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# Gaussian Prime Labeling of Some Product Graphs

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**Abstract:** A graph G on vertices is said to have prime labeling if there exists a labeling from the vertices of G to the first n natural numbers such that any two adjacent vertices have relatively prime labels.Gaussian integers are the complex numbers of the form $a + bi$ where $a, b \in \mathbb{Z}$ and $i^2 = -1$ and it is denoted by $\mathbb{Z}[i]$. A Gaussian prime labeling on G is a bijection from the vertices of G to $[\psi_n]$, the set of the first n Gaussian integers in the spiral ordering such that if $uv \in E(G)$, then $\psi_{(u)}$ and $\psi_{(v)}$ are relatively prime. Using the order on the Gaussian integers, we discuss the Gaussian prime labeling of product graphs.

**MSC:** 05C78.

**Keywords:** Gaussian integers, Gaussian prime labeling, product graphs.


## 1. Introduction

In this paper we discuss the Gaussian prime labeling of product graphs.In the first section we introduce Gaussian integers and their properties.The prime labeling was extended to Gaussian prime labeling by defining the first n Gaussian integers in spiral ordering introduced by Hunter Lehmann and Andrew Park [4]. The spiral ordering allows us to linearly order the Gaussian integers.In the second section we discuss Gaussian prime labeling of some product graphs.

## 2. Gaussian Integers

The complex numbers of the form a+bi,where $a, b \in \mathbb{Z}[i]$ and $i^2 = -1$ is called the Gaussian integers and it is denoted by $\mathbb{Z}[i]$. Any one of complex numbers $\pm 1, \pm i$ are the units in the Gaussian integers.For any Gaussian integer $\alpha$, the associate is $u.\alpha$ where u is the Gaussian unit. $N(a + bi)$ denote the norm of the Gaussian integer a+bi and it is given by $a^2 + b^2$. An even Gaussian integer is a Gaussian integer which is divisible by $1 + i$ and odd otherwise. The Gaussian integers are not totally ordered. So we take the spiral ordering on n Gaussian integers defined by Hunter Lehmann and Andrew Park [4] as follows.

**Definition 2.1.** *The ordering of the Gaussian integers is called spiral ordering which is a recursively defined ordering of the Gaussian integers. We denote the $n^{th}$ Gaussian integer in the spiral ordering by $\psi_n$. The ordering is defined beginning*

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*with* $\psi_1 = 1$ *and continuing as:*

$$\psi_{n+1} = \begin{cases} \psi_n + i, & \text{if } Re(\psi_n) \cong 1 \ (mod\ 2), Re\ (\psi_n) > Im\ (\psi_n) + 1 \\ \psi_n - 1, & \text{if } Im(\psi_n) \cong 0 \ (mod\ 2), Re\ (\psi_n) \leq Im\ (\psi_n) + 1, Re\ (\psi_n) > 1 \\ \psi_n + 1, & \text{if } Im(\psi_n) \cong 1 \ (mod\ 2), Re\ (\psi_n) < Im\ (\psi_n) + 1 \\ \psi_n + i, & \text{if } Im(\psi_n) \cong 0 \ (mod\ 2), Re\ (\psi_n) = 1 \\ \psi_n - i, & \text{if } Re(\psi_n) \cong 0 \ (mod\ 2), Re\ (\psi_n) \geq Im\ (\psi_n) + 1, Im(\psi_n) > 0 \\ \psi_n + 1, & \text{if } Re(\psi_n) \cong 0 \ (mod\ 2), Im\ (\psi_n) = 0 \end{cases}$$

The following figure shows the spiral ordering of Gaussian integers.

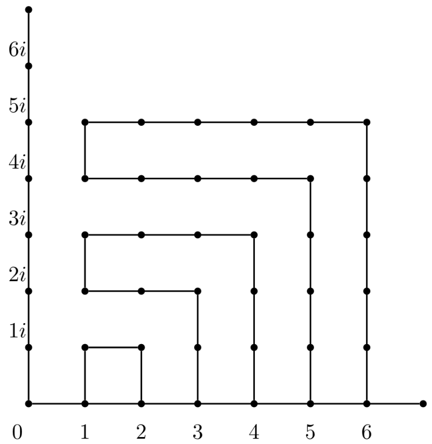


**Figure 1.**

The first 10 Gaussian integers in the spiral ordering are 1, $1+i$, $2+i$, 2, 3, $3+i$, $3+2i$, $2+2i$, $1+2i$, $1+3i$. The set of first n Gaussian integers in the spiral ordering is denoted by $[\psi_n]$. Hunter Lehmann and Andrew Park [4] discussed the following properties of Gaussian integers

(1). A Gaussian integer $\rho$ is called a prime Gaussian integer if its only divisors are $\pm 1$, $\pm i$, $\pm \rho$, or $\pm \rho i$.

(2). Two Gaussian integers $\alpha$ and $\beta$ are relatively prime if their only common divisors are units in $Z[i]$.

(3). Let $\rho$ be a Gaussian integer and u be a unit. Then $\rho$ and $\rho + u$ are relatively prime.

(4). In the spiral ordering,consecutive Gaussian integers are relatively prime.

(5). Let $\rho$ be an odd Gaussian integer, let t be a positive integer and u be a unit.Then $\rho$ and $\rho + u.(1+i)^t$ are relatively prime.

(6). In the spiral ordering, consecutive odd Gaussian integers are relatively prime.

(7). In the spiral ordering, consecutive even Gaussian integers are relatively prime.

(8). Let $\alpha$ be a prime Gaussian integer and $\rho$ be a Gaussian integer. $\rho$ and $\rho + \alpha$ are relatively prime if and only if $\alpha \nmid \rho$.

## 3. Gaussian Prime Labeling of Product Graphs

Meenakshi Sundaram and Nellaimurugan [6] discuss the prime labeling of product graphs. They prove the prime labeling of $(P_n * K_{1,2})^k$ and $(P_n * K_{1,3})^k$ . We denote $P_n * K_{1,m}$ be the graph obtained from $P_n$ by identifying one end vertex $P_n$ with center vertex of $K_{1,m}$. A regular bamboo tree $(P_n * K_{1,m})^k$ is the one point union of n copies of $P_n * K_{1,m}$. We discuss the gaussian prime labeling of $(P_n * K_{1,2})^k$,$(P_n * K_{1,3})^k$ and $(P_n * K_{1,4})^k$.

**Definition 3.1** ([7]). *Let G be a graph on n vertices.A bijection $f : V(G) \to [\psi_n]$ is called a Gaussian prime labeling if for every edge $uv \in E(G)$, f(u) and f(v) are relatively prime.A graph which admits Gaussian prime labeling is called a Gaussian prime graph.*

**Illustartion 3.2.** *The following Figure 2 shows Gaussian prime labeling of complete graph $K_4$.*

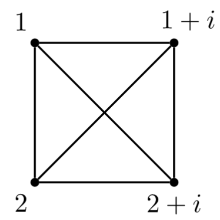


**Figure 2.**

**Theorem 3.3.** *$(P_n * K_{1,2})^k$ is Gaussian prime graph for $n \in \mathbb{N}$ and $k \in \mathbb{N}$.*

*Proof.* There are $kn + 1$ vertices and kn edges in the graph $(P_n * K_{1,2})^k$. The vertices set are denoted by $V = \{u, v_t^s, x_s, y_s, 1 \leq t \leq n-2, 1 \leq s \leq k\}$ and the edges are denoted by $E = \{u, v_1^s, 1 \leq s \leq k\} \cup \{v_t^s v_{t+1}^s, 1 \leq t \leq n-3, 1 \leq s \leq k\} \cup \{v_{n-2}^s x_s, 1 \leq s \leq k\} \cup \{v_{n-2}^s y_s, 1 \leq s \leq k\}$. Define a labeling $f : V \to \{\psi_1, \psi_2, \psi_3, ..., \psi_{kn+1}\}$ by

$$f(u) = \psi_1$$
$$f(v_t^s) = \psi_{(n(s-1)+t+1)},\ 1 \leq t \leq n-2,\ 1 \leq s \leq k$$
$$f(x_s) = \psi_{(n(s-1)+n)},\ 1 \leq s \leq k$$
$$f(y_s) = \psi_{(n(s-1)+n+1)},\ 1 \leq s \leq k$$

The neighbourhood vertices of every vertex in the labeling is either consecutive odd Gaussian integers or consecutive even Gaussian integers or consecutive Gaussian integers. Consecutive Gaussian integers in the spiral ordering are relatively prime. Consecutive odd Gaussian integers in the spiral ordering are relatively prime and Consecutive even Gaussian integers in the spiral ordering are relatively prime. Clearly the labelling is Gaussian prime. □

**Theorem 3.4.** *$(P_n * K_{1,3})^k$ is Gaussian prime graph for $n \in \mathbb{N}$ and $k \in \mathbb{N}$.*

*Proof.* There are k(n+1)+1 vertices and k(n+1) edges in the graph $(P_n * K_{1,3})^k$. The vertices set are denoted by $V = \{u, v_t^s, x_s, y_s, z_s, 1 \leq t \leq n-2, 1 \leq s \leq k\}$ and the edges are denoted by $E = \{u, v_1^s, 1 \leq s \leq k\} \cup \{v_t^s v_{t+1}^s, 1 \leq t \leq n-3, 1 \leq s \leq k\} \cup \{v_{n-2}^s x_s, 1 \leq s \leq k\} \cup \{v_{n-2}^s y_s, 1 \leq s \leq k\} \cup \{v_{n-2}^s z_s, 1 \leq s \leq k\}$. Define a labeling $f : V \to \{\psi_1, \psi_2, \psi_3, ...\psi_{k(n+1)+1}\}$ by

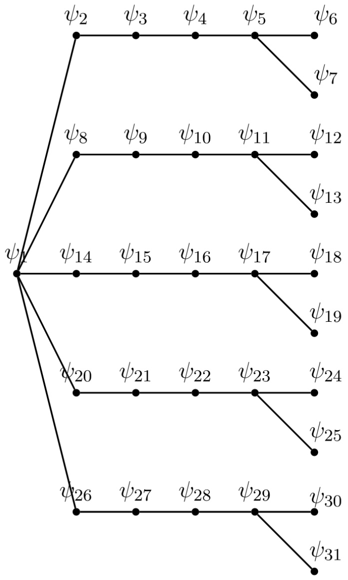


**Figure 3.** **The Gaussian prime labeling of** $(P_6 * K_{1,2})^k$

$$f(u) = \psi_1$$
$$f(v_t^s) = \psi_{((n+1)(s-1)+t+1)},\ 1 \le t \le n-3,\ 1 \le s \le k$$
$$f(v_{n-2}^s) = \psi_{((n+1)(s-1)+n)},\ 1 \le s \le k$$
$$f(x_s) = \psi_{((n+1)(s-1)+n-1)},\ 1 \le s \le k$$
$$f(y_s) = \psi_{((n+1)(s-1)+n+1)},\ 1 \le s \le k$$
$$f(z_s) = \psi_{((n+1)(s-1)+n+2)},\ 1 \le s \le k$$

The neighbourhood vertices of every vertex in the labeling is either consecutive odd Gaussian integers or consecutive even Gaussian integers or consecutive Gaussian integers. Consecutive Gaussian integers in the spiral ordering are relatively prime. Consecutive odd Gaussian integers in the spiral ordering are relatively prime and Consecutive even Gaussian integers in the spiral ordering are relatively prime. Clearly the labeling is Gaussian prime. □

**Theorem 3.5.** $(P_n * K_{1,4})^k$ *is Gaussian prime graph for* $n \in \mathbb{N}$ *and* $k \in \mathbb{N}$.

*Proof.* There are $k(n+2)+1$ vertices and $k(n+2)$ edges in the graph $(P_n \otimes K_{1,4})^k$. The vertices set are denoted by $V = \{u, v_t^s, x_s, y_s, z_s, w_s, 1 \le t \le n-2, 1 \le s \le k\}$ and the edgs are denoted by $E = \{u, v_1^s, 1 \le s \le k\} \cup \{v_t^s v_{t+1}^s, 1 \le t \le n-3, 1 \le s \le k\} \cup \{v_{n-2}^s x_s, 1 \le s \le k\} \cup \{v_{n-2}^s y_s, 1 \le s \le k\} \cup \{v_{n-2}^s z_s, 1 \le s \le k\} \cup \{v_{n-2}^s z_s, 1 \le s \le k\}$. Define a labeling $f : V \to \{\psi_1, \psi_2, \psi_3, ..., \psi_{k(n+2)+1}\}$ by

$$f(u) = \psi_1$$
$$f(v_t^s) = \psi_{((n+2)(s-1)+t+1},\ 1 \le t \le n-3,\ 1 \le s \le k$$
$$f(v_{n-2}^s) = \psi_{((n+2)(s-1)+n+1)},\ 1 \le s \le k$$
$$f(x_s) = \psi_{((n+2)(s-1)+n-1)},\ 1 \le s \le k$$
$$f(y_s) = \psi_{((n+2)(s-1)+n)},\ 1 \le s \le k$$

$$f(z_s) = \psi_{((n+2)(s-1)+n+2)},\ 1 \leq s \leq k$$

$$f(w_s) = \psi_{((n+2)(s-1)+n-2)},\ 1 \leq s \leq k$$

Consider the vertices $f(v^s_{n-2})$ and $f(v^s_{n-3})$, $1 \leq s \leq k$. The differences of labeling on these vertices are any one element in the set $\{1+2i, 2+i, 1, 3\}$ or their associates. They are Gaussian prime integers and using property (8) the labeling on the vertices are relatively prime. The labeling on the adjacent vertices of other vertices are consecutive Gaussian integers,consecutive odd Gaussian integers or consecutive even Gaussian integers. Consecutive Gaussian integers in the spiral ordering are relatively prime. Consecutive odd Gaussian integers in the spiral ordering are relatively prime and Consecutive even Gaussian integers in the spiral ordering are relatively prime. So the labeling is Gaussian prime. □

## References


[1] G.S.Bloom and S.W.Golomb, *Applications of numbered undireted graphs*, Proceedings of IEEE., 165(4)(1977), 562-570.

[2] H.L.Fu and K.C.Huang, *On Prime labeling*, Discrete Math., 127(1994), 181-186.

[3] F.Harary, *Graph Theory*, Addision-Wesley, Reading, Massachusetts, (1969).

[4] Hunter Lehmann and Andrew Park, *Prime labeling of small trees with the Gaussian integers*, Rose-Hulman Undergrad. Math. J., (2015).

[5] Joseph A.Gallian, *A dynamic survey of Graph Labeling*, The Electronic Journal of Combinatorics, 17(2014).

[6] L.Meenakshi Sundaram, A.Nagarajan, S.Navaneethakrishnan and A.Nellaimurugan, *Prime Labeling of Some Product Graphs*, Ultra Scientist 25(2A)(2013).

[7] Steven Lee, Hunter Lehmann and Andrew Park, *Prime labeling of families of trees with the Gaussian integers*, AKCE international Journal of Graphs and Combinatorics, 13(2)(2016), 165-176.

[8] A.Tout, A.N.Dabboucy and K.Howalla, *Prime labeling of graphs*, Nat. Acad. Sci. Letters, 11(1982), 365-368.