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Generalized Odd-Even Sum Labeling and Some $\alpha-$Odd-Even Sum Graphs

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Abstract: A (p, q) graph G is said to be an α-odd-even sum graph if it admits an odd-even sum labeling f defined by Monika and Murugan [9] by adding an addition condition that there is a positive integer $k(0 < k < 2q - 1)$ such that for every edge $uv \in E(G)$, $\min(f(u), f(v)) < k < \max(f(u), f(v))$. In this paper, we study α-odd-even sum labeling of $C_n(n \equiv 0 \pmod 4)$, $S(x_1, x_2, \dots, x_n)$, $K_{m,n}$ $(m, n \geq 2)$, $P_n \Box P_m(m, n \geq 2)$, step grid graph $St_n(n \geq 3)$ and splitting graph of $K_{1,n}$.

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Keywords: α-odd-even sum labeling, Grid graph, Step grid graph, Splitting graph.

1. Introduction

α-labeling and β-valuation (graceful labeling) was introduced by Rosa [11] in 1967. Acharya and Gill[1] have investigated α-labeling for the grid graph $P_n \Box P_m$. Makadia and Kaneria [7] introduced step grid graph St_n and proved that it is graceful $(n \geq 3)$. Harary [5] introduced a notation of sum graph. A (p, q) graph G is said to be an odd-even sum graph if it admits an injective function $f : V(G) \longrightarrow \{\pm 1, \pm 3, \pm 5, \dots, \pm(2q - 1)\}$ such that its edge induced function $f^* : E(G) \longrightarrow \{2, 4, 6, \dots, 2q\}$ define by $f^*(uv) = f(u) + f(v)$, $\forall\ uv \in E(G)$ is bijective, which introduced by Monika and and Murugan [9]. These results motivated us and we introduced here a new concept called α-odd-even sum labeling which is an odd even sum labeling for a graph G and one additional condition that there is a positive integer $k(0 < k < 2q - 1)$ such that $\min\{f(u), f(v)\} < k < \max\{f(u), f(v)\}$, $\forall\ uv \in E(G)$. Every α-odd-even sum graph is always a bipartite graph.

2. Main Results

Theorem 2.1. *Every cycle $C_n(n \equiv 0$ (mod 4)) is an α-odd-even sum graph.*

Proof. Let $V(C_n) = \{v_1, v_2, v_3 \dots, v_n\}$ and $E(C_n) = \{v_i v_{i+1}/1 \leq i < n\} \cup \{v_n v_1\}$.It is obvious that $p = q = n$ for C_n.

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Define $f : V(G) \longrightarrow \{\pm1, \pm3, \pm5, \ldots, \pm(2q-1)\}$ as follows.

$$f(x) = \begin{cases} 3-i, & \forall i = 2,4,6...,n; \\ 2q-1, & \forall i = 1,3,..., \frac{n}{2}-1; \\ 2q-(i+2), & \forall i = \frac{n}{2}+1, \frac{n}{2}+3, ...n-1. \end{cases}$$

Above defined labeling pattern give rise

$$A = \{f(v_i)/i = 2,4,6,\ldots,n\} = \{1,-1,-3,\ldots,-(n-3)\},$$
$$B = \{2q-i/i = 1,3,\ldots,\frac{n}{2}-1\} = \{2n-1, 2n-3, \ldots, \frac{3n}{2}+1\}$$
$$C = \{2q-(i+2)/i = \frac{n}{2}+1, \frac{n}{2}+3, \ldots, n-1\} = \{\frac{3n}{2}-3, \frac{3n}{2}-5, \ldots, n-1)\}.$$

i.e. domain of f is $AUBUC \subseteq \{\pm1, \pm3, \pm5, ..., \pm(2n-1)\}$. Further we see that $f^*(v_1v_n) = n+2$ and

$$f^*(v_iv_{i+1}) = \begin{cases} 2q-2i+2, & i < \frac{n}{2} \\ 2q-2i, & \frac{n}{2} \leq i < n. \end{cases}$$

Therefore, $D = \{f(v_1v_n)\} = \{n+2\}$ and $E = \{f^*(v_iv_{i+1})/1 \leq i < \frac{n}{2}\} = \{n+4, n+6, n+8, \ldots, 2n-2, 2n\}$ and $F = \{f^*(v_iv_{i+1})/\frac{n}{2} \leq i < n\} = \{2,4,6,\ldots,n\}$ i.e. domain of f^* is $DUEUF = \{2,4,6\ldots,2n\}$ = range of f^* and so, f^* is bijective map. Therefore, f is an odd-even sum labeling for $C_n(n \equiv 0 \pmod 4)$. By taking k equal to one of the integer from the set $\{2,3,\ldots,n-2\}$, it is observed that for every $uv \in E(C_n)$, we have $\min\{f(u), f(v)\} < k < \max\{f(u), f(v)\}$. Hence $C_n(n \equiv 0 \pmod 4)$ is an α-odd-even sub graph. □

Theorem 2.2. *$K_{m,n}$ $(m, n \geq 2)$ is an α-odd-even sub graph.*

Proof. Let $V(K_{m,n}) = \{u_1, u_2, u_3 \ldots, u_m\} \cup \{v_1, v_2, v_3 \ldots, v_n\}$ and $E(K_{m,n}) = \{u_iv_j/1 \leq i \leq m, 1 \leq j \leq n\}$. It is obvious that $p = m+n, q = mn$ for $K_{m,n}$. Define $f : V(K_{m,n}) \rightarrow \{\pm1, \pm3, \pm5, \ldots, \pm(2q-1)\}$ as follows.

$$f(v_j) = 3-2j, \ \forall 1 \leq j \leq n;$$
$$f(u_i) = 2(q+n-ni)-1, \ \forall 1 \leq i \leq m;$$

Above defined labeling pattern shows that f is an injective map and f^* is a bijective map as

$$f^*(u_iu_j) = \begin{cases} 2q+2n-ni-1+3-2j \\ 2q+2-2n(i-1)-2j, \end{cases}$$

$\forall\ j = 1,2,...,n,\ \forall\ i = 1,2,...,m$ i.e. range of f^* is equal to domain of f. Therefore f is an odd-even sum labeling for $K_{m,n}$. By taking $k \in \{2,3,\ldots,2n-2\}$, it is observed that for every $uv \in E(K_{m,n})$, we have $\min\{f(u), f(v)\} < k < \max\{f(u), f(v)\}$. Hence, $K_{m,n}(m, n \geq 2)$ is an α-odd-even sum graph. □

Theorem 2.3. *Grid graph $P_n \Box P_m(m, n \geq 2)$ is an α-odd-even sum graph.*

Proof. Let $G = P_n \Box P_m$ and $V(G) = \{u_{i,j}/1 \leq i \leq n, 1 \leq j \leq m\}$. Take $E(G) = \{u_{i,j}u_{i+1,j}/1 \leq i \leq n, 1 \leq j \leq m\} \cup \{u_{i,j}u_{i,j+1}/1 \leq i \leq n, 1 \leq j \leq m\}$. In $G = (P_n \Box P_m)$, it is obvious that $p = mn, q = 2mn - (m+n)$, where $m, n \geq 2$.

Kaneria, Makadia and Viradia [8] defined follwing labeling pattern f for a grid graph $P_n \Box P_m$, which is a graceful labeling for $G = P_n \Box P_m$. $f : V(G) \to \{0, 1, 2, \dots, q\}$ defined by

$$f(u_{i,1}) = \begin{cases} q - \left(\frac{i-1}{2}\right), & i = 2n-1, n \in N; \\ \left(\frac{i-2}{2}\right) & i = 2n, n \in N; \\ & \forall i = 1, 2, ...n \end{cases}$$

$$f(u_{i,2}) = \begin{cases} (n-1) + \left(\frac{i-1}{2}\right), & i = 2n-1, n \in N; \\ (q-n+1) - \left(\frac{i}{2}\right) & i = 2n, n \in N; \\ & \forall i = 1, 2, ...n \end{cases}$$

$$f(u_{i,j}) = \begin{cases} f(u_{i,j-2}) - (2n-1), & f(u_{i,j-2}) > \frac{q}{2}; \\ f(u_{i,j-2}) - (2n-1), & f(u_{i,j-2}) < \frac{q}{2}; \\ & \forall j = 3, 4, ..., m, \forall i = 1, 2, ..., n. \end{cases}$$

Define $g : V(G) \to \{\pm 1, \pm 3, \pm 5, ..., \pm(2q-1)\}$ as follows.

$$g(u_{i,j}) = \begin{cases} 1 - 2f(u_{i,j}), & \text{when} \quad f(u_{i,j}) \le \left\lceil \frac{q-2}{2} \right\rceil; \\ 2f(u_{i,j}) - 1, & \text{when} \quad f(u_{i,j}) \ge \left\lceil \frac{q}{2} \right\rceil \end{cases}$$

Above defined labeling pattern give rise g is an injective map, as $\{g(u_{i,j})/f(u_{i,j}) \ge \lceil \frac{q}{2} \rceil\} \subseteq \{2q-1, 2q-3, ...2\lceil \frac{q}{2} \rceil - 1\}$ and

$$g(u_{i,j}) \le \left\lceil \frac{q-2}{2} \right\rceil\} \subseteq \{-2\left\lceil \frac{q-2}{2} \right\rceil + 1, -2\left\lceil \frac{q-2}{2} \right\rceil + 3, ..., -1, 1\}$$

Moreover $g^* : E(G) \longrightarrow \{2, 4, \dots, 2q\}$ is a bijective map, as $g^*(uv) = 2\,|f(u) - f(v)| = 2f^*(uv)$ and f is a bijection. Therefore, g is an odd-even sum labeling for G. By taking k from $\{2, 3, ..., 2\lceil \frac{q}{2} \rceil - 2\}$. It is observed that for every $uv \in E(G)$, we have min $\{g(u), g(v)\} < k < \max\{g(u), g(v)\}$ and so, G is an α-odd-even sum graph. □

Theorem 2.4. *Step Grid graph $St_n (n \ge 3)$ is an α-odd-even sum graph.*

Proof. Kaneria and Makadia [7] defined step grid graph $St_n (n \ge 3)$ and they have proved that it is a bipartite graceful graph with the following graceful labeling f for St_n.They have defined St_n by taking $u_{1,j} (1 \le j \le n)$ vertices of n^{th} column, $u_{2,j} (1 \le j \le n)$ vertices of $(n-1)^{th}$ column, $u_{3,j} (1 \le j \le n-1)$ vertices of $(n-2)^{th}$ column, $u_{4,j} (1 \le j \le n-2)$ vertices of $(n-3)^{th}$ column and so on. In this manner, $u_{n,j} (j = 1, 2)$ are the vertices of first column of St_n. It is obvious that $p = \frac{1}{2}(n^2 + 3n - 2)$, $q = n^2 + n - 2$ in St_n, where $n \ge 3$. The graceful labeling function $f : V(St_n) \to \{0, 1, 2, \dots, q\}$ defined as follows.

$$\begin{aligned} f(u_{i,j}) &= \tfrac{q}{2} - \tfrac{1}{8} + (-1)^{j+1} \left[\tfrac{j^2}{4} - \tfrac{1}{8} \right], & \forall j = 1, 2, ..., n; \\ f(u_{i,j}) &= f(u_{i-1,j-1}) + (-1)^j, & \forall i = 2, 3, ... \left\lfloor \tfrac{n}{2} \right\rfloor, \forall j = 1, 2, ..., n+i-1; \\ f(u_{i,1}) &= (n-i+1)^2 + 1, & \forall i = n, n-1, ..., \left\lceil \tfrac{n}{2} \right\rceil; \\ f(u_{i,2}) &= q - (n-i+1)(n-i), & \forall i = n, n-1, ..., \left\lceil \tfrac{n}{2} \right\rceil; \\ f(u_{i,j}) &= f(u_{i+1,j-2}) + (-1)^{j-1} & \forall i = n-1, n-2, ...2, \forall j = 3, 4, ...n+2-i \end{aligned}$$

Now define $g : V(St_n) \to \{\pm 1, \pm 2, \dots, \pm(2q-1)\}$ as follows.

$$g(u_{i,j}) = \begin{cases} 3 - 2f(u_{i,j}), & when f(u_{i,j}) < \frac{q}{2}; \\ 2f(u_{i,j}) - 3, & when f(u_{i,j}) \ge \frac{q}{2}; \end{cases}$$

Above defined labeling pattern give rise g is an injective map, as $\{g(u)/f(u) < \frac{q}{2}\} \subseteq \{3, 1, -1, -3, ..., -(q-4)\}$ and $\{g(u)/f(u) \geq \frac{q}{2}\} \subseteq \{2q-3, 2q-5, ...q-3\}$. Moreover

$$g^*(uv) = \begin{cases} g(u) + g(v) \\ 2\,|f(u) - f(v)| \\ 2f^*(uv) \end{cases}$$

Which gives g is bijective map,as f is a bijection. Therefore, g is an odd-even sum labeling for St_n. By taking positive integer k from $\{4, 5, \ldots, q-4\}$, it is observed that for any $uv \in E(St_n)$, $\min\{g(u), g(v)\} < k < \max\{g(u), g(v)\}$. Therefore, $St_n(n \geq 3)$ is an α-odd-even sum graph. □

Theorem 2.5. *Splitting graph of $K_{1,n}$ is an α-odd-even sum graph.*

Proof. For each vertex v of a graph G, take a new vertex u and join u to all the vertices of G,which are adjacent to v. Thus, obtained new graph is called the splitting graph of G. Let G be the splitting graph of $K_{1,n}$ and $V(K_{1,n}) = \{v, v_1, v_2, v_3, \ldots, v_n\}$. It is obvious that $p = |V(G)| = 2n+2, q = |E(G)| = 3n$. Take $V(G) = V(K_{1,n}) \cup \{u, u_1, u_2, ..., u_n\}$, where $u, u_1, u_2, ..., u_n$ be the added vertices corresponding to $v, v_1, v_2, ..., v_n$ to obtained the splitting graph G of $K_{1,n}$. It is observed that $E(G) = E(K_{1,n}) \cup \{(uv_i, vu_i)/1 \leq i \leq n\}$. Define $f : V(G) \to \{\pm 1, \pm 3, \pm 5, ..., \pm(2q-1)\}$ as follows.

$$\begin{aligned} f(v) &= 1, \quad f(v_i) = -1 + 4i, \quad \forall 1 \leq i \leq n; \\ f(u) &= -1, \quad f(u_i) = 4n - 1 + 2i, \quad \forall 1 \leq i \leq n. \end{aligned}$$

Above defined labeling pattern gives rise f is an injective map. Moreover, $f^*(uv_i) = 4i-2$, $f^*(vui) = 2(2n+i)$, $f^*(vv_i) = 4i$, $\forall i = 1, 2, \ldots n$. i.e. $\{f^*(uv_i/1 \leq i \leq n)\} \cup \{f^*(vu_i)/1 \leq i \leq n\} \cup \{f^*(vv_i)/1 \leq i \leq n\} = \{2, 6, 10, \ldots, 4n-2\} \cup \{4n+2, 4n+4, \ldots, 6n\} \cup \{4, 8, 12, \ldots, 4n\}$. Thus, f^* is a bijective map and so, G admits an odd-even sum labeling. By taking $k = 2$, it is observed that for each $w_1w_2 \in E(G)$, we have $\min\{f(w_1), f(w_2)\} < k < \max\{f(w_1), f(w_2\}$. Therefore, G is an α-odd-even sum graph. □

Theorem 2.6. *Caterpillar $S(x_1, x_2, x_3, \ldots, x_n)$ is an α-odd-even sum graph, where $n > 2$.*

Proof. Let $G = S(x_1, x_2, x_3, \ldots, x_n)$, where $n > 2$ and $x_1, x_2, x_3, \ldots, x_n$ all are non-negative integers. It is obvious that $p = x_1, x_2, x_3, \ldots, x_n + n$ and $q = p-1$ in the caterpillar G. Let $V(G) = \{u_i/1 \leq i \leq n\} \cup \{u_{i,j}/1 \leq j \leq x_i, 1 \leq i \leq n\}$ and $E(G) = \{u_iu_{i+1}/1 \leq i < n\} \cup \{u_iu_{i,j}/1 \leq j \leq x_i, 1 \leq i \leq n\}$. Define $f : V(G) \to \{\pm 1, \pm 3, \pm 5, \ldots, \pm(2q-1)\}$ as follows.

$$\begin{aligned} f(u_1) &= 2q-1, \\ f(u_{2i-1}) &= f(u_1) - 2(x_2 + x_4 + ... + x_{2i-2} + i - 1), \quad 2 \leq i \leq \left\lceil \tfrac{n}{2} \right\rceil; \\ f(u_{2i}) &= 1 - 2(x_1 + x_3 + ... + x_{2i-1} + i - 1), \quad 1 \leq i \leq \left\lfloor \tfrac{n}{2} \right\rfloor; \\ f(u_{1,j}) &= 3 - 2j \quad 1 \leq j \leq x_1; \\ f(u_{i,j}) &= f(u_{i-1}) - 2j \quad 1 \leq j \leq x_i; \quad 2 \leq i \leq n. \end{aligned}$$

Above defined labeling pattern give rise f is an injective map and f^* is a bijective map, as $f(u_i, u_{i+1}) = 2q - 2(x_1 + x_3 + ... + x_{i-1} + i - 1), \forall 1 \leq i \leq n-1$ and

$$\begin{aligned} f(u_i, u_{i,j}) &= f(u_i) + f(u_{i,j}) \\ &= f(u_i) + f(u_{i-1}) - 2j \\ &= f^*(u_i, u_{i-1}) - 2j \\ &= 2q - 2(x_1 + ... + x_{i-2} + i - 2) - 2j, \quad \forall 1 \leq j \leq x_1 \quad \forall 1 \leq i \leq n. \end{aligned}$$

Therefore, f is an odd-even sum labeling for G and so, G is an odd-even sum graph. By taking k equal to one of integer from $\{2, 3, \ldots, max\{f(u_{n-1}, f(u_n\}-1\}$, it is observed that for every $uv \in E(G)$, we have $\min\{f(u), f(v)\} < k < \max\{f(u), f(v)\}$. Hence, G is an α-odd-even sum graph. □

Corollary 2.7.

(1). $P_n(n \geq 3)$ is an α-odd-even sum graph.

(2). Star $K_{1,n} = S(0, n-1, 0)$ is an α-odd-even sum graph,when $n \geq 2$

(3). Bistar $B_{m,n} = S(0, m-1, n)$ is an α-odd-even sum graph.

(4). The graph $B(m, n, k) = S(m, 0, 0, \ldots, 0, n$ is an α-odd-even sum graph.

(5). Coconut tree is an α-odd-even sum graph.

(6). comb $(S(1, 1, 1, \ldots, 1))$ is an α-odd-even sum graph.

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