Generalized Odd-Even Sum Labeling and Some α–Odd-Even Sum Graphs

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Abstract: A \((p,q)\) graph \(G\) is said to be an \(\alpha\)-odd-even sum graph if it admits an odd-even sum labeling \(f\) defined by Monika and Murugan \cite{9} by adding an addition condition that there is a positive integer \(k(0 < k < 2q - 1)\) such that for every edge \(uv \in E(G)\), \(\min\{f(u), f(v)\} < k < \max\{f(u), f(v)\}\). In this paper, we study \(\alpha\)-odd-even sum labeling of \(C_n(n \equiv 0 \pmod 4)\), \(S(x_1, x_2, \ldots, x_n)\), \(K_{m,n}\) \((m, n \geq 2)\), \(P_n \square P_m\) \((m, n \geq 2)\), step grid graph \(S_{tn}(n \geq 3)\) and splitting graph of \(K_{1,n}\).

MSC: 05C78.

Keywords: \(\alpha\)-odd-even sum labeling, Grid graph, Step grid graph, Splitting graph.

\section{Introduction}

\(\alpha\)-labeling and \(\beta\)-valuation (graceful labeling) was introduced by Rosa \cite{11} in 1967. Acharya and Gill\cite{1} have investigated \(\alpha\)-labeling for the grid graph \(P_n \square P_m\). Makadia and Kaneria \cite{7} introduced step grid graph \(S_{tn}\) and proved that it is graceful \((n \geq 3)\). Harary \cite{5} introduced a notation of sum graph. A \((p,q)\) graph \(G\) is said to be an odd-even sum graph if it admits an injective function \(f : V(G) \rightarrow \{\pm 1, \pm 3, \pm 5, \ldots, \pm (2q - 1)\}\) such that its edge induced function \(f^*: E(G) \rightarrow \{2, 4, 6, \ldots, 2q\}\) define by \(f^*(uv) = f(u) + f(v), \forall uv \in E(G)\) is bijective, which introduced by Monika and Murugan \cite{9}. These results motivated us and we introduced here a new concept called \(\alpha\)-odd-even sum labeling which is an odd even sum labeling for a graph \(G\) and one additional condition that there is a positive integer \(k(0 < k < 2q - 1)\) such that \(\min\{f(u), f(v)\} < k < \max\{f(u), f(v)\}\), \(\forall uv \in E(G)\). Every \(\alpha\)-odd-even sum graph is always a bipartite graph.

\section{Main Results}

\textbf{Theorem 2.1.} Every cycle \(C_n(n \equiv 0 \pmod 4)\) is an \(\alpha\)-odd-even sum graph.

\textit{Proof.} Let \(V(C_n) = \{v_1, v_2, v_3, \ldots, v_n\}\) and \(E(C_n) = \{v_iv_{i+1}/1 \leq i < n\} \cup \{v_nv_1\}\). It is obvious that \(p = q = n\) for \(C_n\).
Define \( f : V(G) \rightarrow \{\pm 1, \pm 3, \pm 5, \ldots, \pm (2q - 1)\} \) as follows.

\[
    f(x) = \begin{cases} 
        3 - i, & \forall i = 2, 4, 6, \ldots, n; \\
        2q - 1, & \forall i = 1, 3, \ldots, \frac{n}{2} - 1; \\
        2q - (i + 2), & \forall i = \frac{n}{2} + 1, \frac{n}{2} + 3, \ldots, n - 1.
    \end{cases}
\]

Above defined labeling pattern give rise

\[
    A = \{ f(v_i)/i = 2, 4, 6, \ldots, n \} = \{1, -1, -3, \ldots, -(n - 3)\}, \\
    B = \{ 2q - i/i = 1, 3, \ldots, \frac{n}{2} - 1 \} = \{2n - 1, 2n - 3, \ldots, \frac{3n}{2} + 1\} \\
    C = \{ 2q - (i + 2)/i = \frac{n}{2} + 1, \frac{n}{2} + 3, \ldots, n - 1 \} = \{\frac{3n}{2} - 3, \frac{3n}{2} - 5, \ldots, n - 1\}.
\]

i.e. domain of \( f \) is \( A \cup B \cup C \subseteq \{\pm 1, \pm 3, \pm 5, \ldots, \pm (2n - 1)\} \). Further we see that \( f^*(v_i,v_n) = n + 2 \) and

\[
    f^*(v_i,v_n + 1) = \begin{cases} 
        2q - 2i + 2, & i < \frac{n}{2} \\
        2q - 2i, & \frac{n}{2} \leq i < n.
    \end{cases}
\]

Therefore, \( D = \{ f(v_1,v_n) \} = \{n + 2\} \) and \( E = \{ f^*(v_i,v_n + 1)/1 \leq i < \frac{n}{2} \} = \{n + 4, n + 6, n + 8, \ldots, 2n - 2, 2n\} \) and \( F = \{ f^*(v_i,v_n + 1)/\frac{n}{2} \leq i < n \} = \{2, 4, 6, \ldots, n\} \) i.e. domain of \( f^* \) is \( DUE UF = \{2, 4, 6, \ldots, 2n\} \) range of \( f^* \) and so, \( f^* \) is bijective map. Therefore, \( f \) is an odd-even sum labeling for \( C_n(n \equiv 0 \pmod{4}) \). By taking \( k \) equal to one of the integer from the set \( \{2, 3, \ldots, n - 2\} \), it is observed that for every \( uv \in E(C_n) \), we have \( \min\{f(u), f(v)\} < k < \max\{f(u), f(v)\} \).

Hence \( C_n(n \equiv 0 \pmod{4}) \) is an odd-even sub graph.

\[\square\]

**Theorem 2.2.** \( K_{m,n} (m, n \geq 2) \) is an \( \alpha \)-odd-even sub graph.

**Proof.** Let \( V(K_{m,n}) = \{u_1, u_2, u_3, \ldots, u_m\} \cup \{v_1, v_2, v_3, \ldots, v_n\} \) and \( E(K_{m,n}) = \{u_i,v_j/1 \leq i \leq m, 1 \leq j \leq n\} \). It is obvious that \( p = m + n, q = mn \) for \( K_{m,n} \). Define \( f : V(K_{m,n}) \rightarrow \{\pm 1, \pm 3, \pm 5, \ldots, \pm (2q - 1)\} \) as follows.

\[
    f(v_i) = 3 - 2j, \ \forall 1 \leq j \leq n; \\
    f(u_i) = 2(q + n - ni) - 1, \ \forall 1 \leq i \leq m;
\]

Above defined labeling pattern shows that \( f \) is an injective map and \( f^* \) is a bijective map as

\[
    f^*(u_i,u_j) = \begin{cases} 
        2q + 2n - ni - 1 + 3 - 2j, & \forall j = 1, 2, \ldots, n, \ \forall i = 1, 2, \ldots, m; \\
        2q + 2n(i - 1) - 2j, & \forall j = 1, 2, \ldots, n, \ \forall i = 1, 2, \ldots, m.
    \end{cases}
\]

\( \forall j = 1, 2, \ldots, n, \ \forall i = 1, 2, \ldots, m \) i.e. range of \( f^* \) is equal to domain of \( f \). Therefore \( f \) is an odd-even sum labeling for \( K_{m,n} \). By taking \( k \in \{2, 3, \ldots, 2n - 2\} \), it is observed that for every \( uv \in E(K_{m,n}) \), we have \( \min\{f(u), f(v)\} < k < \max\{f(u), f(v)\} \). Hence, \( K_{m,n}(m, n \geq 2) \) is an \( \alpha \)-odd-even sum graph.

\[\square\]

**Theorem 2.3.** Grid graph \( P_m \square P_n (m, n \geq 2) \) is an \( \alpha \)-odd-even sum graph.

**Proof.** Let \( G = P_n \square P_m \) and \( V(G) = \{u_{i,j}/1 \leq i \leq n, 1 \leq j \leq m\} \). Take \( E(G) = \{u_{i,j},u_{i+1,j}/1 \leq i \leq n, 1 \leq j \leq m\} \cup \{u_{i,j},u_{i+1,j}/1 \leq i \leq n, 1 \leq j \leq m\} \). In \( G = (P_n \square P_m) \), it is obvious that \( p = mn, q = 2mn - (m + n) \), where \( m, n \geq 2 \).
Kaneria, Makadia and Viradia [8] defined following labeling pattern $f$ for a grid graph $P_n \Box P_m$, which is a graceful labeling for $G = P_n \Box P_m$. $f : V(G) \to \{0, 1, 2, \ldots, q\}$ defined by

$$f(u, i) = \begin{cases} 
q - (\frac{n+1}{2}), & i = 2n - 1, n \in \mathbb{N}; \\
(\frac{i^2}{2}) + 1, & i = 2n, n \in \mathbb{N}; \\
\forall i = 1, 2, \ldots n 
\end{cases}$$

Moreover, $u$ Step Grid graph

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Proof. Now define $g : V(G) \to \{\pm 1, \pm 3, \pm 5, \ldots, \pm (2q - 1)\}$ as follows.

$$g(u, i) = \begin{cases} 
1 - 2f(u, i), & \text{when } f(u, i) \leq \left\lfloor \frac{2q - 2}{2} \right\rfloor; \\
2f(u, i) - 1, & \text{when } f(u, i) \geq \left\lfloor \frac{2q}{2} \right\rfloor 
\end{cases}$$

Above defined labeling pattern give rise $g$ is an injective map, as $\{g(u, i)/f(u, i)\geq \left\lfloor \frac{2q}{2} \right\rfloor \} \subseteq \{2q - 1, 2q - 3, \ldots, 2q - 1\} \text{ and}$

$$g(u, i) \leq \left\lfloor \frac{2q - 2}{2} \right\rfloor \subseteq \{-2, \frac{q - 2}{2}, 1, -2, \frac{q - 2}{2}, 3, \ldots, -1, 1\}$$

Moreover $g^* : E(G) \to \{2, 4, \ldots, 2q\}$ is a bijective map, as $g^*(uv) = 2 |f(u) - f(v)| = 2f^*(uv)$ and $f$ is a bijection. Therefore, $g$ is an odd-even sum labeling for $G$. By taking $k$ from $\{2, 3, \ldots, 2 \left\lfloor \frac{q}{2} \right\rfloor - 2\}$. It is observed that for every $uv \in E(G)$, we have $\min \{g(u), g(v)\} < k < \max \{g(u), g(v)\}$ and so, $G$ is an $\alpha$-odd-even sum graph. \hfill \Box

Theorem 2.4. Step Grid graph $St_n(n \geq 3)$ is an $\alpha$-odd-even sum graph.

Proof. Kaneria and Makadia [7] defined step grid graph $St_n(n \geq 3)$ and they have proved that it is a bipartite graceful graph with the following graceful labeling $f$ for $St_n$. They have defined $St_n$ by taking $u_{i,j}(1 \leq j \leq n)$ vertices of $n^{th}$ column, $u_{i,j}(1 \leq j \leq n)$ vertices of $(n - 1)^{th}$ column, $u_{i,j}(1 \leq j \leq n - 1)$ vertices of $(n - 2)^{th}$ column, $u_{i,j}(1 \leq j \leq n - 2)$ vertices of $(n - 3)^{th}$ column and so on. In this manner, $u_{n,j}(j = 1, 2)$ are the vertices of first column of $St_n$. It is obvious that $p = \frac{1}{2}(n^2 + 3n - 2)$, $q = n^2 + n - 2$ in $St_n$, where $n \geq 3$. The graceful labeling function $f : V(St_n) \to \{0, 1, 2, \ldots, q\}$ defined as follows.

$$f(u, i, j) = \begin{cases} 
q - \frac{j^2}{2} + (-1)^j + 1, & \forall j = 1, 2, \ldots, n; \\
f(u, i, j) = f(u_{i-1,j-1}) + (-1)^j, & \forall i = 2, 3, \ldots, \left\lfloor \frac{n}{2} \right\rfloor, \forall j = 1, 2, \ldots, n + i - 1; \\
f(u, i, j = 1) = n - i + 1, & \forall n, n - 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor; \\
f(u, i, j = 2) = q - (n - i + 1)(n - i), & \forall n, n - 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor; \\
f(u, i, j) = f(u_{i+1,j-2}) + (-1)^{j-1}, & \forall n - 1, n - 2, \ldots, \forall j = 3, 4, \ldots, n + 2 - i
\end{cases}$$

Now define $g : V(St_n) \to \{\pm 1, \pm 2, \ldots, \pm (2q - 1)\}$ as follows.

$$g(u, i, j) = \begin{cases} 
3 - 2f(u, i, j), & \text{when } f(u, i, j) < \frac{q}{2}; \\
2f(u, i, j) - 3, & \text{when } f(u, i, j) \geq \frac{q}{2}; 
\end{cases}$$

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Above defined labeling pattern give rise $g$ is an injective map, as \( \{g(u)/f(u) < \frac{q}{2}\} \subseteq \{3, 1, -3, ..., -(q - 4)\} \) and \( \{g(u)/f(u) \geq \frac{q}{2}\} \subseteq \{2q - 3, 2q - 5, ..., q - 3\} \). Moreover

\[
g^*(uv) = \begin{cases} 
  g(u) + g(v) & \text{if } u, v \neq u_1, u_2, \ldots, u_n \\
  2|f(u) - f(v)| & \text{if } u = v, u \neq u_1, u_2, \ldots, u_n \\
  2f^*(uv) & \text{if } u, v = u_1, u_2, \ldots, u_n
\end{cases}
\]

Which gives $g$ is bijective map, as $f$ is a bijection. Therefore, $g$ is an odd-even sum labeling for $St_n$. By taking positive integer $k$ from \{4, 5, ..., $q - 4$\}, it is observed that for any $uv \in E(St_n)$, $\min\{g(u), g(v)\} < k < \max\{g(u), g(v)\}$. Therefore, $St_n(n \geq 3)$ is an $\alpha$-odd-even sum graph. \qed

**Theorem 2.5.** Splitting graph of $K_{1,n}$ is an $\alpha$-odd-even sum graph.

**Proof.** For each vertex $v$ of a graph $G$, take a new vertex $u$ and join $u$ to all the vertices of $G$, which are adjacent to $v$. Thus, obtained new graph is called the splitting graph of $G$. Let $G$ be the splitting graph of $K_{1,n}$ and $V(K_{1,n}) = \{v, u_1, u_2, u_3, \ldots, u_n\}$. It is obvious that $p = |V(G)| = 2n + 2q = |E(G)| = 3n$. Take $V(G) = V(K_{1,n}) \cup \{u, u_1, u_2, \ldots, u_n\}$, where $u, u_1, u_2, \ldots, u_n$ be the added vertices corresponding to $v$, $v_1, v_2, \ldots, v_n$ to obtained the splitting graph $G$ of $K_{1,n}$. It is observed that $E(G) = E(K_{1,n}) \cup \{(uv_i, uv_i)/1 \leq i \leq n\}$. Define $f : V(G) \rightarrow \{\pm 1, \pm 3, \pm 5, \ldots, \pm (2q - 1)\}$ as follows.

- $f(v) = 1$, $f(v_i) = -1 + 4i$, $\forall 1 \leq i \leq n$;
- $f(u) = -1$, $f(u_i) = 4n - 1 + 2i$, $\forall 1 \leq i \leq n$.

Above defined labeling pattern gives rise $f$ is an injective map. Moreover, $f^*(uv_i) = 4i - 2$, $f^*(vui) = 2(2n + i)$, $f^*(vui) = 4i$, $\forall i = 1, 2, \ldots, n$, i.e. $\{f^*(uv_i)/1 \leq i \leq n\} \cup \{f^*(vui)/1 \leq i \leq n\} \cup \{f^*(vui)/1 \leq i \leq n\} = \{2, 6, 10, \ldots, 4n - 2\} \cup \{4n + 2, 4n + 4, \ldots, 6n\} \cup \{4, 8, 12, \ldots, 4n\}$. Thus, $f^*$ is a bijective map and so, $G$ admits an odd-even sum labeling. By taking $k = 2$, it is observed that for each $w_1, w_2 \in E(G)$, we have $\min\{f(w_1), f(w_2)\} < k < \max\{f(w_1), f(w_2)\}$. Therefore, $G$ is an $\alpha$-odd-even sum graph. \qed

**Theorem 2.6.** Caterpillar $S(x_1, x_2, x_3, \ldots, x_n)$ is an $\alpha$-odd-even sum graph, where $n > 2$.

**Proof.** Let $G = S(x_1, x_2, x_3, \ldots, x_n)$, where $n > 2$ and $x_1, x_2, x_3, \ldots, x_n$ all are non-negative integers. It is obvious that $p = x_1, x_2, x_3, \ldots, x_n + n$ and $q = p - 1$ in the caterpillar $G$. Let $V(G) = \{u_i/1 \leq i \leq n\} \cup \{u_{i,j}/1 \leq j \leq x_i, 1 \leq i \leq n\}$ and $E(G) = \{u_i, u_{i+1}/1 \leq i < n\} \cup \{u_{i,j}, u_{i+1}/1 \leq j \leq x_i, 1 \leq i \leq n\}$. Define $f : V(G) \rightarrow \{\pm 1, \pm 3, \pm 5, \ldots, \pm (2q - 1)\}$ as follows.

- $f(u_1) = 2q - 1$,
- $f(u_{2i-1}) = f(u_1) - 2(x_2 + x_4 + \ldots + x_{2i-2} + i - 1)$, $2 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$;
- $f(u_2) = 1 - 2(x_1 + x_3 + \ldots + x_{2i-1} + i - 1)$, $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$;
- $f(u_{1,j}) = 3 - 2j$, $1 \leq j \leq x_i$;
- $f(u_{i,j}) = f(u_{i-1}) - 2j$, $1 \leq j \leq x_i$, $2 \leq i \leq n$.

Above defined labeling pattern give rise $f$ is an injective map and $f^*$ is a bijective map, as $f(u_{i, u_{i+1}}) = 2q - 2(x_1 + x_3 + \ldots + x_{i-1} + i - 1)$, $\forall 1 \leq i \leq n - 1$ and

\[
f(u_{i, u_{i,j}}) = f(u_i) + f(u_{i,j})
\]

\[
= f(u_i) + f(u_{i-1}) - 2j
\]

\[
= f^*(u_i, u_{i-1}) - 2j \\
= 2q - 2(x_1 + \ldots + x_{i-2} + i - 2) - 2j, \forall 1 \leq j \leq x_1 \forall 1 \leq i \leq n.
\]
Therefore, $f$ is an odd-even sum labeling for $G$ and so, $G$ is an odd-even sum graph. By taking $k$ equal to one of integer from $\{2, 3, \ldots, \max\{f(u_{n-1}, f(u_n)) - 1\}\}$, it is observed that for every $uv \in E(G)$, we have $\min\{f(u), f(v)\} < k < \max\{f(u), f(v)\}$. Hence, $G$ is an $\alpha$-odd-even sum graph.

\textbf{Corollary 2.7.}

(1). $P_n(n \geq 3)$ is an $\alpha$-odd-even sum graph.

(2). Star $K_{1,n} = S(0, n - 1, 0)$ is an $\alpha$-odd-even sum graph, when $n \geq 2$.

(3). Bistar $B_{m,n} = S(0, m - 1, n)$ is an $\alpha$-odd-even sum graph.

(4). The graph $B(m, n, k) = S(m, 0, \ldots, 0, n)$ is an $\alpha$-odd-even sum graph.

(5). Coconut tree is an $\alpha$-odd-even sum graph.

(6). comb $(S(1, 1, \ldots, 1))$ is an $\alpha$-odd-even sum graph.

\textbf{References}


