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Some Fixed Point Theorems in Complex Valued b-metric Space Using Weak Commuting Maps

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Abstract: The aim of the present paper is to establish fixed point theorems for self mappings under weak ϕ- commuting maps in Complex Valued b-metric space.

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1. Introduction

In 1922, Banach proved the following fixed point theorem. Suppose that (X, d) is a Complete metric space and a self-map T of X satisfies $d(Tx, Ty) \leq \lambda d(x, y)$ for all $x, y \in X$ where $\lambda \in [0, 1)$. That is T is a Contractive mapping. Then T has a unique fixed point. Afterward, several mathematicians considered various definitions of contractive mappings and proved several fixed point theorems [2]. In 2011, Azam et.al [1] introduced complex valued metric space Banach Contraction Principle gives appropriate and simple conditions to establish the existence and uniqueness of a solution of an operator equation $Tx = x$. Later a number of papers were published to generalisation of result. Most of these results deal with the generalisation of contractive conditions in metric spaces.There have been a number of generalisations of metric spaces such as G-metric spaces, Pseudo metric spaces and modular metric spaces. Chen and Li [3] have introduced the notion of Banach operator pairs as a new class of non-commuting maps and have proved various best approximation results using some common fixed point theorems for f-non expansive mappings. In 2013,Rao et.al [11] introduced complex valued b-metric space and proved fixed point theorem in this space. After that complex valued b-metric space is developed as a main area of research. In 2011 Azam et al[1] established some fixed point results for contraction condition for pair of mappings satisfying certain condition satisfying a rational expression.The idea intended to define rational expressions which are not meaningful in cone metric spaces and thus many such results of analysis cannot be generalised to cone metric spaces but to complex valued metric space. Several authors studied fixed point theorems in Complex valued metric spaces. The purpose of this paper is to extend weak-ϕ commuting in Complex Valued b-metric space and to obtain fixed point theorems for self mappings satisfying the weakly commuting conditions.

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2. Preliminary Notes

Here we recall the definitions, examples and results which will be used in the following section

Definition 2.1. *Let C set of complex numbers and let $z, w \in C$. Define a partial order relation $\leq$ on C; $Rez \leq Rew$ and $Imz \leq Imw$.*

Definition 2.2. *Let X be a non-empty set.Suppose that the mapping $d : X \times X \to Y$ is called a complex valued b-metric on X if it satisfies the following conditions are satisfied*

(CM1) $d(x, y) > 0$ and $d(x, y) = 0$ if and only if $x = y$

(CM2) $d(x, y) = d(y, x)$ for all $x, y \in X$

(CM3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y \in X$.

Then(X,d) is called a complex valued metric space.

Example 2.3. *Let X be a Complex valued b-metric space. Consider $d(z, w) = 3i|z - w|$. Then d is a complex valued metric.*

Definition 2.4. *Let X be a non-empty set and let $s \geq 1$. Suppose that the mapping $d : X \times X \to C$ is called a complex valued metric on X if it satisfies the following conditions are satisfied*

(CM1) $d(x, y) > 0$ and $d(x, y) = 0$ if and only if $x = y$

(CM2) $d(x, y) = d(y, x)$ for all $x, y \in X$

(CM3) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y \in X$.

Then (X, d) is called a complex valued b-metric space.

If $b = 1$ we get complex valued metric space

Example 2.5. *Let X be a Complex valued b-metric space. Consider $d(x, y) = |x - y|^2 + i|x - y|^2$ for $x, y \in X$ with $s = 2$. Then d is a complex valued b-metric.*

Definition 2.6.

(1). If x_n be a sequence in X and $x \in X$. Then $\{x_n\}$ is said to be cauchy sequence, if $\{x_n\}$ converges to $x \in X$ we denote this by $\lim_{n\to\infty} x_n = x$;

(2). If $c \in X$ with $0 \leq c$ there exist $n \in N$; $d(x_n + m, x_n) \leq c$, where $m \in N$, then x_n is said to be a Cauchy sequence;

(3). If for every Cauchy sequence in X is convergent then (X, d) is said to be a Complete Complex valued b-metric space.

Definition 2.7. *Let (X, d) be a Complex valued b-metric space and $S : X \to X$. Then*

(1). S is said to be sequentially convergent if we have for any sequence $\{x_n\}$, if $\{Sx_n\}$ is Convergent, $\{x_n\}$ also is convergent;

(2). S is said to be subsequentially convergent if, for every sequence $\{x_n\}$ that $\{Sx_n\}$ is convergent, $\{x_n\}$ has a convergent subsequence;

(3). is said to be continuous if $\lim_{n\to\infty} x_n$ implies that $\lim_{n\to\infty} Sx_n = Sx$ for all $\{x_n\}$ in X.

Definition 2.8. *Let f and g be self-maps on a set X, if $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.*

Definition 2.9. *Let f and g are self-maps defined on a set X. Then f and g said to be weakly compatible if they commute at their coincidence points.*

Definition 2.10. *Let X be a non-empty set $f, g : X \to X$ be mappings. A pair (f, g) is called weakly compatible if $x \in X$; $fx = gx$ implies $fgx = gfx$.*

Definition 2.11. *Two self maps A and S of a Complex valued metric space (X, d) is called weakly commuting if $d(ASx, SAx) \leq d(Ax, Sx)$ for all $x \in X$ and $t \geq 0$.*

Definition 2.12. *Two maps A and S of a Complex valued b-metric space (X, d) is called R-weakly commuting if there exist $R > 0$ such that $d(ASx, SAx) \leq Rd(Ax, Sx)$.*

Definition 2.13. *Let (X, d) be a Complex Valued b-metric space and $T, F : X \to X$ be two functions. A mapping S is said to be a F-Contraction if there is $a \in [0, 1)$ such that $d(FTx, FTy) \leq ad(Fx, Fy)$ for all $x, y \in X$.*

Lemma 2.14. *Let (X, d) be a Complex valued b-metric space. A sequence $\{x_n\}$ is said to be convergent to x if $|d(x_n, x)| \to 0$.*

Lemma 2.15. *Let (X, d) be a Complex valued b-metric space. A sequence $\{x_n\}$ is said to be a Cauchy sequence if for all $n \geq m$; $|d(x_n, x_n)| \to 0$.*

3. Main Results

Definition 3.1. *Let (X, d) be a Complex Valued b-metric space with a metric d and for all and $x, y \in X$, the operators $S, T : X \to X$ is called a weakly ϕ commuting maps if it satisfies the condition $d(STx, TSy) \leq d(Sx, Tx) - \phi d(Sx, Tx)$.*

Theorem 3.2. *Let (X, d) be a complete Complex valued b-metric space with $b \geq 1$ and $x, y \in X$. Two maps S and T are weakly ϕ commuting. Then S and T have a common fixed point.*

Proof. Construct a sequence $\{x_n\}$ in X such that $Sx_0 = Tx_1$ for all $n \in N$. $y_n = Sx_{2n}$ and $y_{2n+1} = Sx_{2n+1}$ and $Tx_{2n} = y_{2n}$. Consider

$$\begin{aligned} d(y_{2n+1}, y_{2n}) &= d(STx_{2n}, TSx_{2n}) \\ &\leq d(Sx_{2n}, Tx_{2n}) - \phi[d(Sx_{2n}, Tx_{2n}] \\ &\leq [d(y_{2n+1}, y_{2n}) - \phi[d(y_{2n+1}, y_{2n})]] \end{aligned}$$

$y_{2n} = y_{2n+1}$ for some n. Therefore (y_{2n}) is a Cauchy sequence. If $y_{2n} \neq y_{2n+1}$

$$\begin{aligned} d(y_{2n+1}, y_{2n}) &= d(STx_{2n}, TSx_{2n}) \\ &\leq d(Sx_{2n}, Tx_{2n}) - \phi[d(Sx_{2n}, Tx_{2n})] \\ &\leq [d(y_{2n+1}, y_{2n}) - \phi[d(y_{2n+1}, y_{2n})]] \\ &< d(y_{2n}, y_{2n+1}) \end{aligned}$$

, Which is a contradiction, hence (y_n) is a Cauchy sequence. Next we have $d(y_n, y_{n+1}) \leq k^n d(y_0, y_1)$. We have

$$\begin{aligned} d(y_n, y_m) &\leq s[d(y_n, y_{n+1}) + d(y_{n+1}, y_m)] \\ &\leq sd(y_n, y_{n+1}) + s^2 d(y_{n+1}, y_{n+2}) + s^3 d(y_{n+2}, y_{n+3}) + \cdots + s^n d(y_{m-1}, y_m) \\ &\leq sk^n d(y_0, y_1) + s^2 k^{n+1} d(y_0, y_1) + s^3 k^{n+2} d(y_0, y_1) + \cdots + s^n k^{m-1} d(y_0, y_1) \\ &\leq sk^n [1 + sk + s^2 k^2 + \cdots + s^{n-1} k^{m-n-1}] d(y_0, y_1) \end{aligned}$$

Therefore

$$|d(y_n, y_m)| \leq k^n sd(y_0, y_1)(1 - sk)^{-1} d(y_0, y_1)$$

$d(y_n, y_m) \to 0$ as $n \to \infty$. Thus $\{y_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exist an $u \in X$ such that $\{x_n\} \to u$ as $n \to \infty$. Assume if $\{x_n\} \to u$

$$\begin{aligned} z = d(u, Tu) &\leq s[d(u, y_{2n+1} + d(y_{2n+1}, Tu) \\ &\leq sd(u, y_{2n+1}) + sd(y_{2n+1}, Tu) \\ &\leq sd(u, y_{n+1} + s[d(TSx_{2n}, STx_u)] \\ &\leq sd(u, y_{2n+1}) + s[d(Sx_2n, Su) - \phi(d(Sx_{2n}, Su))] \\ &\leq sd(u, Tu) \end{aligned}$$

Hence(1-s)d(u,Tu)$\leq$ 0. So that u is a fixed point of T. Similarly u is a fixed point of S. Next we have to prove that fixed point is a unique. For that consider two fixed points u and v

$$\begin{aligned} d(u, v) &\leq s[d(u, y_{n+1}) + d(y_{n+1}, v)] \\ &\leq s[d(STu, TSx_{2n}) + d(TSx_{2n+1}, STu)] \\ &\leq 0 \end{aligned}$$

which is a contradiction. Therefore $u = v$, so fixed point is unique. □

Example 3.3. *Let (X, d) be a Complex valued b-metric space with a metric $d(x, y) = |x_1 - x_2| + i|y_1 - y_2|$ Consider $\phi X \to X$ be an increasing map defined by $\phi(x) = x$. Let S and T be two self maps $Sz = \frac{z}{5}$ and $Tz = \frac{z+1}{5}$, where $z = x + iy$. Here S and T satisfies all the conditions of Theorem 3.2, S and T have a unique common fixed point $0 \in X$.*

Definition 3.4. *Let (X, d) be a Complex Valued b-metric space with a metric d and for all and $x, y \in X$, the operators $S, T : X \to X$ is called a R-weakly ϕ commuting maps if it satisfies the condition $d(STx, TSy) \leq R[d(Sx, Tx) - \phi[d(Sx, Tx)]]$.*

Theorem 3.5. *Let (X, d) be a Complex valued b-metric space with a metric d and $b \geq 1$. For $x, y \in X$. Two maps S and T are R-weakly ϕ-commuting, then S and T have a common fixed point.*

Proof. Construct a sequence $\{x_n\}$ in X such that $Sx_0 = Tx_1$ for all $n \in N$. $y_n = Sx_{2n}$ and $y_{2n+1} = Sx_{2n+1}$ and $Tx_{2n} = y_{2n}$. Consider

$$\begin{aligned} d(y_{2n+1}, y_{2n}) &= d(STx_{2n}, TSx_{2n}) \\ &\leq R[d(Sx_{2n}, Tx_{2n}) - \phi[d(Sx_{2n}, Tx_{2n}]] \\ &\leq R[d(y_{2n+1}, y_{2n}) - \phi[d(y_{2n+1}, y_{2n})]] \end{aligned}$$

$y_{2n} = y_{2n+1}$ for some n. Therefore (y_{2n}) is a Cauchy sequence. If $y_{2n} \neq y_{2n+1}$,

$$\begin{aligned} d(y_{2n+1}, y_{2n}) &= d(STx_{2n}, TSx_{2n}) \\ &\leq Rd(Sx_{2n}, Tx_{2n}) - \phi[d(Sx_{2n}, Tx_{2n})] \end{aligned}$$

$$\leq R[d(y_{2n+1},y_{2n})-\phi[d(y_{2n+1},y_{2n})]]$$
$$< Rd(y_{2n},y_{2n+1})$$

Therefore $(1-R)d(y_{2n+1},y_{2n})\leq 0$, where $|R|<1$. Which is a contradiction, hence (y_n) is a Cauchy sequence. Next we have $d(y_n,y_{n+1})\leq k^n d(y_0,y_1)$. We have

$$\begin{aligned}
d(y_n,y_m) &\leq s[d(y_n,y_{n+1})+d(y_{n+1},y_m)]\\
&\leq sd(y_n,y_{n+1})+s^2d(y_{n+1},y_{n+2})+s^3d(y_{n+2},y_{n+3})+\cdots+s^nd(y_{m-1},y_m)\\
&\leq sk^nd(y_0,y_1)+s^2k^{n+1}d(y_0,y_1)+s^3k^{n+2}d(y_0,y_1)+\cdots+s^nk^{m-1}d(y_0,y_1)\\
&\leq sk^n[1+sk+s^2k^2+\cdots+s^{n-1}k^{m-n-1}]d(y_0,y_1)
\end{aligned}$$

Therefore

$$|d(y_n,y_m)|\leq k^nsd(y_0,y_1)(1-sk)^{-1}d(y_0,y_1)$$

$d(y_n,y_m)\to 0$ as $n\to\infty$. Thus $\{y_n\}$ is a Cauchy sequence. Since X is a complete metric space,there exist an $u\in X$ such that $\{x_n\}\to u$ as $n\to\infty$. Assume if $\{x_n\}\to u$,

$$\begin{aligned}
z=d(u,Tu) &\leq s[d(u,y_{2n+1}+d(y_{2n+1},Tu)\\
&\leq sd(u,y_{2n+1})+sd(y_{2n+1},Tu)\\
&\leq sd(u,y_{n+1}+s[d(TSx_{2n},STx_u)]\\
&\leq sR[d(Sx_2n,Tu)-\phi(d(Sx_{2n},Tu))]+sR[d(Tx_2n,Su)-\phi(d(Tx_{2n},Su))
\end{aligned}$$

Hence $(1-sR)d(u,Tu)\leq 0$. So that u is a fixed point of T. Similarly u is a fixed point of S. Next we have to prove that fixed point is a unique. For that consider two fixed points u and v

$$\begin{aligned}
d(u,v) &\leq s[d(u,y_{n+1})+d(y_{n+1},v)]\\
&\leq s[d(STu,TSx_{2n})+d(TSx_{2n+1},STu)]\\
&\leq s[d(STu,TSu)+d(TSu,STu)]\\
&\leq sR[d(Su,Tu)-\phi(d(Su,Tu)]+sR[d(Su,Tu)-\phi(d(Su,Tu)]
\end{aligned}$$

which is a contradiction. Therefore $u=v$, so fixed point is unique. □

Example 3.6. *Let (X,d) be a Complex valued b-metric space with a metric $d(x,y)=|x_1-x_2|+i|y_1-y_2|$ Also $R=\frac{i}{3}$ Consider $\phi X\to X$ be an increasing map defined by $\phi(x)=x$. Let S and T be two self maps $Sz=\frac{z}{5}$ and $Tz=\frac{z+1}{5}$, where $z=x+iy$. Here S and T satisfies all the conditions of Theorem 3.2, S and T have a unique common fixed point $0\in X$.*

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