A_s}(x_1,x_2), \mu_{A_s}(y_1,y_2)\} \\ &= \min\{\min\{\mu(x_1),\mu(x_2)\}, \min\{\mu(y_1),\mu(y_2)\}\} \end{aligned} \tag{25}$$

$\Rightarrow \mu(x_1+y_1) \geq \min\{\min\{\mu(x_1),\mu(x_2)\}, \min\{\mu(y_1),\mu(y_2)\}\}$. Similarly,

$$\begin{aligned} \max\{\gamma(x_1+y_1), \gamma(x_2+y_2)\} &= \gamma_{A_s}(x_1+y_1, x_2+y_2) \\ &\leq \max\{\gamma_{A_s}(x_1,x_2), \gamma_{A_s}(y_1,y_2)\} \\ &= \max\{\max\{\gamma(x_1),\gamma(x_2)\}, \max\{\gamma(y_1),\gamma(y_2)\}\} \end{aligned} \tag{26}$$

$\Rightarrow \gamma(x_1+y_1) \leq \max\{\max\{\gamma(x_1),\gamma(x_2)\}, \max\{\gamma(y_1),\gamma(y_2)\}\}$. In this inequality, we choose the values of x_1, x_2, y_1 and y_2 as follows: $x_1 = x$, $x_2 = 0$, $y_1 = y$ and $y_2 = 0$. Then we have

$$\mu(x+y) \geq \min\{\min\{\mu(x),\mu(0)\}, \min\{\mu(y),\mu(0)\}\} = \min\{\mu(x),\mu(y)\}$$
$$\gamma(x+y) \leq \max\{\max\{\gamma(x),\gamma(0)\}, \max\{\gamma(y),\gamma(0)\}\} = \max\{\gamma(x),\gamma(y)\}$$

by using Theorem 3.1. Next, we have

$$\begin{aligned} \min\{\mu(x_1y_1), \mu(x_2y_2)\} &= \mu_{A_s}(x_1y_1, x_2y_2) \\ &= \mu_{A_s}((x_1,x_2)(y_1,y_2)) \\ &\geq \mu_{A_s}(y_1,y_2) \\ &= \min\{\mu(y_1),\mu(y_2)\} \end{aligned} \tag{27}$$

$$\begin{aligned} \max\{\gamma(x_1y_1), \gamma(x_2y_2)\} &= \gamma_{A_s}(x_1y_1, x_2y_2) \\ &= \gamma_{A_s}((x_1,x_2)(y_1,y_2)) \\ &\leq \gamma_{A_s}(y_1,y_2) \\ &= \max\{\gamma(y_1),\gamma(y_2)\} \end{aligned} \tag{28}$$

and so $\mu(x_1y_1) \geq \min\{\mu(y_1),\mu(y_2)\}$. Taking $x_1 = x, y_1 = y$ and $y_2 = 0$ and using Theorem 3.1, we get $\mu(xy) \geq \min\{\mu(y),\mu(0)\} = \mu(y)$; $\gamma(xy) \leq \max\{\gamma(y),\gamma(0)\} = \gamma(y)$. Hence A is an intuitionistic fuzzy left ideal of S. Let $a, b, x \in S$ be such that $x + a = b$. Then $(x,0) + (a,0) = (b,0)$. Since A_s is an intuitionistic fuzzy left k-ideal of $S \times S$, it follows from Theorem 3.1 that

$$\begin{aligned} \mu(x) &= \min\{\mu(x),\mu(0)\} \\ &= \mu_{A_s}(x,0) \\ &\geq \min\{\mu_{A_s}(a,0), \mu_{A_s}(b,0)\} \\ &= \min\{\min\{\mu(a),\mu(0)\}, \min\{\mu(b),\mu(0)\}\} \\ &= \min\{\mu(a),\mu(b)\} \end{aligned} \tag{29}$$

$$\begin{aligned}\gamma(x) &= \max\{\gamma(x), \gamma(0)\} \\ &= \gamma_{A_s}(x, 0) \\ &\leq \max\{\gamma_{A_s}(a, 0), \gamma_{A_s}(b, 0)\} \\ &= \max\{\max\{\gamma(a), \gamma(0)\}, \max\{\gamma(b), \gamma(0)\}\} \\ &= \max\{\gamma(a), \gamma(b)\}\end{aligned} \tag{30}$$

Consequently, A is an intuitionistic fuzzy left k-ideal of S. This completes the proof. □

4. Conclusion

A semi-group is an algebraic structure consisting of a non-empty set S together with an associative binary operation. The formal study of semi-groups began in the early 20th century. Semi-groups are important in many areas of mathematics, for example, coding and language theory, automata theory, combinatorics and mathematical analysis. Semi-ring plays an important role in studying matrices and determinants. In this paper, we proved that the Cartesian product of two intuitionistic fuzzy k-ideals of semi-ring S is also an intuitionistic fuzzy k-ideal. Conversely, if $A \times B$ is an intuitionistic fuzzy left k-ideal of $S \times S$, then either A or B is an intuitionistic fuzzy left k-ideal of S. Also We proved that an intuitionistic fuzzy set A in a semi-ring S is a fuzzy left k-ideal of S if and only if the strongest fuzzy relation A_s on S is an intuitionistic fuzzy left k-ideal of $S \times S$.

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