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On Proper 2-rainbow Domination in Graphs

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Abstract: For a graph G, let $f : V(G) \to \mathcal{P}(\{1, 2, ..., k\})$ be a function. If for each vertex $v \in V(G)$ such that $f(v) = \phi$ we have $\cup_{u \in N(v)} f(u) = \{1, 2, ..., k\}$, then f is called a k-rainbow dominating function (or simply kRDF) of G. The weight $w(f)$, of a kRDF f is defined as $w(f) = \sum\limits_{v \in V(G)} |f(v)|$. The minimum weight of a kRDF of G is called the k-rainbow domination number of G, and is denoted by $\gamma_{rk}(G)$. In this paper we define and study a new domination called proper k-rainbow domination. A k-rainbow dominating function is called a proper k-rainbow dominating function if for every pair of adjacent vertices u and v, $f(u) \not\subseteq f(v)$ and $f(v) \not\subseteq f(u)$. The weight, $w(f)$, of a proper kRDF f is defined as $w(f) = \sum\limits_{v \in V(G)} |f(v)|$. The minimum weight of a proper kRDF of G is called the proper k-rainbow domination number of G, and is denoted by $\gamma_{prk}(G)$. The bounds for 2-rainbow domination and proper 2-rainbow domination for different classes of graphs namely cycles, complete multipartite graph, $P_n \times P_m$ and Harary graph are found.

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Keywords: Rainbow domination, proper rainbow domination, Harary graph.

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph of order n. We denote the open neighborhood of a vertex v of G by $N_G(v)$, or just $N(v)$, and its closed neighborhood by $N[v]$. For a vertex set $S \subseteq V(G)$, we let $N(S) = \cup_{v \in S} N(v)$ and $N[S] = \cup_{v \in S} N[v]$. A set of vertices S in G is a dominating set if $N[S] = V(G)$. The domination number of G, $\gamma(G)$, is the minimum cardinality of a dominating set of G. For a graph G, let $f : V(G) \to \mathcal{P}(\{1, 2, ..., k\})$ be a function. If for each vertex $v \in V(G)$ such that $f(v) = \phi$ we have $\cup_{u \in N(v)} f(u) = \{1, 2, ..., k\}$, then f is called a k-rainbow dominating function (or simply kRDF) of G. The weight $w(f)$, of a kRDF f is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight of a kRDF of G is called the k-rainbow domination number of G, and is denoted by $\gamma_{rk}(G)$. We denote cartesian product of two graphs G and H by $G \times H$. The Harary graph denoted by $H_{k,n}$ is a graph on the n vertices $\{v_1, v_2, ..., v_n\}$ defined by the following construction:

- If k is even, then each vertex v_i is adjacent to $v_{i\pm1}, v_{i\pm2}, ..., v_{i\pm\frac{k}{2}}$, where the indices are subjected to the wraparound convention that $v_i \cong v_{i+n}$.
- If k is odd and n is even, then $H_{k,n}$ is $H_{k-1,n}$ with additional adjacencies between each v_i and $v_{i+\frac{n}{2}}$ for each i.
- If k and n are both odd, then $H_{k,n}$ is $H_{k-1,n}$ with additional adjacencies $\{v_1, v_{1+\frac{n-1}{2}}\}$, $\{v_1, v_{1+\frac{n+1}{2}}\}$, $\{v_2, v_{2+\frac{n+1}{2}}\}$, $\{v_3, v_{3+\frac{n+1}{2}}\}$, . . ., $\{v_{\frac{n-1}{2}}, v_n\}$.

The concept of rainbow domination was first introduced and studied in [2]. The exact values of 2-rainbow domination

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numbers of several classes of graphs namely paths, cycles, suns and trees are found in [3] and [4]. The bounds of 2-rainbow domination for generalized petersen graphs are discussed in [3] and [6]. Some bounds for classes of graphs namely Harary graph, k-regular graph, $P_1 \times P_m$ are estimated in [1]. The critical concept for 2-rainbow domination in graphs was studied in [5].

2. Proper 2-rainbow Domination and 2-rainbow Domination

2.1. Proper k-rainbow Domination

Assume that there are k different types of weapons. Our aim is that each vertex/location that is not occupied by any weapon has in its neighborhood all the k weapons and adjacent vertices/locations store different weapons so that defence becomes strong when an attack happens at a particular vertex/location. This leads us to the following definition. Let G be a graph and let $f : V(G) \to \mathcal{P}(\{1, 2, ..., k\})$ be a function. Then f is called a proper k- rainbow dominating function if, (i) for each vertex $v \in V(G)$ such that $f(v) = \phi$ we have $\cup_{u \in N(v)} f(u) = \{1, 2, ..., k\}$ and (ii) for every pair of adjacent vertices u and v, $f(u) \nsubseteq f(v)$ and $f(v) \nsubseteq f(u)$ (except ϕ). The weight $w(f)$, of a proper kRDF f is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight of a proper kRDF of G is called the proper k-rainbow domination number of G, and is denoted by $\gamma_{prk}(G)$. Clearly, when $k = 1$ this concept coincides with the ordinary domination and rainbow domination. In this paper we consider the 2-rainbow domination and proper 2-rainbow domination of graphs. The following are some observations on proper 2-rainbow domination.

Observation 2.1. $\gamma_{rk}(G) \leq \gamma_{prk}(G)$.

Observation 2.2. $\gamma_{pr2}(P_n) = \lfloor \frac{n}{2} \rfloor + 1$; $n \geq 1$.

Observation 2.3. $\gamma_{pr2}(C_n) = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$; $n = 2k$, $k \geq 2$.

Observation 2.4. $\gamma_{pr2}(C_n) = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$; $n = 4k - 1$, $k \geq 1$.

Observation 2.5. $\gamma_{pr2}(K_n) = 2; n \geq 1$.

Observation 2.6. $\gamma_{pr2}(F_n) = 2; n \geq 1$, *where* F_n *is the friendship graph.*

2.2. Bounds for 2-rainbow Domination and Proper 2-rainbow Domination

Theorem 2.7. *For* $n_1, n_2, ..., n_m > 1$, $\gamma_{pr2}(K_{n_1,n_2,...,n_m}) = min\{n_1, n_2, ..., n_m\}$.

Proof. Let G be a complete multipartite graph with m partitions V_1, V_2, . . ., V_m of sizes n_1, n_2,. . ., n_m respectively. Without loss of generality, assume that V_1 has least number of vertices. Now we define a proper 2-rainbow dominating function as follows:

Assign $\{1\}$ to u_1 and $\{2\}$ to u_2 where u_1, $u_2 \in V_1$. Since u_1, u_2 are adjacent to all vertices in V_2, . . ., V_m, the vertices in V_2, . . ., V_m can be assigned ϕ. The remaining $n_1 - 2$ vertices in V_1 can be assigned $\{1\}$ or $\{2\}$. Now this is is a proper 2-rainbow dominating function with weight n_1. Hence, $\gamma_{pr2}(G) \leq n_1$. It suffices to show that $\gamma_{pr2}(G) \nless n_1$. Let f be a proper 2-rainbow dominating function with minimum weight. If suppose we assign $\{1\}$ to a vertex u_1 in V_1. Then no vertex in V_2, . . ., V_m can receive the same label, as u_1 is adjacent to all vertices in V_2, . . ., V_m. Therefore, assign $\{2\}$ to a vertex u_2 in V_2. Now all vertices in V_3, . . ., V_m can be assigned ϕ. But the remaining vertices cannot assign $\{2\}$ in V_1 and $\{1\}$ in in V_2. $w(f) \geq n_1 + n_2$. This is a contradiction, since $\gamma_{pr2}(G) \leq n_1$. If suppose $\{1, 2\}$ is assigned to a vertex u_1 in V_1. Then all vertices in the partitions V_2, . . ., V_m can be assigned ϕ. Since V_1 is an independent set, the remaining vertices cannot be assigned ϕ. Therefore, $w(f) \geq n_1 + 1$. This is not possible, since $\gamma_{pr2}(G) \leq n_1$. □

Theorem 2.8. *$\gamma_{r2}(K_{n_1,n_2,...,n_m}) \leq 4$, where max $\{n_1, n_2, ..., n_m\} > 1$ and $m \geq 3$.*

Proof. Let G be a complete multipartite graph with m ($m > 3$) partitions V_1, V_2, . . ., V_m of sizes n_1, n_2,. . ., n_m respectively. Without loss of generality, assume that $n_1 > 1$. Now we construct a 2-rainbow dominating function as follows: Assign $\{1\}$ to u_1 and $\{2\}$ to u_2 where u_1, $u_2 \in V_1$. Assign $\{1\}$ to u_3 and $\{2\}$ to u_4 where $u_3 \in V_2$ and $u_4 \in V_3$. Now we can assign ϕ to all other vertices in G, as these vertices has $\{1,2\}$ in its neighborhood. Clearly this is a 2-rainbow dominating function whose weight is 4. □

Theorem 2.9. *For $k \geq 1$, $\gamma_{pr2}(C_{4k+1}) = \gamma_{r2}(C_{4k+1}) + 1$.*

Proof. Let C_n be the cycle $v_1v_2. . .v_nv_1$, where $n = 4k+1$. Let $f : V(C_n) \to \mathcal{P}(\{1,2\})$ be a function defined as follows: For $1 \leq i \leq n-2$,

$$f(v_i) = \begin{cases} \{1\} & \text{if } i \cong 1 \pmod 4 \\ \{2\} & \text{if } i \cong 3 \pmod 4 \\ \phi & \text{otherwise} \end{cases}$$

$$f(v_{n-1}) = \{1\}$$

$$f(v_n) = \{2\}$$

Now, f is a proper 2-rainbow dominating function of C_n. Therefore,

$$\begin{aligned} w(f) &= 2 + \gamma_{pr2}(P_{n-2}) \\ &= 2 + \lfloor \frac{n-2}{2} \rfloor + 1 \\ &= \lceil \frac{n}{2} \rceil + 1 \\ &= \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor \\ &= \gamma_{r2}(C_n) + 1. \end{aligned}$$

It suffices to prove that $\gamma_{pr2}(C_n) \geq \gamma_{r2}(C_n) + 1$. Let f be a proper 2-rainbow dominating function of C_n with minimum weight. If suppose there is a vertex $x \in C_n$ with $f(x) = \{1,2\}$. Then,

$$\begin{aligned} w(f) \geq 2 + \gamma_{pr2}(P_{n-3}) &= 2 + \lfloor \frac{n-3}{2} \rfloor + 1 \\ &= \lceil \frac{n}{2} \rceil + 1 \\ &= \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor + 1 \\ &= \gamma_{r2}(C_n) + 1. \end{aligned}$$

Assume that $|f(x)| \leq 1 \ \forall x \in C_n$. Then for any pair of adjacent vertices x and y, to at least one of them f assigns a non empty value where x, $y \notin \{v_{n-1}, v_n\}$. Therefore,

$$\begin{aligned} w(f) \geq 2 + \lceil \frac{n-2}{2} \rceil &= \lceil \frac{n}{2} \rceil + 1 \\ &= \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor + 1 \\ &= \gamma_{r2}(C_n) + 1. \end{aligned}$$

□

Theorem 2.10. *$\gamma_{pr2}(P_n \times P_m) \leq \frac{nm}{2}$, where n is even and $m > 1$.*

Proof. It suffices to construct a proper 2-rainbow dominating function of $P_n \times P_m$ with weight $\frac{nm}{2}$. In $P_n \times P_m$, there are nm vertices. Let

$$v_{11}, v_{12}, \ldots, v_{1n}$$
$$v_{21}, v_{22}, \ldots, v_{2n}$$
$$\ldots\ldots\ldots$$
$$v_{m1}, v_{m2}, \ldots, v_{mn}$$

be vertices in $P_n \times P_m$. We define a proper 2-rainbow dominating function $f : V(P_n \times P_m) \to \mathcal{P}(\{1, 2\})$ as follows:
For $1 \leq i \leq m$, $1 \leq j \leq n$,

$$f(v_{ij}) = \begin{cases} \{1\} & \text{if } i \text{ and } j \text{ are odd} \\ \{2\} & \text{if } i \text{ and } j \text{ are even} \\ \phi & \text{otherwise} \end{cases}$$

Since nm is even and every vertex with indices i, j of same parity are labeled with a non empty singleton set, $w(f) = \frac{nm}{2}$. □

Theorem 2.11. *$\gamma_{pr2}(P_n \times P_m) \leq \frac{(n-1)m}{2} + \lfloor \frac{m}{2} \rfloor$, where n is odd and $m > 1$.*

Proof. In $P_n \times P_m$, there are nm vertices. Let

$$v_{11}, v_{12}, \ldots, v_{1n}$$
$$v_{21}, v_{22}, \ldots, v_{2n}$$
$$\ldots\ldots\ldots$$
$$v_{m1}, v_{m2}, \ldots, v_{mn}$$

be vertices in $P_n \times P_m$. We define a proper 2-rainbow dominating function of $P_n \times P_m$ with weight $\frac{(n-1)m}{2} + \lfloor \frac{m}{2} \rfloor$ as follows:
For $1 \leq i \leq m$, $1 \leq j \leq n$,

$$f(v_{ij}) = \begin{cases} \{1\} & \text{if } i \text{ is even and } j \text{ is odd} \\ \{2\} & \text{if } i \text{ is odd and } j \text{ is even} \\ \phi & \text{otherwise} \end{cases}$$

Case (1): Assume that m is even.
Since nm is even and every vertex with indices i, j of opposite parity are labeled with a non empty singleton set, $w(f) = \frac{nm}{2} = \frac{(n-1)m}{2} + \lfloor \frac{m}{2} \rfloor$.
Case (2): Suppose that m is odd.
Since there are $(\frac{m-1}{2})(\frac{n+1}{2})$ vertices with indices i as even, j as odd and $(\frac{m+1}{2})(\frac{n-1}{2})$ vertices with indices i as odd, j as even, $w(f) = (\frac{m-1}{2})(\frac{n+1}{2}) + (\frac{m+1}{2})(\frac{n-1}{2}) = \frac{mn-1}{2} = \frac{(n-1)m}{2} + \lfloor \frac{m}{2} \rfloor$. □

Theorem 2.12. *$\gamma_{pr2}(H_{3,n}) \leq \frac{n}{2}$, where n is even and $n \geq 4$.*

Proof. Clearly it suffices to construct a proper 2-rainbow dominating function of $H_{3,n}$ with weight $\frac{n}{2}$. We define a proper 2-rainbow dominating function $f : V(H_{3,n}) \to \mathcal{P}(\{1, 2\})$ as follows:

For $1 \leq i \leq n$,

$$f(v_i) = \begin{cases} \{1\} & \text{if } i \cong 1 \pmod 4 \\ \{2\} & \text{if } i \cong 3 \pmod 4 \\ \phi & \text{otherwise} \end{cases}$$

Since n is even and every odd indexed vertices are labeled with a non empty singleton set, $w(f) = \frac{n}{2}$. □

Theorem 2.13. *$\gamma_{r2}(H_{k,n}) \leq 2\lceil \frac{n}{k+1} \rceil$, where n is even, k is odd, $k \geq 5$ and $n \neq k+3$.*

Proof. It suffices to define a 2-rainbow dominating function of $H_{k,n}$ with weight $2\lceil \frac{n}{k+1} \rceil$. We define a 2-rainbow dominating function $f : V(H_{k,n}) \to \mathcal{P}(\{1,2\})$ as follows:

Case (1): Suppose that $n \cong 0, k-1 \pmod{k+1}$

For $1 \leq i \leq n$,

$$f(v_i) = \begin{cases} \{1\} & \text{if } i \cong 1 \pmod{k+1} \\ \{2\} & \text{if } i \cong \frac{k+3}{2} \pmod{k+1} \\ \phi & \text{otherwise} \end{cases}$$

If $n \cong 0 \pmod{k+1}$, then $n = p(k+1)$. $w(f) = 2p = \frac{2n}{k+1}$. When $n \cong k-1 \pmod{k+1}$, $\frac{k+3}{2} < k-1$ for $k \geq 5$. Hence, there are exactly $\lceil \frac{n}{k+1} \rceil$ vertices that receive the label $\{1\}$ and $\lceil \frac{n}{k+1} \rceil$ vertices that receive the label $\{2\}$. Therefore, $w(f) = 2\lceil \frac{n}{k+1} \rceil$.

Case (2): Assume that $n \ncong 0, k-1 (\text{mod } k+1)$

For $1 \leq i \leq n-1$,

$$f(v_i) = \begin{cases} \{1\} & \text{if } i \cong 1 \pmod{k+1} \\ \{2\} & \text{if } i \cong \frac{k+3}{2} \pmod{k+1} \\ \phi & \text{otherwise} \end{cases}$$

$$f(v_n) = \{2\}$$

Since there are exactly $\lceil \frac{n}{k+1} \rceil$ vertices that receive the label $\{1\}$ and $\lceil \frac{n}{k+1} \rceil$ vertices that receive the label $\{2\}$, $w(f) = \lceil \frac{n}{k+1} \rceil$. □

Proposition 2.14. *$\gamma_{r2}(H_{k,n}) \leq 3$, where n is even, k is odd, $k \geq 5$ and $n = k+3$.*

Proof. We construct a 2-rainbow dominating function of $H_{k,n}$ with weight 3. We define a 2-rainbow dominating function $f : V(H_{k,n}) \to \mathcal{P}(\{1,2\})$ as follows:

For $1 \leq i \leq n$,

$$f(v_i) = \begin{cases} \{1\} & \text{if } i \cong 1 \pmod{k+1} \\ \{2\} & \text{if } i \cong \frac{k+3}{2} \pmod{k+1} \\ \phi & \text{otherwise} \end{cases}$$

Since $n = k+3$ and $k \geq 5$, there are exactly 2 vertices whose indices are congruent to 1 modulo $k+1$ and 1 vertex whose index is congruent to $\frac{k+3}{2}$ modulo $k+1$. Therefore, $w(f) = 3$. □

Proposition 2.15. *$\gamma_{pr2}(H_{k,n}) = 2$, where $n = k+2$ and n, k are odd.*

Proof. Since there is a vertex of degree $n-1$, assigning $\{1,2\}$ to that vertex will result all other vertices receiving the label ϕ. Hence $w(f) = 2$. It suffices to show that $\gamma_{pr2}(H_{k,n}) \neq 1$. Let f be a proper 2-rainbow dominating function with minimum weight. If suppose $\gamma_{pr2}(H_{k,n}) = 1$. Then only one vertex has $|f(v_i)| = 1$ and all other vertices should be assigned ϕ, which is not possible. □

3. Conclusion

In this paper we defined and discussed on proper 2-rainbow domination in graphs. Also we have found bounds for proper 2-rainbow domination number and 2-rainbow domination number for various classes of graphs namely complete multipartite graphs, Harary graphs, $P_n \times P_m$ and cycles. Further works can be done in this area by finding proper 2-rainbow domination numbers for other classes of graphs and by characterizing graphs G such that $\gamma_{pr2}(G) = \gamma_{r2}(G)$.

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