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Roman Labeling in Digraphs

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Abstract: Let G be a connected graph. Roman labeling of G is a function $f : V(G) \to \{0, 1, 2\}$ such that any vertex with label 0 is adjacent to a vertex with label 2. The function value $f(v)$ of a vertex v of the graph G is called the label of v. Weight of a Roman labeling, f is defined as the sum of all vertex labels.That is, w $(f) = \sum\limits_{v \in V(G)} f(v)$. Roman number of a graph G is defined as the minimum weight of a Roman labeling on G and is denoted by $S(G)$. It was introduced by Satheesh and Suresh Kumar [3]. In this paper, we extend the idea of Roman labeling and Roman number to Digraphs.

Keywords: Roman labeling, Roman number, Digraphs.

1. Introduction

Kamaraj and Jakkammal [1] extended the idea of Roman domination to Digraphs, which is followed by the study of Sheikholeslami and Volkmann [2]. A Roman labeling on a digraph $D = (V, A)$ is a function $f : V \to \{0, 1, 2\}$, which satisfies the condition that every vertex v for which $f(v) = 0$ has an in-neighbor u, such that $f(u) = 2$. The weight of labeling is the sum of all the vertex labels. The minimum weight of Roman labeling on D (denoted by $S(D)$) is the Roman number of a digraph D. A Roman labeling is a Roman dominating function of D with weight $S(D)$. A Roman labeling $f : V \to \{0, 1, 2\}$, canbe represented by the ordered partition (V_0, V_1, V_2), where $V_i = \{v \in V(D) : |f(v) = i\}$. In this representation, its weight is $|V_1| + 2|V_2|$. For the terms and definitions not defined explicitly here, refer Harary [4].

2. Main Results

If a cycle, which contains directed edges, has r vertices of in-degree 2, then it has at least r vertices having out-degree 2. This fact can be used to reduce the upper bound for many oriented cycles.

Proposition 2.1. *If a directed cycle C of order n has r vertices of in degree 2, then $S(C) = n - r$.*

Proof. All vertices having out-degree 2 can be included in V_2 and other vertices which are not in the out neighbourhood of the vertices in V_2 are included in V_1 to obtain a Roman labeling. This result can be used to reset the upper bound for digraphs which contains a Hamiltonian cycle in the underlying graph and orientation of the edges on the Hamiltonian circuit contains r vertices of in degree 2. Other arcs present in the graph, except those in the Hamiltonian circuit, can only decrease the Roman domination in D. So the following result is evident. □

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Proposition 2.2. *If a directed graph D of order n contains a Hamiltonian cycle C, which has r vertices of in-degree 2 together with its orientation, then $S(C) = n - r$.*

Proposition 2.3. *Let V_1 and V_2 be a partition of $V(D)$ into two subsets, such that D doesn't contain any arc from $\langle V_1 \rangle$ to $\langle V_2 \rangle$. Then, $S(D) = S(\langle V_1 \rangle) + S(\langle V_2 \rangle)$, where $\langle F \rangle$ denote the induced subgraph of D induced by the vertices, $F \subseteq V(D)$.*

Proof. Let f_1 and f_2 be the Roman labelings of the subgraphs $\langle V_1 \rangle$ and $\langle V_2 \rangle$ respectively. Since all the arcs between $\langle V_1 \rangle$ and $\langle V_2 \rangle$ are directed from $\langle V_1 \rangle$ to $\langle V_2 \rangle$, union of the minimal Roman labelings of the subgraphs acts as a Roman labeling of the whole graph. We define an orientation of a connected digraph based on a point. An orientation of a digraph is a vertex based orientation if there exists a vertex v such that all edges incident with v are oriented outward from v. All remaining edges are oriented conditionally step by step. We call v the vertex at level 0. Also we define, $V_0 = \{v\}$. All vertices in $N(v)$ are vertices at level 2.Correspondingly we define

$$V_1 = N[v] - V_0,$$
$$V_r = \bigcup_{v_i \in V_{r-1}} N[v_i] - (V_1 \cup V_2 \cup \cdots \cup V_{r-1}) \quad \text{for } r < 2.$$

If there is an edge between V_{r-1} and V_r, then it is oriented from V_{r-1} to V_r. An edge connecting two vertices in V_r is arbitrarily oriented. All the vertices in V_r are called vertices at r^{th} level. □

Lemma 2.4. *The point orientation of a digraph with respect to a vertex is unique excluding the directed edges at the same level.*

Proof. The procedure of orienting the edges starting from the given vertex is exact excluding the edges at a particular level. □

Corollary 2.5. *The point orientation of a tree with respect to a vertex is unique.*

Proof. Tree contains no cycle. So there is no edge connecting two vertices at a particular level. So the result follows from Theorem 2.4. □

Lemma 2.6. *If the point orientation of a digraph with respect to a vertex has s edges at the same level, then the total number of different orientations is 2s.*

Proof. Each edge at the same level can contribute two different orientations. Thus all together we have 2s orientations. Given a vertex based orientation of a graph and let n_i be the number of vertices in V_i which are adjacent to at least one vertex in V_{i+1} and m_i be the number of remaining vertices in the i^{th} level set. Then, $S(D) \leq 2(n_0 + n_2 + \ldots n_s) + m_0 + m_2 + \ldots m_s$, where s is the highest possible even index. □

Theorem 2.7. *Let D be a digraph and its underlying undirected graph be a complete bipartite graph. Also let (V_1, V_2) be a bipartition of G and $|V_1| = n_1$ and $|V_2| = n_2$. Then, if all edges are directed from V_1 to V_2, then $S(D) = n_1 + 1$.*

Proof. We set $f(v) = 2$ for one vertex in V_1 and $f(x) = 1$ for all other vertices in V_1. $f(y) = 0$ for all $y \in V_2$. The value of the function $f(V) = n_1 + 1$. There is no function having value less than $n_1 + 1$. So, $S(D) = n_1 + 1$. □

Theorem 2.8. *Let D be a digraph and its underlying undirected graph be a complete bipartite graph. Also let (V_1, V_2) be a bipartition of G and $|V_1| = n_1$ and $|V_2| = n_2$. If G is oriented starting with a vertex in V_1, then*

(1). $S(D) = 4$ *if* $|V_1| > 2$.

(2). If $|V_1| = 2$, then $S(D) = 3$.

(3). If $|V_1| = 1$, then $S(D) = 2$.

Proof. If $|V_1| > 2$, let v be the vertex that we take as the base of orientation. Put the labels as follows: $f(v) = 2$, $f(u) = 2$ for exactly one vertex $u \in V_2$ and put 0 to all other vertices. This function, f is a minimum Roman Labeling. So, $S(D) = 2$. Now, if $|V_1| = 2$, then we set $f(v) = 2$, $f(x) = 1$ where x is the other vertex in V_1 and $f(y) = 0$ for all remaining vertices. So $S(D) = 3$. If $|V_1| = 1$, this case is same as Theorem 2.7, when $n_1 = 1$. □

In the following discussion, we deal with deletion of arcs from a directed graph and the change it causes on the value of $S(D)$. Let f be a minimal Roman labeling of a digraph with the corresponding vertex set partition Vf_0, Vf_1 and Vf_2. If an arc is deleted from Vf_2 to Vf_0, then the Roman domination number of the graph D_0 may be affected. On the other hand, if we remove any other arc, the Roman number of the new graph is same as that of the initial graph. In this discussion graphs include both connected and disconnected graphs because removal of arcs makes the graph disconnected.

Proposition 2.9. *If D_0 is a new digraph obtained by deleting an arc other than one from Vf_2 to Vf_0, then $S(D) = S(D_0)$.*

Proof. Let f be a R function of D. It remains as a Roman dominating function in the new graph D_0. Suppose that D_0 has a Roman dominating function g with $g(V) < f(V)$, the function g can become Roman labeling of D as an addition of arc doesn't undermine the state of a function as a Roman labeling. This contradicts the minimality of the Roman labeling, f. As a direct consequence of the above result, we get a set of graphs having the same order and the same Roman number but varying number of arcs. If we denote the set of all digraphs having order n and Roman number r by $S_{n,r}$ and the set of all digraphs on n vertices by U_n, then, $S_{n,r} \sqsubseteq U_n$. □

3. Conclusion and Future Work

For a given orientation of a digraph, how many new digraphs can be obtained by deleting arcs and still have $S(D) = S(D_0)$? Can we establish any relationship between the number of digraphs and the structure of the digraph? Can we extend research in the same line, deleting vertices? These are few questions that may motivate us for further research.

Next we proceed to define a generalization of Roman labelings. Its motivation comes directly from the definition of Roman labeling and indirectly from the facility location problem. A Roman labeling associates function values either 0, 1 or 2 to the vertices in a graph. This is equivalent to the assignment of same number of persons at the vertex. If 2 is assigned to a vertex, it is assumed that the first person can take care of the home vertex and the second person keeps all vertices adjacent to the home vertex. An assignment is a Roman labeling, if all vertices are either directly guarded by a person at the vertex or indirectly guarded by a second person at a neighboring vertex, provided that the vertex is empty.

Ordinary Roman labeling is an assignment of single person to some vertices in a graph so that each person can guard the vertex where he is assigned and all the adjacent vertices. We can view it in a very general setting. We can say that $f(v_i)$ persons are assigned to the vertex v_i. The first person will guard the home vertex. The second person will guard all vertices in $N_1(v_i)$. In general, the i^{th} person will guard all vertices in $N_{i-1}(v)$, where $i = 1, 2, \ldots, r$.

An N-Roman Labeling is an arrangement of a set of r or lesser number of people at the vertices of a graph, such that all vertices are guarded by at least one person. The N–Roman number of a graph G is the minimum of the sum of the function values, taken over all N-Roman lebeling of G. We can allow a person to guard all the vertices in many adjacent levels. For example, suppose in an N-Roman lebeling, first person guards all the vertices in $N_0(v) \cup N_1(v) \cup N_2(v)$ and the second person guards only those vertices in $N_3(v)$, we call the Roman lebeling a $[\{0,1,2\},\{3\}]$-Roman lebeling. In this sense, the Roman lebeling is a $[\{0\},\{1\}]$-Roman lebeling.

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