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A New Class of Meromorphic Multivalent Functions Involving an Extended Linear Derivative Operator of Ruscheweyh

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Abstract: In this paper,we introduce and discuss a new subclass $A_p(\lambda, \alpha)$ of meromorphic multivalent functions in the punctured unit disk $U^* = \{z : z \epsilon C, 0 < |z| < 1\}$ defined by a certain extended linear derivative operator of Ruscheweyh. Coefficients inequality,distortion theorems,closure theorem for this class are obtained.We also obtain radius of starlikeness and radius of convexity for this class.

MSC: 30C45.

Keywords: Meromorphic Multivalent Functions, Ruscheweyh Derivatives, Meromorphic Starlike Functions.

1. Introduction

Let A_p denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{k=p+1}^{\infty} a_{k-p} z^{k-p}; \quad (p = 1, 2, 3.....), \tag{1}$$

which are analytic and p-valent in the punctured unit disk $U^* = \{z \in C : 0 < |z| < 1\}$. For $f(z) \in A_p$ given by (1) and $g(z) \in A_p$ given by

$$g(z) = z^{-p} + \sum_{k=p+1}^{\infty} b_{k-p} z^{k-p}; \quad (p = 1, 2, 3.....). \tag{2}$$

Some classes related to meromorphic functions are studied by Morga [2], Xu and Yang [6], Raina and srivastava [3] etc. The Hadamard product (or convolution product) of f and g is defined by

$$(f * g)(z) = z^{-p} + \sum_{k=p+1}^{\infty} a_{k-p} b_{k-p} z^{k-p} \quad = (g * f)(z). \tag{3}$$

The extended linear derivative operator of Ruscheweyh type for the functions of the class A_p

$$D_*^{\lambda,p} : A_p \to A_p,$$

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is defined by

$$D_*^{\lambda,p} f(z) = \frac{1}{z^p(1-z)^{\lambda+1}} * f(z); \qquad (\lambda > -1 : f \in A_p). \tag{4}$$

In terms of binomial coefficients, (4) can be written as

$$D_*^{\lambda,p} f(z) = z^{-p} + \sum_{k=p+1}^{\infty} \begin{pmatrix} \lambda+k \\ k \end{pmatrix} a_{k-p} z^{k-p} \quad (\lambda > -1 : f \in A_p). \tag{5}$$

In particular when $\lambda = n (n \in N)$, it is easily observed from (4) and (5) that

$$D_*^{\lambda,p} f(z) = \frac{z^{-p}(z^{n+p} f(z))^{(n)}}{n!} \qquad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{6}$$

The Equation (5) of linear operator $D_*^{\lambda,p}$ is motivated by Ruscheweyh operator D^λ [5]. Some linear operators analogous to $D_*^{\lambda,p}$ are considered by Raina and Srivastava [4] and Liu and Srivastava [1]. Using the operator $D_*^{\lambda,p}(\lambda > -1)$, we now introduce a new subclass $A_p(\lambda, \alpha)$ of meromorphically p-valent analytic functions defined as follows:

Definition 1.1. *A function $f(z) \in A_p$ is said to be a member of the class $A_p(\lambda, \alpha)$ if and only if*

$$\left| \frac{z(D_*^{\lambda,p} f(z))''}{(D_*^{\lambda,p} f(z))'} + p + 1 \right| < p - \alpha; \qquad (z \in D, p \in N, \lambda > -1, 0 \le \alpha < p). \tag{7}$$

The main aim of this paper to obtain some properties as coefficients inequality,distortion theorem and closure theorem for the functions of this class. Radii of starlikeness and convexity are also obtained for this class.

2. Coefficients Estimates

Theorem 2.1. *A function $f(z) \in A_p$ is in the class $A_p(\lambda, \alpha)$ if and only if*

$$\sum_{k=p+1}^{\infty} \begin{pmatrix} \lambda+k \\ k \end{pmatrix} (k-p)(k+p-\alpha) a_{k-p} \le p(p-\alpha). \tag{8}$$

The result is sharp.

Proof. Assuming that the inequality (8) holds true. Then from (7), we find that

$$\begin{aligned}
&\left| z(D_*^{\lambda,p} f(z))'' + (p+1)(D_*^{\lambda,p} f(z))' \right| - (p-\alpha) \left| (D_*^{\lambda,p} f(z))' \right| \\
&= \left| \begin{matrix} \left\{ p(p+1)z^{-p-1} + \sum_{k=p+1}^{\infty} \begin{pmatrix} \lambda+k \\ k \end{pmatrix} (k-p)(k-p-1) a_{k-p} z^{k-p-1} \right\} \\ +(p+1) \left\{ -pz^{-p-1} + \sum_{k=p+1}^{\infty} \begin{pmatrix} \lambda+k \\ k \end{pmatrix} (k-p) a_{k-p} z^{k-p-1} \right\} \end{matrix} \right| \\
&= \left| \sum_{k=p+1}^{\infty} \begin{pmatrix} \lambda+k \\ k \end{pmatrix} k(k-p) a_{k-p} z^{k-p-1} \right| \\
&\quad - (p-\alpha) \left| \left\{ pz^{-p-1} - \sum_{k=p+1}^{\infty} \begin{pmatrix} \lambda+k \\ k \end{pmatrix} (k-p) a_{k-p} z^{k-p-1} \right\} \right|
\end{aligned}$$

$$\leq \sum_{k=p+1}^{\infty}\begin{pmatrix}\lambda+k\\ k\end{pmatrix}k(k-p)a_{k-p}z^{k-p-1}-p(p-\alpha)$$

$$+(p-\alpha)\sum_{k=p+1}^{\infty}\begin{pmatrix}\lambda+k\\ k\end{pmatrix}(k-p)a_{k-p}$$

$$\leq \sum_{k=p+1}^{\infty}\begin{pmatrix}\lambda+k\\ k\end{pmatrix}(k-p)(k+p-\alpha)a_{k-p}-p(p-\alpha)\leq 0.$$

Hence $f(z)\in A_p(\lambda,\alpha)$.

Conversely, suppose that $f(z)\in A_p(\lambda,\alpha)$. Then from (7), we have

$$\left|\frac{z(D_*^{\lambda,p}f(z))''}{(D_*^{\lambda,p}f(z))'}+p+1\right| = \left|\frac{\sum_{k=p+1}^{\infty}\begin{pmatrix}\lambda+k\\ k\end{pmatrix}k(k-p)a_{k-p}z^{k-p-1}}{-pz^{-p-1}+\sum_{k=p+1}^{\infty}\begin{pmatrix}\lambda+k\\ k\end{pmatrix}(k-p)a_{k-p}z^{k-p-1}}\right|$$

$$=\left|\frac{\sum_{k=p+1}^{\infty}\begin{pmatrix}\lambda+k\\ k\end{pmatrix}k(k-p)a_{k-p}z^{k-p-1}}{pz^{-p-1}-\sum_{k=p+1}^{\infty}\begin{pmatrix}\lambda+k\\ k\end{pmatrix}(k-p)a_{k-p}z^{k-p-1}}\right|<(p-\alpha).$$

or

$$\sum_{k=p+1}^{\infty}\begin{pmatrix}\lambda+k\\ k\end{pmatrix}k(k-p)a_{k-p}<p(p-\alpha)-(p-\alpha)\sum_{k=p+1}^{\infty}\begin{pmatrix}\lambda+k\\ k\end{pmatrix}(k-p)a_{k-p}$$

or

$$\sum_{k=p+1}^{\infty}\begin{pmatrix}\lambda+k\\ k\end{pmatrix}(k-p)(k+p-\alpha)a_{k-p}\leq p(p-\alpha).$$

The result is sharp.The extremal function is given by

$$f(z)=z^{-p}+\frac{p(p-\alpha)}{\begin{pmatrix}\lambda+k\\ k\end{pmatrix}(k-p)(k+p-\alpha)}z^{k-p}\ (k>p) \tag{9}$$

□

3. Distortion Theorem

Theorem 3.1. *If* $f(z)\in A_p(\lambda,\alpha)$*, then for* $0<|z|=r<1$,

$$r^{-p}-\frac{p(p-\alpha)}{\begin{pmatrix}\lambda+p+1\\ p+1\end{pmatrix}(1+2p-\alpha)}r\leq|f(z)|$$

$$\leq r^{-p}+\frac{p(p-\alpha)}{\begin{pmatrix}\lambda+p+1\\ p+1\end{pmatrix}(1+2p-\alpha)}r \tag{10}$$

Proof. Since $f(z) \in A_p(\lambda, \alpha)$, then from (8) it follows that

$$\begin{pmatrix} \lambda+p+1 \\ p+1 \end{pmatrix}(1+2p-\alpha)\sum_{k=p+1}^{\infty} a_{k-p} \leq \sum_{k=p+1}^{\infty}\begin{pmatrix} \lambda+k \\ k \end{pmatrix}(k-p)(k+p-\alpha)a_{k-p} \leq p(p-\alpha)$$

which yields

$$\sum_{k=p+1}^{\infty} a_{k-p} \leq \frac{p(p-\alpha)}{\begin{pmatrix} \lambda+p+1 \\ p+1 \end{pmatrix}(1+2p-\alpha)} \tag{11}$$

Then

$$|f(z)| \geq |z|^{-p} - \sum_{k=p+1}^{\infty} |a_{k-p}|\,|z|^{k-p} \geq |z|^{-p} - |z|\sum_{k=p+1}^{\infty} |a_{k-p}|$$

or

$$|f(z)| \geq r^{-p} - \frac{p(p-\alpha)}{\begin{pmatrix} \lambda+p+1 \\ p+1 \end{pmatrix}(1+2p-\alpha)}r \tag{12}$$

and

$$|f(z)| \leq |z|^{-p} + \sum_{k=p+1}^{\infty} |a_{k-p}|\,|z|^{k-p} \leq |z|^{-p} + |z|\sum_{k=p+1}^{\infty} |a_{k-p}|$$

or

$$|f(z)| \leq r^{-p} + \frac{p(p-\alpha)}{\begin{pmatrix} \lambda+p+1 \\ p+1 \end{pmatrix}(1+2p-\alpha)}r \tag{13}$$

Inequalities (11) and (12) provide (**??**). The above bounds are sharp. Equalities are attained for the function

$$|f(z)| = z^{-p} + \frac{p(p-\alpha)}{\begin{pmatrix} \lambda+p+1 \\ p+1 \end{pmatrix}(1+2p-\alpha)}z, \quad z = \pm r. \tag{14}$$

□

Theorem 3.2. *If* $f(z) \in A_p(\lambda, \alpha)$, *then for* $0 < |z| = r < 1$,

$$pr^{-p-1} - \frac{p(p-\alpha)}{\begin{pmatrix} \lambda+p+1 \\ p+1 \end{pmatrix}(1+2p-\alpha)} \leq |f'(z)| \leq pr^{-p-1} + \frac{p(p-\alpha)}{\begin{pmatrix} \lambda+p+1 \\ p+1 \end{pmatrix}(1+2p-\alpha)} \tag{15}$$

Proof. Since $f(z) \in A_p(\lambda, \alpha)$, then from (8) it follows that

$$\begin{pmatrix} \lambda+p+1 \\ p+1 \end{pmatrix}(1+2p-\alpha)\sum_{k=p+1}^{\infty} (k-p)a_{k-p} \leq \sum_{k=p+1}^{\infty}\begin{pmatrix} \lambda+k \\ k \end{pmatrix}(k-p)(k+p-\alpha)a_{k-p} \leq p(p-\alpha),$$

which yields

$$\sum_{k=p+1}^{\infty}(k-p)a_{k-p} \leq \frac{p(p-\alpha)}{\begin{pmatrix}\lambda+p+1\\ p+1\end{pmatrix}(1+2p-\alpha)} \tag{16}$$

Then

$$\left|f'(z)\right| \geq p\left|z\right|^{-p-1} - \sum_{k=p+1}^{\infty}(k-p)\left|a_{k-p}\right|\left|z\right|^{k-p-1} \geq p\left|z\right|^{-p-1} - \sum_{k=p+1}^{\infty}(k-p)\left|a_{k-p}\right|$$

or

$$\left|f'(z)\right| \geq pr^{-p-1} - \frac{p(p-\alpha)}{\begin{pmatrix}\lambda+p+1\\ p+1\end{pmatrix}(1+2p-\alpha)} \tag{17}$$

and

$$\left|f'(z)\right| \leq p\left|z\right|^{-p-1} + \sum_{k=p+1}^{\infty}(k-p)\left|a_{k-p}\right|\left|z\right|^{k-p-1} \leq p\left|z\right|^{-p-1} + \sum_{k=p+1}^{\infty}(k-p)\left|a_{k-p}\right|$$

or

$$\left|f'(z)\right| \leq pr^{-p-1} + \frac{p(p-\alpha)}{\begin{pmatrix}\lambda+p+1\\ p+1\end{pmatrix}(1+2p-\alpha)} \tag{18}$$

Inequalities (15) and (16) provide (14). The above bounds are sharp. Equalities are attained for the function given by (13). □

4. Closure Theorem

Theorem 4.1. *Let*

$$f_p(z) = z^{-p} \quad \text{and } f_k(z) = z^{-p} + \frac{p(p-\alpha)}{\begin{pmatrix}\lambda+k\\ k\end{pmatrix}(k-p)(k+p-\alpha)}z^{k-p} \quad (k \geq p+1). \tag{19}$$

Then $f(z) \in A_p(\lambda,\alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=p}^{\infty}\mu_k f_k(z) \text{ where } \mu_k \geq 0 \text{ and } \sum_{k=p}^{\infty}\mu_k = 1. \tag{20}$$

Proof. Suppose that $f(z)$ can be expressed in the form (20). Then

$$\begin{aligned} f(z) &= \sum_{k=p}^{\infty}\mu_k f_k(z) = \mu_p f_p(z) + \sum_{k=p+1}^{\infty}\mu_k f_k(z) \\ &= z^{-p} + \sum_{k=p+1}^{\infty}\frac{p(p-\alpha)\mu_k}{\begin{pmatrix}\lambda+k\\ k\end{pmatrix}(k-p)(k+p-\alpha)}z^{k-p} \end{aligned}$$

Since

$$\sum_{k=p+1}^{\infty}\begin{pmatrix}\lambda+k\\ k\end{pmatrix}(k-p)(k+p-\alpha) \times \frac{p(p-\alpha)\mu_k}{\begin{pmatrix}\lambda+k\\ k\end{pmatrix}(k-p)(k+p-\alpha)} = \sum_{k=p+1}^{\infty}p(p-\alpha)\mu_k$$

$$= p(p-\alpha)\sum_{k=p+1}^{\infty}\mu_k = p(p-\alpha)(1-\mu_p) \leq p(p-\alpha)$$

So by Theorem 2.1, we have $f(z) \in A_p(\lambda, \alpha)$.

Conversely, let $f(z) \in A_p(\lambda, \alpha)$. Since

$$a_{k-p} \leq \frac{p(p-\alpha)}{\begin{pmatrix} \lambda+k \\ k \end{pmatrix}(k-p)(k+p-\alpha)} \quad (k \geq p+1)$$

Setting

$$\mu_k = \frac{\begin{pmatrix} \lambda+k \\ k \end{pmatrix}(k-p)(k+p-\alpha)}{p(p-\alpha)} \quad (k \geq p+1)$$

and

$$\mu_p = 1 - \sum_{k=p+1}^{\infty}\mu_k$$

It follows that

$$f(z) = \sum_{k=p}^{\infty}\mu_k f_k(z)$$

□

5. Radius of Starlikeness

Theorem 5.1. *Let the function f(z) defined by (1) be in the class $A_p(\lambda,\alpha)$. Then f is meromorphically p-valent starlike in the disk $|z| < r_1$, where*

$$r_1 = r_1(p,\lambda,\alpha) = \overset{\inf}{k > p}\left(\frac{\begin{pmatrix} \lambda+k \\ k \end{pmatrix}(k+p-\alpha)}{(p-\alpha)}\right)^{\frac{1}{k}} \tag{21}$$

Proof. It is enough to highlight that

$$\left|\frac{zf'(z)}{f(z)} + p\right| \leq p, |z| < r_1.$$

Thus we have

$$\left|\frac{zf'(z)}{f(z)} + p\right| = \left|\frac{\sum_{k=p+1}^{\infty} k a_{k-p} z^{k-p}}{z^{-p} + \sum_{k=p+1}^{\infty} a_{k-p} z^{k-p}}\right| \leq p.$$

or

$$\sum_{k=p+1}^{\infty}\left(\frac{k-p}{p}\right)a_{k-p}\left|z^k\right| \leq 1.$$

In view of Theorem 2.1, the last inequality is true if

$$\left(\frac{k-p}{p}\right)\left|z^k\right| \leq \left(\frac{\begin{pmatrix} \lambda+k \\ k \end{pmatrix}(k+p-\alpha)}{(p-\alpha)}\right) \quad (k \geq p+1)$$

which when solved for $|z|$ yields (21). □

6. Radius of Convexity

Theorem 6.1. *Let the function f(z) defined by* (1) *be in the class* $A_p(\lambda, \alpha)$. *Then f is meromorphically p-valent convex in the disk* $|z| < r_2$, *where*

$$r_2 = r_2(p, \lambda, \alpha) = k \overset{\inf}{>} p \left(\frac{\begin{pmatrix} \lambda + k \\ k \end{pmatrix} (k + p - \alpha)}{(k+p)(p-\alpha)} \right)^{\frac{1}{k}} \tag{22}$$

Proof. Upon noting the fact that $f(z)$ is convex if and only if $zf'(z)$ is starlike, the Theorem 6.1 follows. □

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