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Approximation Weights of Gauss Quadrature Method

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Abstract: To find the exact area of Definite Integral of continuous function on the Interval $[a, b]$ is very problematic. In Numerical methods, the most popular method to find the area of finite Definite Integral is Gauss Legendre Quadrature Method (GLQM). In this GLQM, the weights are very difficult to find. In this paper, the new method is obtained using the New Weights which are nearest to GLQM weights. Also, the order of GLQM depends on the number of nodes, whereas the order of this new method is always two.

MSC: 65D30, 65D32.

Keywords: Gauss Quadrature method, Numerical method, Newton-Cotes method, Numerical Integration, Lobatto quadrature method.

1. Introduction

In numerical analysis, a **quadrature rule** is an approximation of the definite integral of a function, usually stated as a weighted sum of function values at specified points within the domain of integration. An n-point **Gaussian quadrature rule**, named after **Carl Friedrich Gauss**, is a quadrature rule constructed to yield an exact result for polynomials of degree 2n -1 or less by a suitable choice of the points x_i and weights w_i for i = 1, ..., n. The domain of integration for such a rule is conventionally taken as [1, 1], so the rule is stated as

$$\int_{-1}^{1} f(x)dx = \sum_{i=0}^{n} w_i f(x_i) + R_n[f] \tag{1}$$

Gaussian quadrature as above will only produce good results if the function $f(x)$ is well approximated by a polynomial function within the range [1, 1]. The method is not, for example, suitable for functions with singularities. However, if the integrated function can be written as $f(x) = w(x)g(x)$, where $g(x)$ is approximately polynomial and $w(x)$ is known, then alternative weights w_i and points xi that depend on the weighting function $w(x)$ may give better results, where

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} w(x)g(x) = \sum_{i=0}^{n} w_i' g(x_i') \tag{2}$$

Common weighting functions include $w(x) = 1/\sqrt{1-x^2}$ (**Chebyshev Gauss**) and $w(x) = e^{x^2}$ (**GaussHermite**). For the simplest integration problem stated above, i.e. with $w(x) = 1$, the associated polynomials are Legendre polynomials,

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \tag{3}$$

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and the method is usually known as **Gauss-Legendre quadrature(GLQM)**. With the n'th polynomial normalized to give $P_n(1) = 1$, the i'th Gauss node, x_i, is the i'th root of P_n; its weight is given by [2]

$$w_i = \frac{2}{(1 - x_i^2)[P_n'(x_i)]^2} \tag{4}$$

Some type of calculation of w_i are give [3], [5] and [7]. Swarztrauber, Paul N [6] are computed nodes and weights of GLQM and Hale, Nicholas, and Alex Townsend [4] are fond fast and accurate computation of Gauss–Legendre and Gauss–Jacobi quadrature nodes and weights. From 4 weights are depending on nodes. But we are assuming weight are depending on length of the interval, i.e weights of GLQM on [a, b] are same, if length of interval (b-a) is constant. In this way we found approximate weights in this paper which are not depending on nodes. The error terns can be obtained in the following manner [8]:

Since the method on [a, b] is exact for polynomial of degree $\leqslant n$, we have $R_n[f] = 0$ when $f(x) = x^i$, $i = 0, 1, ..., n$; $R_n[f] \neq 0$ when $f(x) = x^{n+1}$. Thus, we can write the error terms of the form

$$R_n[f] = \frac{C}{(n+1)!} f^{(n+1)}(\xi) \tag{5}$$

where $\xi \in$[a, b] and

$$C = \int_a^b w(x) x^{n+1} dx - \sum_{i=0}^{n} w_i f(x_i) \tag{6}$$

is called error constant. If C is zero for $f(x) = x^{n+1}$, then we take the next term x^{n+2}.

2. Preliminaries and Results

Definition 2.1. *The Quadrature method is called Open Type method If the nodes $x_i \in (a\ b)$, $\forall i = 0, 1, ..., n-1$. and is called Closed Type method if the nodes $x_0 = a$, and $x_{n-1} = b$.*

Definition 2.2. *Order of accuracy, or precision, of a Quadrature formula is the largest positive integer n such that the formula is exact for x^k, for each $k = 0, 1, ..., n$.*

Next we define an important function on [a, b], which useful for further results.

Definition 2.3. *Let $f(x, n)$, $\forall n \in N$ be sequence of differentiable function on [a, b] and satisfy the following properties*

(1). $f(x, n)$ has exactly n simple real roots in [a, b],

(2). $df/dx(x, n)$ has exactly $n - 1$ simple real roots in]a, b[,

then f(x, n) is called ***Real-Root Function*** *on [a, b].*

For example $Sin((n-1)x)$ and $\frac{d^n}{dx^n}(x^2 - 1)^n$ are real-root functions on [0, π] and [-1, 1], respectively.

Theorem 2.4. *Let the polynomial $p_n(x) = \prod_{i=1}^{n}(x - x_i)$, $1 \neq n \in N$,$x_i \in R, \forall i$ and $x_i < x_{i+1}, \forall i$ is real root function on $[x_1, x_n]$. also $p^{(\eta)}(x), \eta = 1, 2, .., n-2$ are real root functions on $[x_i, x_n]$.*

Theorem 2.5. *If f(x) is integrable function on [a b] then*

$$\int_a^b f(x)dx = \sum_{i=0}^{n} (r_{i+1} - r_i) f(x_i) + R[f] \tag{7}$$

where $x_i s$ are roots of Real-root function $g(x)$ on [a b] with $x_i < x_{i+1}$, $i = 0, 1, .., n-1$ and $r_i s$ are roots of $g'(x)$ with $r_0 = a$, $r_{n+1} = b$ and $r_i < r_{i+1}$, $i = 0, 1, ...n$. $\lambda_i = r_{i+1} - r_i$.the left hand side of (1) is convergence to right hand side of (1) as $n \longrightarrow \infty$. Since it is satisfy the general Quadrature rule. Sum of the weights is same as length of the interval [a b].

2.1. Composite Trapezoidal Rule from Real Root Function

Take the Real-root function $(x-a)(x-b)$ of order 2 on $[a, b]$. The root of $\frac{d}{dx}(x-a)(x-b) = 2x-(a+b)$ is $a+b/2$ in $[a$ $b]$. We have by the theorem 2.5

$$I = \int_a^b f(x)dx = \frac{a+b}{2}(f(a)+f(b))$$

This called Trapezoidal formula. We know error of above rule is $R_2[f] = -\frac{(b-a)^3}{12}f''(\xi)$ where ξ in [a, b]. In general, Composed trapezoidal formula on [a b] is 7 where x_is are roots of the Real-root function $sin((n-1)\pi(x-a)/(b-a))$ and r_is are roots of $cos((n-1)\pi(x-a)/(b-a))$ with $r_o = a$, and $r_n = b$.

2.2. Mid Points Integration Method form Real Root Function

The interval [a, b] can divided into n equal subintervals and take nodes $x_i \in]a\ b[, i = 1, ..., n$ are mid points of each interval with spacing h . That is the nodes are roots of the polynomial

$$f(x) = sin\left(\frac{n\pi(x-a+h/2)}{b-a}\right). \tag{8}$$

Here $f(x)$ is real root function on [a b]. The r_is roots of the polynomial $f'(x)$(derivative of $f(x)$). i.e $r_i - r_{i-1} = h, \forall i.$, then the integration method is

$$\int_a^b f(x)dx = h\sum_{i=0}^{n} f(x_i).$$

2.3. Quadrature method on [a, b] with equispaced points

Let consider the intragral $\int_a^b f(x)dx$. Let take the nodes $x_i, i = 0(1)n$, $x_0 = a$ and $x_n = b$ are equispaced points with spacing $h = x_{i+1} - x_i$. Now construct a root polynomial with same nodes. That is the polynomial $p_n(x) = \prod_{i=0}^{n}(x-x_i)$ is Real-root function on [a b]. The r_is roots of the polynomial dp_n/dx then new weights of Newton-Cotes Formulas are given in table 1. Now some cases arise: **For** $n = 1$ the quadrature method is same as trapezoidal rule. **For** $n = 2$ the roots of dp_2/dx are $a + h\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right)$ and $a + h\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right)$. Then quadrature rule is

$$\int_a^b f(x)dx = \frac{h}{\sqrt{3}}\left((\sqrt{3}-1)f(x_0) + 2f(x_1) + (\sqrt{3}-1)f(x_2)\right).$$

This method is exact for polynomial degree 2 and error constant C from equation 6 is

$$C = -\frac{1}{12}(\sqrt{3}-1)(a-b)(2\sqrt{3}ab - \sqrt{3}a - \sqrt{3}b + 2a^2 + 2ab + 2b^2 - 3a - 3b)$$

and error $R_3[f] = \frac{C}{3!}f^{(2)}(\xi)$ where ξ in [a, b]. Generally the order of this type of quadrature is two. The weights λ_i of the integral method of 1 with equispaced points are given in table 1.

n, λ	λ_0	λ_1	λ_2	λ_3	λ_4
1	1	1	-	-	-
2	$\frac{\sqrt{3}-1}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	$\frac{\sqrt{3}-1}{\sqrt{3}}$	-	-
3	$\frac{3-\sqrt{5}}{2}$	$\frac{\sqrt{5}}{2}$	$\frac{\sqrt{5}}{2}$	$\frac{3-\sqrt{5}}{2}$	-
4	0.355567	1.100521	1.087825	1.100521	0.3555671

Table 1. **New weight of Newton-cote rules**

2.4. New Weights of GLQM on [-1, 1] (open-type)

Nodes of GLQM are roots of the polynomial $\frac{d^n}{dx^n}(x^2-1)^n$(or $P_n(x)$) and r_i's are taken from roots of the polynomial $\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n$ (Nodes of Lobatto Integration Method, the roots of $\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n$ and $\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n$ are same except -1 and 1). Here we given nodes and new weights of GLQM in table 2. Lets we have to define a weight real root function(i.e it's gives exactly weights of GLQM) on [-1, 1]. We have equation (4) , then define

$$w_3(x) = (x^2-1)(x^2-(-1-w_i)) \tag{9}$$

$$w_4(x) = x(x^2-1)(x^2-(-1-w_i)) \tag{10}$$

and

$$w_n(x) = \begin{cases} (x^2-1)(x^2-(-1-w_i))\prod_{i=1}^{(n-1)/2}(x^2-(w_{i+1}-w_i)) & \text{if } n>4 \text{ is odd} \\ x(x^2-1)(x^2-(-1-w_i))\prod_{i=2}^{(n-2)/2}(x^2-(w_{i+1}-w_i)) & \text{if } n>4 \text{ is even.} \end{cases} \tag{11}$$

where w_i's are root of equation (3) and the $w_n(x), n=3,...n$ are real root function on [-1, 1]. Again we define a real root function

$$f_n(x) = \int P_n(x)dx = \frac{1}{n(n+1)}(x^2-1)P_n'(x) \tag{12}$$

The values of 4 can be written as

$$w_i = \frac{2}{n(n+1)f_n(x_i)P_n'(x_i)} \quad \text{or} \quad w_i = \frac{2}{n(n+1)\int P_n(x)dxP_n'(x)}$$

We are showing plots of $w_n(x)$ and $f_n(x)$ (see figure 1), that roots are approximately equal.

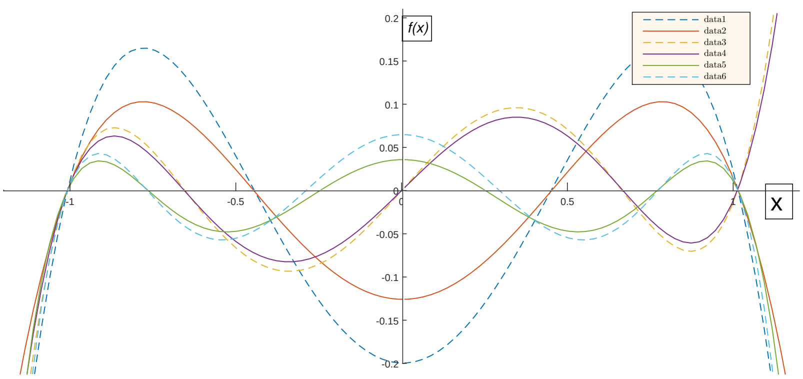

Figure 1. Error between $w_n(x)$ and $f_n(x)$ with n=3,4,5. data 1,2,3,4,5 and 6, are plots of $w_3(x), f_3(x), w_4(x), f_4(x), w_5(x)$ and $f_5(x)$, respectively

n	Nodes	Old weights (O)	New weights (N)	Error=O-N
1	0	2	2	0
2	±0.5773502692	1	1	0
3	0	0.8888888889	0.8944271910	0.0055383021
	±0.7745966692	0.5555555556	0.5527864045	-0.0027691511
4	±0.3399810436	0.6521451549	0.6546536707	0.0025085158
	±0.8611363116	0.3478548451	0.3453463293	- 0.0025085158

n	Nodes	Old weights (O)	New weights (N)	Error=O-N
5	0	0.5688888889	0.5704630330	-0.0015741441
	±0.5384693101	0.4786286705	0.4798238075	-0.0011951370
	±0.9061798459	0.2369268851	0.2349446761	0.00198220903
6	±0.2386191861	0.4679139346	0.4688487935	-0.0009348589
	±0.6612093865	0.3607615730	0.3613751028	-0.0006135298
	±0.9324695142	0.1713244924	0.1697761037	0.00154838868

Table 2. **New weight of GLQM on [-1 1]**

2.5. New Weights of Lobatto Quadrature Method on [-1 1] (closed-type)

Also called Radau quadrature (Chandrasekhar 1960). A Gaussian quadrature with weighting function $w(x) = 1$ in which the endpoints of the interval [-1, 1] are included in a total of n abscissas, giving $r = n - 2$ free abscissas. Abscissas are symmetrical about the origin, and the general formula is

$$\int_{-1}^{1} f(x)dx = w_1 f(-1) + w_n f(1) + \sum_{i=2}^{n-1} w_i f(x_i)$$

the nodes x_i of lobatto quadrature method are roots of polynomial $\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n$ (or $P_n'(x)$) The weights of the free abscissas are

$$w_i = -\frac{2}{n(n-1)[P_{n-1}(x_i)]^2}$$

and of the end points are $w_{1,n} = 2/(n^2-n)$.Let r_i are be the roots of polynomial $\frac{d^n}{dx^n}(x^2-1)^n$ (nodes of legendary polynomial) then the new numerical weights $r_{i+1} - r_i$ of Lobatto quadrature method are given table 3.

n	Nodes	Old weights (O)	New weights (N)	Error=O-N
2	0	1.33333333	1.1547005380	0.178632795
	±1	0.33333333	0.4226497308	-0.089316398
3	±0.44721360	0.83333333	0.7745966692	0.0587366661
	±1	0.33333333	0.4226497308	-0.089316398
4	0	0.71111111	0.6799620872	0.0311490228
	±0.65465367	0.54444444	0.5211552680	0.0232891764
	±1	0.10000000	0.1388636884	-0.0388636884 [H]
5	±0.28523152	0.55485837	0.5384693101	0.0163890599
	±0.76505532	0.37847496	0.3677105358	0.0107644242
	±1	0.06666667	0.0938201541	-0.0271534881
6	0	0.48761905	0.4772383722	0.0103806754
	±0.46884879	0.43174538	0.4225902004	0.0091551808
	±0.83022390	0.27682605	0.2712601277	0.0055659196
	±1	0.04761905	0.0675304858	-0.0199114382

Table 3. **New weight of labboto Quadrature on [-1 1]**

Note: New Weights of GLQM on [a, b] (open-type): the nodes of this quadrature method are roots of polynomial $\frac{d^n}{dx^n}(x-a)^n(x-b)^n$ and let r_i are be the roots of polynomial $\frac{d^n}{dx^n}(x-a)^{n+1}(x-b)^{n+1}$.

2.6. Other forms

Andrews, George E., Richard Askey, and Ranjan Roy are introduce some special functions in [1] The integration problem can be expressed in a slightly more general way by introducing a positive weight function w into the integrand, and allowing an interval other than [1, 1]. That is, the problem is to calculate$\int_a^b w(x)f(x)$ for some choices of a, b, and . For $a = -1, b = 1$, and $w(x) = 1$, the problem is the same as that considered above. Other choices lead to other integration rules. Some of these are tabulated (Table 4) below. Nodes are root of the polynomial and $r_i's$ are roots of derivative of polynomial, respectively. So, we will trying to real root function, it's give weights of particular quadreture method. Next section we introduced a real root function, it's give weights of GLQM on [-1, 1].

Interval	$w(x)$	Roots of the Polynomial (Nodes)
[-1, 1]	1	Legendre polynomial
(1, 1)	$(1-x)^a(1+x^b)$, $a, b > -1$	Jacobi polynomials
(1, 1)	$1/\sqrt{(1-x^2)}$	Chebyshev polynomials (first kind)
[-1, 1]	$\sqrt{(1-x^2)}$	Chebyshev polynomials (second kind)
[0, ∞)	e^{-x}	Laguerre polynomials
[0,∞)	$x^a e^{-x}$, $a > -1$	Generalized Laguerre polynomials
(∞, ∞)	e^{-x^2}	Hermite polynomials

Table 4. **Nodes of Quadrature method with weight function $w(x)$ on given interval**

3. Exact Weights of GLQM From Real Root Function

We are researched on some real root function $f(x)$ on [-1, 1] which function gives length of the two roots(nearest) is weight of GLQM on [-1, 1] and we get some results. The below real root function $f_{n,a}(x), n \in N, a \geq 1$ which gives weights of GLQM.

$$f_{n,a}(x) = (x^2-1)\frac{d^n}{dx^n}(1-x^2)^a \tag{13}$$

$f_{2,1}(x)$ gives weight of 1-point Legendre rule; $f_{3,2}(x)$ gives weights of 2-points Legendre rule; $f_{4,3.017047625}(x)$ gives weights of 3-points Legendre rule; $f_{5,4.015875840}(x)$ gives weights of 4-points Legendre rule. There is no fixed a for 5-point Legendre rule.

4. Conclusions

In this paper, new weights of GLQM on [-1, 1] and Lobatto Quadrature Method on [-1 1] have been introduced. This method is slow convergence and the order is two. The new concept of finding weights and obtaining approximate weights of GLQM and Lobatto Quadrature Method on [-1 1] is presented . Finally, it is showed that there exist a real root function on [-1, 1] such that this function give weights of GLQM upto $n = 4$. But there is no real root function for $n = 5$, I think atleast a real root function exist which give weights of GLQM on [-1, 1]. We can apply this concept to polynomial $P_n(x)$ of degree n and $P'_n(x)$ has exactly $n-1$ real roots. i.e $P'_n(x) = \prod_{i=1}^{n-1}(x-a_i), x_i \in R$ then

$$\int_{a_1}^{a_{n-1}} P_n(x)dx = \sum_{i=1}^{n-2}(a_{i+1}-a_i)P_n(b_i)$$

where $a_1 < a_2 < ... < a_{n-1}$ and $b_1 < b_2 < ... < b_{n-2}$ are root of the polynomial $P'_n(x)$ and $P''_n(x)$, respectively. That integration is exact for all polynomial $P_n(x)$(real root function). That's why, I'm applying this concept to all methods.

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