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Discrete q-heat Equation Model with Shift Values

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Abstract: In this paper, we investigate the generalized q-partial difference operator and propose a model of it in discrete q-heat equation with shift values. The diffusion of heat is studied in dimensions up to three and several solutions are postulated for the same. Through numerical simulations using MATLAB, solutions are validated and applications are derived.

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1. Introduction

In 2011, M.Maria Susai Manuel, et.al, [9], extended the definition of Δ_α to $\underset{\alpha(\ell)}{\Delta}$ defined as $\underset{\alpha(\ell)}{\Delta} v(k) = v(k+\ell) - \alpha v(k)$ for the real valued function v(k), $\ell > 0$. In 2014, the authors in [5], have applied q-difference operator defined as $\Delta_q v(k) = v(qk) - v(k)$ and obtained finite series formula for logarithmic function. The difference operator $\underset{k(q)}{\Delta}$ with variable coefficients defined as

$$\underset{k(q)}{\Delta} v(k) = v(kq) - kv(k)$$

is established in [5]. Here, we extend the q-difference operator Δ_q to a partial difference operator with several parameters as

$$\underset{(q)}{\Delta} v(k) = v(k_1q_1, k_2q_2, ..., k_nq_n) - v(k_1, k_2, ..., k_n). \tag{1}$$

This operator $\underset{(q)}{\Delta}$ becomes generalized partial difference operator if some $q_i = 0$. The equations involving $\underset{(q)}{\Delta}$ with atleast one $q_i = 0$ is called generalized partial difference equation. A linear generalized partial difference equation is of the form,

$$\underset{(q)}{\Delta} v(k) = u(k), \tag{2}$$

where $\underset{(q)}{\Delta}$ is as given in (1), $q_i = 0$ for some i and $u(k) : \mathbb{R}^n \to \mathbb{R}$ is a given function. A function $v(k) : \mathbb{R}^n \to \mathbb{R}$ satisfying (2) is called a solution of the equation (2). The equation (2) has a numerical solution of the form,

$$v(k) - v\left(\frac{k}{q^m}\right) = \sum_{r=1}^{m} u\left(\frac{k}{q^r}\right), \tag{3}$$

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where $\dfrac{k}{q^r} = \left(\dfrac{k_1}{{q_1}^r}, \dfrac{k_2}{{q_2}^r}, ..., \dfrac{k_m}{{q_m}^r}\right)$, m is any positive integer. Relation (3) is the basic inverse principle with respect to $\underset{(q)}{\Delta}$ [5, 9, 11]. For example, the basic inverse principle with respect to $\underset{(0,q_2)}{\Delta}$ is given by

$$v(k_1, k_2) - v\left(k_1, \frac{k_2}{{q_2}^m}\right) = \sum_{r=1}^{m} u\left(k_1, \frac{k_2}{{q_2}^r}\right), \tag{4}$$

where $v(k_1, k_2) = \underset{(0,q_2)}{\Delta^{-1}} u(k_1, k_2)$. From the theory of generalized difference equation, we have two types of solutions to (2), namely closed form and summation form solutions [5, 9, 11]. Similarly, the partial difference equation (2) has two types of solutions. Here we form partial difference equation for the heat flow and apply Fourier cooling law and obtain solution of heat equation with several variables and shift values.

2. Heat Equation for Medium, when γ is Constant

Consider homogeneous diffusion medium in $\Re^3$. Let γ be heat diffusion constant and $v(k_1, k_2, k_3, k_4, k_5)$ be the temperature at position (k_1, k_2, k_3), at time k_4 with density (or pressure) k_5. The proportional amount of heat flows from left to right at $(k_1, k_2, k_3, k_4, k_5)$ is $\underset{(\frac{1}{q_1},0,0)}{\Delta} v(k)$, right to left $\underset{(q_1,0,0)}{\Delta} v(k)$, top to bottom $\underset{(0,q_2,0)}{\Delta} v(k)$, bottom to top $\underset{(0,\frac{1}{q_2},0)}{\Delta} v(k)$, rear to back $\underset{(0,0,q_3)}{\Delta} v(k)$, back to rear $\underset{(0,0,\frac{1}{q_3})}{\Delta} v(k)$. By the Fourier law of cooling, the heat equation for medium in $\Re^3$ is

$$\underset{(q_4,q_5)}{\Delta} v(k) = \gamma \underset{(q^{\pm}_{(1,2,3)})}{\Delta} v(k), \tag{5}$$

where $\underset{(q^{\pm}(1,2,3))}{\Delta} = \underset{(q_1)}{\Delta} + \underset{(\frac{1}{q_1})}{\Delta} + \underset{(q_2)}{\Delta} + \underset{(\frac{1}{q_2})}{\Delta} + \underset{(q_3)}{\Delta} + \underset{(\frac{1}{q_3})}{\Delta}$ and $k = (k_1, k_2, k_3, k_4, k_5)$.

Theorem 2.1. *Assume that $v(k_1, k_2, k_3, \frac{k_4}{{q_4}^m}, \frac{k_5}{{q_5}^m})$ and the partial differences $\underset{(q^{\pm}_{(1,2,3)})}{\Delta} v(k) = \underset{(q^{\pm}_{(1,2,3)})}{u}(k)$ are known functions, then the heat equation (5) has a solution,*

$$v(k) = v(k_1, k_2, k_3, \frac{k_4}{{q_4}^m}, \frac{k_5}{{q_5}^m}) + \gamma \sum_{r=1}^{m} \underset{(q^{\pm}_{(1,2,3)})}{u}(k_1, k_2, k_3, \frac{k_4}{{q_4}^r}, \frac{k_5}{{q_5}^r}). \tag{6}$$

Proof. Taking $\underset{(q^{\pm}_{(1,2,3)})}{\Delta} v(k) = \underset{(q^{\pm}_{(1,2,3)})}{u}(k)$ in (5), we get

$$v(k) = \gamma \underset{(q_4,q_5)}{\Delta^{-1}} \underset{(q^{\pm}_{(1,2,3)})}{u}(k). \tag{7}$$

The proof follows by applying inverse principle (4) in (7). □

In the following theorem we use the following notations:

$$v(k_{(1,2,3)}(q^{\pm}_{(1,2,3)}), *, *) = v(k_1q_1, k_2, k_3, *, *) + v(\frac{k_1}{q_1}, k_2, k_3, *, *) + v(k_1, k_2q_2, k_3, *, *) + v(k_1, \frac{k_2}{q_2}, k_3, *, *)$$
$$+ v(k_1, k_2, k_3q_3, *, *) + v(k_1, k_2, \frac{k_3}{q_3}, *, *).$$
$$v(*, k_{(2,3),*,*}(*, q^{\pm}_{(2,3)}), *, *) = v(*, k_2q_2, k_3, *, *) + v(*, \frac{k_2}{q_2}, k_3, *, *) + v(*, k_2, k_3q_3, *, *) + v(*, k_2, \frac{k_3}{q_3}, *, *).$$

Theorem 2.2. *If $v(k)$ is a solution of the equation (5) and m is a positive integer then the following relations are equivalent.*

(a) $v(k) = (1-6\gamma)^m v(k_1, k_2, k_3, \frac{k_4}{q_4{}^m}, \frac{k_5}{q_5{}^m})$

$$+\sum_{r=0}^{m-1} \gamma(1-6\gamma)^r \Big[v(k_{(1,2,3)}(q^{\pm}_{(1,2,3)}), \frac{k_4}{q_4^{(r+1)}}, \frac{k_5}{q_5^{(r+1)}})\Big], \tag{8}$$

(b) $v(k) = \frac{1}{(1-6\gamma)^m} v(k_1, k_2, k_3, k_4 q_4{}^m, k_5 q_5{}^m)$

$$-\sum_{r=1}^{m} \frac{\gamma}{(1-6\gamma)^r} \Big[v(k_{(1,2,3)}(q^{\pm}_{(1,2,3)}), k_4 q_4{}^{(r-1)}, k_5 q_5{}^{(r-1)})\Big], \tag{9}$$

(c) $v(k) = \frac{1}{\gamma^m} v(\frac{k_1}{q_1{}^m}, k_2, k_3, k_4 q_4{}^m, k_5 q_5{}^m) - \sum_{r=1}^{m} \frac{1-6\gamma}{\gamma^r} v(\frac{k_1}{q_1{}^r}, k_2, k_3, k_4 q_4{}^{(r-1)}, k_5 q_5{}^{(r-1)})$

$$-\sum_{r=0}^{m-1} \frac{1}{\gamma^r} v(\frac{k_1}{q_1{}^{(r+1)}}, k_{(2,3)}(q^{\pm}_{(2,3)}), k_4 q_4{}^r, k_5 q_5{}^r), \tag{10}$$

(d) $v(k) = \frac{1}{\gamma^m} v(k_1 q_1{}^m, k_2, k_3, k_4 q_4{}^m, k_5 q_5{}^m) - \sum_{r=1}^{m} \frac{1-6\gamma}{\gamma^r} v(k_1 q_1{}^r, k_2, k_3, k_4 q_4{}^{(r-1)}, k_5 q_5{}^{(r-1)})$

$$-\sum_{r=0}^{m-1} \frac{1}{\gamma^r} v(k_1 q_1{}^{(r+1)}, k_{(2,3)}(q^{\pm}_{(2,3)}), k_4 q_4{}^r, k_5 q_5{}^r). \tag{11}$$

Proof. From (5) and (1), we arrive

(i). $v(k) = (1-6\gamma) v\left(k_1, k_2, k_3, \frac{k_4}{q_4}, \frac{k_5}{q_5}\right) + \gamma\left[v(k_{(1,2,3)}(q^{\pm}_{(1,2,3)}), \frac{k_4}{q_4}, \frac{k_5}{q_5})\right]$,

(ii). $v(k) = \frac{1}{(1-6\gamma)} v(k_1, k_2, k_3, k_4 q_4, k_5 q_5) - \frac{\gamma}{(1-6\gamma)}\left[v(k_{(1,2,3)}(q^{\pm}_{(1,2,3)}), k_4, k_5)\right]$,

(iii). $v(k) = \frac{1}{\gamma} v\left(\frac{k_1}{q_1}, k_2, k_3, k_4 q_4, k_5 q_5\right) - \frac{1-6\gamma}{\gamma} v\left(\frac{k_1}{q_1}, k_2, k_3, k_4, k_5\right) - v\left(\frac{k_1}{q_1{}^2}, k_2, k_3, k_4, k_5\right) - v\left(\frac{k_1}{q_1}, k_{(2,3)}(q^{\pm}_{(2,3)}), k_4, k_5\right)$

and

(iv). $v(k) = \frac{1}{\gamma} v(k_1 q_1, k_2, k_3, k_4 q_4, k_5 q_5) - \frac{1-6\gamma}{\gamma} v(k_1 q_1, k_2, k_3, k_4, k_5) - v(k_1 q_1{}^2, k_2, k_3, k_4, k_5) - v(k_1 q_1, k_{(2,3)}(q^{\pm}_{(2,3)}), k_4, k_5)$.

Now the proof of (a), (b), (c), (d) follows by replacing k_4 and k_5 by $\frac{k_4}{q_4}, \frac{k_4}{q_4^2}, ..., \frac{k_4}{q_4^m}$ and $\frac{k_5}{q_5}, \frac{k_5}{q_5^2}, ..., \frac{k_5}{q_5^m}$, k_4 and k_5 by $k_4 q_4, k_4 q_4^2, ..., k_4 q_4^m$ and $k_5 q_5, k_5 q_5^2, ..., k_5 q_5^m$, k_1,k_4 and k_5 by $\frac{k_4}{q_4}, \frac{k_1}{q_1^2}, ..., \frac{k_1}{q_1^m}$, $k_4 q_4, k_4 q_4^2, ..., k_4 q_4^m$ and $k_5 q_5, k_5 q_5^2, ..., k_5 q_5^m$, k_1, k_4 and k_5 by $k_1 q_1, k_1 q_1^2, ..., k_1 q_1^m$, $k_4 q_4, k_4 q_4^2, ..., k_4 q_4^m$ and $k_5 q_5, k_5 q_5^2, ..., k_5 q_5^m$ in (i), (ii), (iii) and (iv) respectively. □

Example 2.3. *The following example shows that the diffusion of medium in three dimensional system can be identified if the solution $v(k_1, k_2, k_3, k_4, k_5)$ of (5) is known and vice versa. Suppose that $v(k_1, k_2, k_3, k_4, k_5) = k_1 k_2 k_3 k_4 k_5$ is a closed form solution of (5), then we have the relation*

$$\underset{(q_4,q_5)}{\Delta} k_1 k_2 k_3 k_4 k_5 = \gamma\left[\underset{(q^{\pm}_{(1,2,3)})}{\Delta} k_1 k_2 k_3 k_4 k_5\right],$$

which yields

$$k_1 k_2 k_3 k_4 k_5 (q_4 q_5 - 1) = \gamma\left[k_1 k_2 k_3 k_4 k_5 (q_1 + \frac{1}{q_1} + q_2 + \frac{1}{q_2} + q_3 + \frac{1}{q_3} - 6)\right].$$

Cancelling $k_1 k_2 k_3 k_4 k_5$ on both sides derives

$$\gamma = \frac{q_4 q_5 - 1}{q_1 + \frac{1}{q_1} + q_2 + \frac{1}{q_2} + q_3 + \frac{1}{q_3} - 6}. \tag{12}$$

For numerical verification, if we assume that $k_1 = 1$, $k_2 = 2$, $k_3 = 3$, $k_4 = 4$, $k_5 = 5$, $q_1 = 1$, $q_2 = 2$, $q_3 = 3$, $q_4 = 4$, $q_5 = 5$, $m = 2$ *then* $v(k_1, k_2, k_3, k_4, k_5) = 120$, $\gamma = \dfrac{4 * 5 - 1}{1 + \frac{1}{1} + 2 + \frac{1}{2} + 3 + \frac{1}{3} - 6}$, *LHS and RHS of (a) of Theorem* (2.2) *is given below respectively.* $120 = 1122.964462 + 487.0909103 - 1490.055371 \Rightarrow 120 = 120$. *LHS and RHS of (b), (c), (d) of Theorem* (2.2) *are as similar as above (a). For matlab coding, if we assume that* $k_1 = 1, k_2 = 2, k_3 = 3, k_4 = 4, k_5 = 5, q_1 = 1$, $q_2 = 2, q_3 = 3, q_4 = 4, q_5 = 5, m = 5$ *then* $1. * 2. * 3. * 4. * 5 = (1 - 6. * (10.36363636)). \wedge (5). * (6. * (4./((4). \wedge 5)). * (5./((5). \wedge 5))) + symsum((10.36363636). * (1 - 6. * (10.36363636)). \wedge r. * ((6. * (4./(4. \wedge (r+1))). * (5./(5. \wedge (r+1)))) + (6. * (4./(4. \wedge (r+1))). * (5./(5. \wedge (r+1)))) + (12. * (4./(4. \wedge (r+1))). * (5./(5. \wedge (r+1)))) + (3. * (4./(4. \wedge (r+1))). * (5./(5. \wedge (r+1)))) + (18. * (4./(4. \wedge (r+1))). * (5./(5. \wedge (r+1)))) + (2. * (4./(4. \wedge (r+1))). * (5./(5. \wedge (r+1))))), r, 0, 4)$

3. conclusion

The study of partial difference operator has numerous applications in discrete fields and heat equation is one of its kind. The nature of propagation of heat through materials of dimensions(up to three) can be postulated. The core theorem (2.2) provide the possibility of predicting the temperature either for the past or the future after getting the know the temperature at few finite points at present time. It also gives us the right knowledge about the choice of the material that has to be considered for propagation of heat.

References

[1] M.J.Ablowitz and J.F.Ladik, *On the solution of a class of nonlinear partial difference equations*, Studies Appl. Math., 57(1977), 1-12.

[2] G.Britto Antony Xavier, T.G.Gerly and H.Nasira Begum, *Finite Series of Polynomials and Polynomial Factorials arising from Generalized q-Difference operator*, Far East Journal of Mathematical Sciences, 94(1)(2014), 47-63.

[3] N.R.O.Bastos, R.A.C.Ferreira and D.F.M.Torres, *Discrete-Time Fractional Variational Problems*, Signal Processing, 91(3)(2011), 513-524.

[4] R.A.C.Ferreira and D.F.M.Torres, *Fractional h-difference equations arising from the calculus of variations*, Applicable Analysis and Discrete Mathematics, 5(1)(2011), 110-121.

[5] G.Britto Antony Xavier, T.G.Gerly and H.Nasira Begum, *Finite Series of Polynomials and Polynomial Factorials arising from Generalized q-Difference operator*, Far East Journal of Mathematical Sciences, 94(1)(2014), 47-63.

[6] S.S.Cheng, *Sturmian comparison theorems for three term recurrence equations*, J. Math. Anal. Appl., 111(1985), 464-474.

[7] Jerzy Popenda and Blazej Szmanda, *On the Oscillation of Solutions of Certain Difference Equations*, Demonstratio Mathematica, XVII(1)(1984), 153-164.

[8] T.Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley-Interscience, New York, NY, USA, (2001).

[9] M.Maria Susai Manuel, V.Chandrasekar and G.Britto Antony Xavier, *Solutions and Applications of Certain Class of α-Difference Equations*, International Journal of Applied Mathematics, 24(6)(2011), 943-954.

[10] K.S.Miller and B.Ross, *Fractional Difference Calculus in Univalent Functions*, Horwood, Chichester, UK(1989), 139-152.

[11] M.Susai Manuel, G.Britto Antony Xavier, V.Chandrasekar and R.Pugalarasu, *Theory and application of the Generalized Difference Operator of the n^{th} kind(Part I)*, Demonstratio Mathematica, 45(1)(2012), 95-106.