Int. J. Math. And Appl., 6(1–B)(2018), 395–401
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of *Mathematics And its Applications*

$I_{\ddot{g}}$-Separation Axioms in Ideal Topological Spaces

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Abstract: This paper mainly concentrates on the study of separation axioms in Ideal Topological spaces. In particular, we delibrate the properties of $\mathrm{I}_{\ddot{g}} - T_0$ space, $\mathrm{I}_{\ddot{g}} - T_1$ space, $\mathrm{I}_{\ddot{g}} - T_2$ space, $\mathrm{I}_{\ddot{g}} - Q_1$ space and $\mathrm{I}_{\ddot{g}} - Q_2$ space.

MSC: 54A05, 54D10.

Keywords: $I_{\ddot{g}} - T_0$ space, $I_{\ddot{g}} - T_1$ space, $I_{\ddot{g}} - T_2$ space, $I_{\ddot{g}} - Q_1$ space and $I_{\ddot{g}} - Q_2$ space.

1. Introduction

Separation axioms constitute a classical topic in general topology. These axioms are statements about richness of topology. These axioms concern, the separation axioms of points, point from closed set and closed set from each other. Separation axioms in ideal topological spaces have implication than corresponding axioms in topological spaces. Ideals in topological space has been considered since 1930 by the author Vaidyanathaswamy [11]. Jankovic [4] and Hamlett [4] introduced new topologies from old via ideals. The main purpose of this paper to study the properties of $\mathrm{I}_{\ddot{g}} - T_0$ space, $\mathrm{I}_{\ddot{g}} - T_1$ space, $\mathrm{I}_{\ddot{g}} - T_2$ space, $\mathrm{I}_{\ddot{g}} - Q_1$ space and $\mathrm{I}_{\ddot{g}} - Q_2$ space.

2. Preliminaries

The present paper throughout by (X, τ) or (Y, σ) denote a topological space with no separation properties assumed. For a subset A of a topological space (X, τ), $cl(A)$, and $Int(A)$ will denote the closure and interior of A in (X, τ) respectively. An ideal I on a topological space (X, τ) is a non-empty collection of subsets of X which satisfies,

(1). $A \in I$ and $B \subset A \implies B \in I$.

(2). $A \in I$ and $B \in I \implies A \cup B \in I$.

An ideal topological space is a topological space (X, τ) with an ideal I on X and is denoted by (X, τ, I). For a subset $A \subset X$, $A^*(I) = \{x \in X : U \cap A \notin I$ for every neighbourhood U of x$\}$ is called the local function of A with respect to I and τ. We simply write A^* instead of $A^*(I)$ to be brief [6]. For every ideal topological space (X, τ, I) there exists a topology $\tau^*(I)$, finer that τ, generated by $\beta(I, \tau) = \{U - i : U \in \tau andi \in I\}$, but in general $\beta(I, \tau)$ is not always a topology. Additionally, $cl^*(A) = A \cup A^*$ defines a kuratowski [6] closure operator for $\tau^*(I)$.

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Definition 2.1. *A space X is said to be ultra-Hausdorff [9] if every two distinct points of X can be separated by disjoint clopen sets.*

Definition 2.2. *A subset A of an ideal topological space (X, τ, I) is said to be [2]*

(1). $I_{\ddot{g}}$-closed if $A^* \subseteq U$ whenever $A \subseteq U$ and U is sg-open.

(2). $I_{\ddot{g}}$-open if its complement is $I_{\ddot{g}}$-closed.

Definition 2.3. *A function $f : (X, \tau) \to (Y, \sigma, J)$ is said to be*

(1). $I_{\ddot{g}}$-continuous($I_g - continuous$) [2] if the inverse image of every closed set in Y is $I_{\ddot{g}}$-closed ($I_g - closed$) in X.

(2). Strongly $I_{\ddot{g}}$-continuous [2] if the inverse image of every $I_{\ddot{g}}$-closed set in Y is closed in X.

(3). Perfectly $I_{\ddot{g}}$-continuous [2] if the inverse image of every $I_{\ddot{g}}$-open set in (Y, σ, J) is both open and closed in (X, τ).

(4). $I_{\ddot{g}}$-totally continuous function [3] if the inverse image of every $I_{\ddot{g}}$-open subset of Y is clopen in X.

(5). totally $I_{\ddot{g}}$-continuity [3] if $f^{-1}(V)$ is $I_{\ddot{g}}$-clopen in X for each open set V in (Y, σ, J).

Lemma 2.4. *Every closed set in (X, τ, I) is $I_{\ddot{g}}$-closed set [2].*

Definition 2.5. *Let (X, τ, I) be a ideal topological space $A \subset X$. The intersection of all $I_{\ddot{g}}$-closed supersets of A is called the closure of A and is denoted by $Cl_{I_{\ddot{g}}}(A)$.*

3. $I_{\ddot{g}} - T_0$ Space

In this section,we introduce the concept of I$_{\ddot{g}} - T_0$ space in Ideal Topological Spaces and their properties are discussed.

Definition 3.1. *An ideal Topological space (X, τ, I) is said to be $I_{\ddot{g}} - T_0$ space if for each pair of distinct points x, y of X, there exists an $I_{\ddot{g}}$ open set containing one of the point but not the other.*

Theorem 3.2. *An ideal topological space (X, τ, I) is an $I_{\ddot{g}} - T_0$ space if and only if $I_{\ddot{g}}$-closures of distinct points are distinct.*

Proof. Let x and y be two distinct points in X and X be an I$_{\ddot{g}} - T_0$ space. Then there exists an I$_{\ddot{g}}$-open set G such that $x \in G$ but $y \notin G$. Also $x \notin G^c$ and $y \in G^c$ where G^c is an I$_{\ddot{g}}$-closed set in X. Since I$_{\ddot{g}} - cl(\{y\})$ is the intersection of all I$_{\ddot{g}}$-closed sets which contain y, $y \in I_{\ddot{g}} - cl(\{y\})$ but $x \notin I_{\ddot{g}} - cl(\{y\})$ as $x \notin G^c$. Thus $I_{\ddot{g}} - cl(\{x\}) \neq I_{\ddot{g}} - cl(\{y\})$.
Conversely, Suppose that for any pair of distinct points x and y in X. $I_{\ddot{g}} - cl(\{x\}) \neq I_{\ddot{g}} - cl(\{y\})$. Then there exists atleast one point $z \in X$ such that $z \in I_{\ddot{g}} - cl(\{x\})$ but $z \notin I_{\ddot{g}} - cl(\{y\})$.
claim: $x \notin I_{\ddot{g}} - cl(\{y\})$. If $x \in I_{\ddot{g}} - cl(\{y\})$ then $\{x\} \subset I_{\ddot{g}} - cl(\{y\}) \Rightarrow I_{\ddot{g}} - cl(\{x\}) \subset I_{\ddot{g}} - cl(I_{\ddot{g}} - cl(\{y\}))$. $\Rightarrow I_{\ddot{g}} - cl(\{x\}) \subset I_{\ddot{g}} - cl(\{y\})$. Therefore $z \in I_{\ddot{g}} - cl(\{x\}) \Rightarrow z \in I_{\ddot{g}} - cl(\{y\})$.which is a contradiction.$\Rightarrow x \notin I_{\ddot{g}} - cl(\{y\})$. Now $x \notin I_{\ddot{g}} - cl(\{y\}) \Rightarrow x \in (I_{\ddot{g}} - cl(\{y\}))^c$,which is $I_{\ddot{g}}$-open. Thus $(I_{\ddot{g}} - cl(\{y\}))^c$ is $I_{\ddot{g}}$-open set containing x but not y. Hence X is $I_{\ddot{g}} - T_0$ space. □

Theorem 3.3 (Hereditary Property)**.** *Every subspace of a $I_{\ddot{g}} - T_0$ space is $I_{\ddot{g}} - T_0$ space.*

Proof. let X be a $I_{\ddot{g}} - T_0$ space and Y be a subset of X. Let x, y be two distinct points of Y. Since $Y \subseteq X$ and X is $I_{\ddot{g}} - T_0$ space, there exists an $I_{\ddot{g}}$-open set G such that $x \in G$ but $y \notin G$. Then there exists an $I_{\ddot{g}}$-open set $G \cap Y$ in Y which contains x but not contain y (By definition of subspace). Hence Y is a $I_{\ddot{g}} - T_0$ space. □

Theorem 3.4. *Every T_0 space is a $I_{\ddot{g}} - T_0$ space.*

Remark 3.5. *The converse of the above theorem need not be true as seen from the following example.*

Example 3.6. *Consider the ideal topological space(X, τ, I), where $X = \{a, b, c\}$ with $\tau = \{\emptyset, X, \{a, b\}\}$ and $I = \{\emptyset, \{a\}\}$. Then X is a $I_{\ddot{g}} - T_0$ but not T_0 space, since a and b are contained by all open sets of X.*

Theorem 3.7. *Let $f : (X, \tau, I) \to (Y, \sigma, J)$ be an injective map and Y is $I_{\ddot{g}} - T_0$. If f is $I_{\ddot{g}}$-totally continuous then X is ultra-Hausdorff.*

Proof. Let x and y be any two disjoint points in X. Since X is injective, $f(x)$ and $f(y) \in Y$ such that $f(x) \neq f(y)$. Since Y is $I_{\ddot{g}} - T_0$ space, there exists an $I_{\ddot{g}}$-open set U containing $f(x)$ but not $f(y)$. We have $x \in f^{-1}(U)$ and but $y \notin f^{-1}(U)$. Thus $x \in f^{-1}(U), y \in (f^{-1}(U))^c$ and $f^{-1}(U)$ is clopen in X because f is $I_{\ddot{g}}$-totally continuous. Implies that every pair of distinct points of X can be separated by distinct clopen sets in X. Therefore X is ultra-Hausdorff. □

Theorem 3.8. *Let $f : (X, \tau, I) \to (Y, \sigma, J)$ be an $I_{\ddot{g}}$-irresolute, bijective map. If Y is an$I_{\ddot{g}-T_0}$ space, then X is $I_{\ddot{g}} - T_0$ space.*

Proof. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since f is bijective, there exists $y_1, y_2 \in X$ with $y_1 \neq y_2$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Implies $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. Since Y is a $I_{\ddot{g}} - T_0$ space, there exists an $I_{\ddot{g}}$-open set M in Y such that $y_1 \in M$ and $y_2 \notin M$. Since f is $I_{\ddot{g}}$-irresolute, $f^{-1}(M)$ is $I_{\ddot{g}}$-open set in X. Now we have $y_1 \in M \Rightarrow f^1(y_1) \in f^{-1}(M) \Rightarrow x_1 \in f^{-1}(M)$ and $y_2 \notin M \Rightarrow f^1(y_2) \notin f^{-1}(M) \Rightarrow x_2 \notin f^{-1}(M)$. Hence for any two disjoint points x_1, x_2 in X, there exists $I_{\ddot{g}}$-open set $f^{-1}(M)$ in X such that $x_1 \in f^{-1}(M)$ and $x_2 \notin f^{-1}(M)$. Hence X is a $I_{\ddot{g}} - T_0$ space. □

Theorem 3.9. *Let $f : (X, \tau, I) \to (Y, \sigma, J)$ be a bijection, $I_{\ddot{g}}$-continuous and Y be a T_0 space, then X is a $I_{\ddot{g}} - T_0$ space.*

Proof. let $f : (X, \tau, I) \to (Y, \sigma, J)$ be bijection, $I_{\ddot{g}}$-continuous and Y is T_0 space. To prove that X is a $I_{\ddot{g}} - T_0$ space. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since f is a bijection, there exists $y_1, y_2 \in Y$ with $y_1 \neq y_2$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$ Implies $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. Since Y is T_0-space, there exists a open set M in X such that $y_1 \in M$ and $y_2 \notin M$. Since f is $I_{\ddot{g}}$-continuous, $f^1(M)$ is a $I_{\ddot{g}}$-open set in Y. Since f is $I_{\ddot{g}}$-continuous, $f^{-1}(M)$ is a $I_{\ddot{g}}$-open set in Y. Now we have $y_1 \in M \Rightarrow x_1 \in f^{-1}(M)$ and $y_2 \notin M \Rightarrow f^{-1}(y_2) \notin f^{-1}(M) \Rightarrow x_2 \notin f^{-1}(M)$.Hence any two distinct point $x_1, x_2 \in X$, there exists an $I_{\ddot{g}}$-open set $f^{-1}(M)$ in X such that $x_1 \in f^{-1}(M)$ but $x_2 \notin f^{-1}(M)$. Hence X is an $I_{\ddot{g}} - T_0$ space. □

4. $I_{\ddot{g}} - T_1$ Space

In this section,we introduce the concept of $I_{\ddot{g}} - T_1$ space in Ideal Topological Spaces and their properties are discussed.

Definition 4.1. *An ideal topological space (X, τ, I) is said to be $I_{\ddot{g}} - T_1$ space if for each pair of distinct points x, y of X, there exists a pair of $I_{\ddot{g}}$-open sets one containing x but not y and the other containing y but not x.*

Theorem 4.2 (Hereditrary Property)**.** *Every subspace of an $I_{\ddot{g}} - T_1$ space is also an $I_{\ddot{g}} - T_1$ space.*

Proof. Let X be an $I_{\ddot{g}} - T_1$ space and Y be a subspace of X. Let $x, y \in Y \subseteq X$ such that $x \neq y$. By hypothesis X is $I_{\ddot{g}} - T_1$ space, then by definition there exists $I_{\ddot{g}}$-open set U, V in X such that $x \in U, y \in V, x \notin V$ and $y \notin U$. By definition of subspace, $U \cap Y$ and $V \cap Y$ are $I_{\ddot{g}}$-open sets in Y. Further $x \in U, x \in Y \Rightarrow x \in U \cap Y$ and $y \in V, y \in Y \Rightarrow y \in V \cap Y$. Then there exists an $I_{\ddot{g}}$-open sets $U \cap Y$ and $V \cap Y$ in Y such that $x \in U \cap Y, y \in V \cap Y$ and $x \notin V \cap Y, y \notin U \cap Y$. Hence Y is a $I_{\ddot{g}} - T_1$ space. □

Theorem 4.3. *Every T_1 space is an $I_{\ddot{g}} - T_1$ space.*

Remark 4.4. *The converse of the above theorem need not be true as seen from the following example.*

Example 4.5. *Consider the ideal topological space(X, τ, I) where $X = \{a, b, c\}$ with $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$ and $I = \{\emptyset, \{a\}, \{a, c\}\}$. Then X is $I_{\ddot{g}} - T_1$ space but not T_0 space, since there is no open set containing a but not containing b.*

Theorem 4.6. *Every $I_{\ddot{g}} - T_1$ space is also an $I_{\ddot{g}} - T_0$ space.*

Proof: Suppose X is an $I_{\ddot{g}} - T_1$ space, then for any pair of distinct points x and y in X, there exists $I_{\ddot{g}}$-open sets G and H such that $x \in G, Y \notin G$ and $x \notin H, y \in H$. Thus there exists an $I_{\ddot{g}}$-open set containing one of the point but not the other. Hence X is an $I_{\ddot{g}} - T_0$ space.

Remark 4.7. *The converse of the above theorem need not be true as seen from the following example.*

Example 4.8. *Consider the ideal topological space(X, τ, I), where $X = \{a, b, c\}$ with $\tau = \{\emptyset, X, \{a, b\}\}$ and $I = \{\emptyset, \{a\}\}$. Then X is $I_{\ddot{g}} - T_0$ but not $I_{\ddot{g}} - T_1$ since for the distinct point of a and b, there exist a pair of $I_{\ddot{g}}$-open sets $\{a\}$ and $\{a, b\}$ one containing a and the other containing both a and b.*

Theorem 4.9. *Let $f : (X, \tau, I) \to (Y, \sigma, J)$ be an injective map and Y be an $I_{\ddot{g}} - T_1$ space. If f is $I_{\ddot{g}}$-irresolute then X is an $I_{\ddot{g}} - T_1$ space.*

Proof. Assume that Y is a $I_{\ddot{g}} - T_1$ space.To Prove X is an $I_{\ddot{g}} - T_1$ space. Let $x, y \in X$ where $x \neq y$. Since f is injective, $f(x) \neq f(y)$. Then there exists an $I_{\ddot{g}}$-open sets U, V in Y such that $f(x) \in U$ and $f(y) \in V, f(x) \notin V, f(y) \notin V$ implies $x \in f^{-1}(U), y \in f^{-1}(V)$ and $x \notin f^{-1}(V), y \notin f^{-1}(U)$. Since f is $I_{\ddot{g}}$-irresolute, $f^{-1}(U), f^{-1}(V)$ are $I_{\ddot{g}}$-open sets in X. Therefore for any two distinct points $x, y \in X$, there exists $I_{\ddot{g}}$-open sets $f^{-1}(U), f^{-1}(V)$ in X such that $x \in f^{-1}(U), y \in f^{-1}(V)$ and $x \notin f^{-1}(V), y \notin f^{-1}(U)$. Hence X is an $I_{\ddot{g}} - T_1$ space. □

Theorem 4.10. *If $f : (X, \tau, I) \to (Y, \sigma, J)$ is $I_{\ddot{g}}$-totally continuous, injective and Y is $I_{\ddot{g}} - T_1$, then X is clopen T_1.*

Proof. Let x and y be any two distinct points in X. Since f is injective, $f(x)$ and $f(y)$ are in Y such that $f(x) \neq f(y)$. Since Y is $I_{\ddot{g}} - T_1$, there exists an $I_{\ddot{g}}$-open sets U and V in Y such that $f(x) \in U, f(y) \notin U$ and $f(y) \in V, f(x) \notin V$. We have, $x \in f^{-1}(U), y \notin f^{-1}(U)$ and $y \in f^{-1}(V), x \notin f^{-1}(V)$. Since f is $I_{\ddot{g}}$-totally continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are clopen subset of X.Implies $x \in f^{-1}(U), y \in f^{-1}(V)$ and $x \notin f^{-1}(V), y \notin f^{-1}(U)$ are clopen in X. Hence X is clopen T_1. □

Theorem 4.11. *$\{x\}$ is $I_{\ddot{g}}$-closed in X, for every $x \in X$ if and only if X is $I_{\ddot{g}} - T_0$ space.*

Proof. Let x, y be two distinct points of X such that $\{x\}$ and $\{y\}$ are $I_{\ddot{g}}$-closed. Then $\{x\}^c$ and $\{y\}^c$ are $I_{\ddot{g}}$-open in X such that $y \in \{x\}^c$ but $x \notin \{x\}^c$ and $x \in \{y\}^c$ but $y \notin \{y\}^c$. Hence X is $I_{\ddot{g}} - T_1$ space.
Conversely, Assume that X is $I_{\ddot{g}} - T_1$. To prove that $\{x\}$ is $I_{\ddot{g}}$-closed in X. Let y be any point distinct from x. Then $x \neq y$.Since X is $I_{\ddot{g}} - T_1$ space, there exists $I_{\ddot{g}}$-open sets U and V such that $x \in U, y \in V$ and $x \notin V, y \notin U$. Therefore V is neighbourhood of y does not contains x. y is not a accumulation point of $\{x\} \Rightarrow D(\{x\}) = \emptyset$ Therefore $\overline{\{x\}} = \{x\} \cup D\{x\} = \{x\} \cup \emptyset = \{x\}$. Therefore $\overline{\{x\}} = \{x\}$. Hence $\{x\}$ is $I_{\ddot{g}}$-closed set. □

5. $I_{\ddot{g}} - T_2$ Space

In this section,we introduce the concept of $I_{\ddot{g}} - T_2$ space in Ideal Topological Spaces and their properties are discussed.

Definition 5.1. *An ideal topological space (X, τ, I) is said to be $I_{\ddot{g}} - T_2$ space (or) Hausdorff space if for each pair of distinct points x, y of X, there exists disjoint $I_{\ddot{g}}$-open sets U and V such that $x \in U$ and $y \in V$.*

Theorem 5.2. *Every T_2 space is an $I_{\ddot{g}} - T_2$ space.*

Remark 5.3. *The converse of the above theorem need not be true as seen from the following example.*

Example 5.4. *Consider the ideal topological space(X, τ, I), where $X = \{a, b, c, d\}$ with $\tau = \{\emptyset, X, \{a\}, \{b, d\}, \{a, b, d\}\}$ and $I = \{\emptyset, \{a\}\}$. Then X is $I_{\ddot{g}} - T_2$ space T_2 space because the intersection of open sets $\{a\}$ and $\{a, b, d\}$is not empty.*

Theorem 5.5. *Every $I_{\ddot{g}} - T_2$ space is $I_{\ddot{g}} - T_1$ space.*

Proof. Suppose X is $I_{\ddot{g}} - T_2$ space, then for distinct points x and y in X, there exists $I_{\ddot{g}}$-open set G and H such that $G \cap H = \emptyset$ and $x \in G, y \in H$. Therefore $x \in G, y \notin G$ and $y \in H, x \notin H$. Thus X is an $I_{\ddot{g}} - T_1$ space. ☐

Remark 5.6. *The converse of the above theorem need not be true as seen from the following example.*

Example 5.7. *Consider the ideal topological space(X, τ, I), where $X = \{a, b, c, d\}$ with $\tau = \{\emptyset, X, \{a\}, \{b, d\}, \{a, b, d\}\}$ and $I = \{\emptyset, \{b\}, \{d\}, \{b, d\}\}$. Then X is $I_{\ddot{g}} - T_1$ space $I_{\ddot{g}} - T_2$ space because the intersection of $I_{\ddot{g}}$-open sets is not empty.*

Theorem 5.8. *Every subspace of an $I_{\ddot{g}} - T_2$ space is also an $I_{\ddot{g}} - T_2$ space.*

Proof. Let X be an $I_{\ddot{g}} - T_2$ space and Y be a subspace of X. Let $a, b \in Y \subseteq X$ with $a \neq b$.By hypothesis, there exists $I_{\ddot{g}}$-open set G, H in X such that $a \in G, b \in H, G \cap H = \emptyset$. By definition of subspace , $G \cap Y$ and $H \cap Y$ are $I_{\ddot{g}}$-open sets in Y. Further $a \in G, a \in Y \Rightarrow a \in G \cap Y$ and $b \in H, b \in Y \Rightarrow b \in H \cap Y$. Consider,$(Y \cap G) \cap (Y \cap H) = Y \cap (G \cap H) = Y \cap \emptyset = \emptyset$(since $G \cap H = \emptyset$). Therefore $(Y \cap G) \cap (Y \cap H) = \emptyset$. Therefore $Y \cap G$ and $Y \cap H$ are disjoint $I_{\ddot{g}}$-open sets in Y such that $a \in G \cap Y$ and $b \in H \cap Y$. Hence Y is an $I_{\ddot{g}} - T_2$ space. ☐

Theorem 5.9. *If $f : (X, \tau, I) \to (Y, \sigma, J)$ is $I_{\ddot{g}}$-totally continuous, injective and Y is $I_{\ddot{g}} - T_2$ space, then X is ultra-Hausdorff.*

Proof. Let x and y be any two disjoint points in X. Since f is injective, $f(x)$ and $f(y) \in Y$ such that $f(x) \neq f(y)$. Since Y is $I_{\ddot{g}} - T_2$ space, there exists $I_{\ddot{g}}$-open sets U and V such that $f(x) \in U, f(y) \in V$ and $U \cap V = \emptyset \Rightarrow x \in f^{-1}(U)$ and $y \in f^{-1}(V)$. Since f is $I_{\ddot{g}}$-totally continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are clopen sets in X. Also $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\emptyset) = \emptyset$. Thus every pair of distinct points of X can be separated by disjoint clopen sets. Thus X is ultra-Hausdorff. ☐

Theorem 5.10. *If $\{x\}$ is an $I_{\ddot{g}}$-closed set in X, for every $x \in X$ then X is an $I_{\ddot{g}} - T_2$ space.*

Proof. Let x and y be two distinct points of X such that $\{x\}$ and $\{y\}$ are $I_{\ddot{g}}$-closed sets in X. Then $\{x\}^c$ and $\{y\}^c$ are $I_{\ddot{g}}$-open in X such that $x \in \{y\}^c$ and $y \in \{x\}^c$. Hence X is $I_{\ddot{g}} - T_2$ space. ☐

Theorem 5.11. *If X is an $I_{\ddot{g}} - T_2$ space, then $y \neq x \in X$, there exists an $I_{\ddot{g}}$-open set G such that $x \in G$ and $y \notin cl_{I_{\ddot{g}}}(G)$.*

Proof. Let $x, y \in X$ with $y \neq x$. Since X is $I_{\ddot{g}} - T_2$ space, there exists disjoint $I_{\ddot{g}}$-open sets G and H in X such that $x \in G$ and $y \in H$. Therefore H^c is $I_{\ddot{g}}$-closed set such that $cl_{I_{\ddot{g}}}(G) \subseteq H^c$. Since $y \in H$, we have $y \notin H^c$. Hence $y \notin cl_{I_{\ddot{g}}}(G)$. ☐

Definition 5.12. *An ideal topological space (X, τ, I) is $I_{\ddot{g}} - Q_1$ space if for any $x, y \in X$ with $cl_{I_{\ddot{g}}}(\{x\}) \neq cl_{I_{\ddot{g}}}(\{y\})$ then there exists $I_{\ddot{g}}$-open sets U and V such that $cl_{I_{\ddot{g}}}(\{x\}) \subseteq U$ and $cl_{I_{\ddot{g}}}(\{y\}) \subseteq V$.*

Theorem 5.13. *If (X, τ, I) is an $I_{\ddot{g}} - T_2$ space then it is an $I_{\ddot{g}} - Q_1$ space.*

Proof. Let x and y be two distinct points of the set X such that $cl_{I_{\ddot{g}}}(\{x\}) \neq cl_{I_{\ddot{g}}}(\{y\})$. Then $\{x\}$ and $\{y\}$ are $I_{\ddot{g}}$-closed set and so $\{x\} = cl_{I_{\ddot{g}}}(\{x\}), \{y\} = cl_{I_{\ddot{g}}}(\{y\})$. Since X is $I_{\ddot{g}} - T_2$ space , there exists disjoint $I_{\ddot{g}}$-open sets U and V such that $x \in U$ and $y \in V$. Therefore $cl_{I_{\ddot{g}}}(\{x\}) \subseteq U$ and $cl_{I_{\ddot{g}}}(\{y\}) \subseteq V$. Hence X is $I_{\ddot{g}} - Q_1$ space. □

Definition 5.14. *An ideal topological space (X, τ, I) is $I_{\ddot{g}} - Q_2$ space if for every $I_{\ddot{g}}$-closed set $F \subseteq X$ and any point $x \in X - F$, there exists disjoint open sets $U, V \subseteq X$ and $x \in U$ and $F \subseteq V$.*

Theorem 5.15. *For any ideal space (X, τ, I) if $x \in G \subseteq X$ and G is $I_{\ddot{g}}$-open set, there exists an $I_{\ddot{g}}$-open set $H \subseteq X$ such that $x \in H \subseteq cl_{I_{\ddot{g}}}(H) \subseteq G$. Then X is $I_{\ddot{g}} - Q_2$ space.*

Proof. Let $F \subseteq X$ be $I_{\ddot{g}}$-closed set with $x \in F^c$. Since F^c is an $I_{\ddot{g}}$-open set by our assumption, choose an $I_{\ddot{g}}$-open set H with $x \in H \subseteq cl_{I_{\ddot{g}}}(H) \subseteq X - F$. Let $K = X - cl_{I_{\ddot{g}}}(H)$ and so K is $I_{\ddot{g}}$-open. Further $F \subseteq X - cl_{I_{\ddot{g}}}(H) = K$ and $H \cap K = \emptyset$. Hence X is $I_{\ddot{g}} - Q_2$ space. □

Theorem 5.16. *If $f : (X, \tau, I) \to (Y, \sigma, J)$ is totally $I_{\ddot{g}}$-continuous, injection and Y is a T_0 space then X is an $I_{\ddot{g}} - T_2$ space.*

Proof. Let x and y be any two distinct points in X. Since f is injective, we have $f(x)$ and $f(y)$ in Y such that $f(x) \neq f(y)$. Since Y is T_0 space, there exists open set U containing $f(x)$ but not $f(y)$. Then $x \in f^{-1}(U)$ and $y \notin f^{-1}(U)$. Since f is totally $I_{\ddot{g}}$-continuous, $f^{-1}(U)$ is an $I_{\ddot{g}}$-clopen subset of X. Also $x \in f^{-1}(U)$ and $y \in (f^{-1}(U))^c$. Therefore X is $I_{\ddot{g}} - T_2$ space. □

Theorem 5.17. *Product of two $I_{\ddot{g}} - T_0$ space is a $I_{\ddot{g}} - T_0$ space.*

Proof. Let X and Y be two ideal topological spaces and let $X \times Y$ be their product space. If x and y be distinct points of X. Since X is $I_{\ddot{g}} - T_0$, there exists an $I_{\ddot{g}}$-open set U in X such that it contains one of these two and not the other. Let (x_1, y_1) and (x_2, y_2) be any two distinct points $X \times Y$ then either $x_1 \neq x_2$ (or) $y_1 \neq y_2$. If $x_1 \neq x_2$ and since X is $I_{\ddot{g}} - T_0$ space, there exists an $I_{\ddot{g}}$-open set U in X such that $x_1 \in U$ and $x_2 \notin U$. Then $U \times Y$ is $I_{\ddot{g}}$-open set containing (x_1, y_1) but not containing (x_2, y_2). Similarly, If $y_1 \neq y_2$ and since Y is $I_{\ddot{g}} - T_0$ space, there exists an $I_{\ddot{g}}$-open set V in Y such that $y_1 \in V$ and $y_2 \notin V$. Then $X \times V$ is $I_{\ddot{g}}$-open set containing (x_1, y_1) but not containing (x_2, y_2). Hence corresponding to distinct points of $X \times Y$, there exists an $I_{\ddot{g}}$-open set containing one but not the other. implies $X \times Y$ is a $I_{\ddot{g}} - T_0$ space. □

Theorem 5.18. *Product of two $I_{\ddot{g}} - T_1$ space is a $I_{\ddot{g}} - T_1$ space.*

Proof. Let X and Y be two ideal topological spaces. Let $X \times Y$ be their product space. Let (x, y) be an arbitrary point of $X \times Y$ such that $x \in X$ and $y \in Y$. Since X and Y are $I_{\ddot{g}} - T_1$ space.(By theorem 5.10) $\{x\}$ and $\{y\}$ is $I_{\ddot{g}}$-closed in X and Y respectively. Hence $X \backslash \{x\}$ and $Y \backslash \{y\}$ is $I_{\ddot{g}}$-open sets in X and Y. Then $(X, Y) \backslash (x, y)$ is $I_{\ddot{g}}$-open set. Thus $\{(x, y)\}$ is $I_{\ddot{g}}$-closed. □

Theorem 5.19. *Product of two $I_{\ddot{g}} - T_2$ space is a $I_{\ddot{g}} - T_2$ space.*

Proof. Let X and Y be two ideal topological spaces and let $X \times Y$ be their product space. If x and y be distinct points of X. Let (x_1, y_1) and (x_2, y_2) be any two distinct points of $X \times Y$ then either $x_1 \neq x_2$ (or) $y_1 \neq y_2$. If $x_1 \neq x_2$ and since X is $I_{\ddot{g}} - T_2$ space, there exists an $I_{\ddot{g}}$-open sets U, V in X such that $x_1 \in U, x_2 \in V$ and $U \cap V = \emptyset$. Hence $U \times Y$ and $V \times Y$ are $I_{\ddot{g}}$-open sets containing (x_1, y_1) and (x_2, y_2) respectively such that $(U \times Y) \cap (V \times Y) = (U \cap V) \times Y = \emptyset \times Y = \emptyset$. Hence $(X \times Y)$ is $I_{\ddot{g}} - T_2$ space. □

6. Conclusion

In this paper we concentrated on $I_{\ddot{g}} - T_0$,$I_{\ddot{g}} - T_1$,$I_{\ddot{g}} - T_2$,$I_{\ddot{g}} - Q_1$ and $I_{\ddot{g}} - Q_2$ spaces and also their properties. Further we proposed to introduced I$_{\ddot{g}}$-compactness,Higher separation axioms in ideal topological spaces.

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