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Coefficient Estimates for Certain Subclasses of Sakaguchi Type Bi-Univalent Functions

Sanjay Issar[1,*], Kamal Kumar Mishra[2] and Amit Soni[3]

1 Department of Mathematics, Government Dungar College, Bikaner, Rajasthan, India.

2 Commissionarete College Education Rajasthan, Jaipur, Rajasthan, India.

3 Department of Mathematics, Government Engineering College, Bikaner, Rajasthan, India.

Abstract: In this paper, we introduced certain Sakaguchi type subclasses of the function class Σ of analytic and bi-univalent functions associated with quasi-subordination in the open unit disk $\mathbb{U}$. Estimates on the first two Taylor-MacLaurin coefficients $|a_2|$ and $|a_3|$ for functions belonging to these classes are determined. Certain special cases are also indicated.

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1. Introduction and Preliminaries

Let $\mathcal{A}$ be the class of all normalized analytic functions of the form

$$f(z) = z + a_2z^2 + a_3z^3 + ..., \tag{1}$$

which are defined in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also let $\mathcal{S}$ denote a subclass of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. Let $\mathcal{S}_R$ be the subclass of $\mathcal{S}$ consisting of functions satisfying

$$\Re\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0 \ \ (z \in \mathbb{U}). \tag{2}$$

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi [1]. Das and Singh [2] introduced another class $\mathcal{C}_S$ namely, convex functions with respect to symmetric points and satisfying the condition

$$\Re\left(\frac{(zf'(z))'}{(f(z) - f(-z))'}\right) > 0 \ \ (z \in \mathbb{U}). \tag{3}$$

Univalent functions belonging to the class $\mathcal{S}$ are invertible but their inverse functions may not be defined on the entire unit disk $\mathbb{U}$. The Koebe one-quarter theorem (see [3]) ensures that the image of $\mathbb{U}$ under every function $f \in \mathcal{S}$ contains a disk of

* E-mail: sanjayissar@gmail.com

radius $\frac{1}{4}$. Thus every function $f \in \mathcal{S}$ has an inverse (say g), satisfying $g(f(z)) = z$ for all $z \in \mathbb{U}$ and $f(g(w)) = w$, where $|\omega| < r_0(f),\ r_0(f) \geq \frac{1}{4}$. In fact, it can be easily verified that the inverse function g is given by

$$g(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \dots . \tag{4}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. The class of all bi-univalent functions defined in $\mathbb{U}$ is denoted by Σ. Lewin [4] investigated the class Σ of bi-univalent functions and showed that $|a_2| \leq 1.51$ for the functions in the class Σ. Subsequently Brannan and Clunie [5] conjectured that $|a_2| \leq \sqrt{2}$. Also, Netanyahu [6] proved that $\max_{f\in\Sigma} |a_2| = \frac{4}{3}$. Still the coefficient estimate problem is open for each $|a_n|, \quad (n = 3, 4...)$. Brannan and Taha [7] (see also [8]) introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses of univalent functions consisting of starlike, convex and strongly starlike functions and obtained estimates for their initial coefficients. Many researchers have recently introduced and investigated several interesting subclasses of the bi-univalent functions class Σ and they have found non-sharp estimates on the first two Taylor-MacLaurin coefficient $|a_2|$ and $|a_3|$. (See [9]-[18]). In the style of Ma and Minda, Ravichandran [19] introduced the following Sakaguchi type subclasses by means of subordination:

$$\mathcal{S}_s^*(\phi) = \left\{ f \in \mathcal{A} : \ \frac{2zf'(z)}{f(z) - f(-z)} \prec \phi(z) \right\}, \tag{5}$$

where ϕ is an analytic function with positive real part on $\mathbb{U}$, with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk $\mathbb{U}$ onto a region starlike with respect to the 1 and symmetric with respect to real axis. A function $f \in \mathcal{S}_s^*(\phi)$ is called starlike function with respect to symmetric points in $\mathbb{U}$. $\mathcal{C}_s(\phi)$ is the class of convex function $f \in \mathcal{A}$ with respect to symmetric points in $\mathbb{U}$ for which

$$\frac{2(zf'(z))'}{(f(z) - f(-z))'} \prec \phi(z). \tag{6}$$

These classes $\mathcal{S}_s^*(\phi)$ and $\mathcal{C}_s(\phi)$ include several well known subclasses of starlike and convex functions respectively as special cases. With subordination, various Ma-Minda type subclasses of the bi-univalent function class Σ are recently introduced and non sharp estimations on $|a_2|$ and $|a_3|$ are found in several investigations (See [9, 11, 12, 14, 16, 17]).

The concept of subordination was generalized in 1970 by Robertson [20] through introducing a new concept of quasi subordination. For two analytic functions f and ϕ, the function f is quasi subordination to ϕ written as

$$f(z) \prec_q \phi(z) \qquad (z \in \mathbb{U}), \tag{7}$$

if there exist analytic functions ψ and ω, with $|\psi(z)| \leq 1$, $\omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$\frac{f(z)}{\psi(z)} \prec \phi(z), \tag{8}$$

which is equivalent to

$$f(z) = \psi(z)\phi(\omega(z)) \qquad (z \in \mathbb{U}).$$

Observe that if $\psi(z) = 1$, then $f(z) = \phi(\omega(z))$, so that $f(z) \prec \phi(z)$ in $\mathbb{U}$, also if $\omega(z) = z$, then $f(z) = \psi(z)\phi(z)$ and it is said that $f(z)$ is majorized by $\phi(z)$ and written as $f(z) \ll \phi(z)$ in $\mathbb{U}$. Hence it is obvious that the quasi-subordination is generalization of the usual subordination as well as majorization.

The work on quasi-subordination is quite extensive. Infact further generalization of Ma-Minda type subclasses belonging to the class Σ are made by several authors including ([21]-[25]) by means of quasi-subordination. In [16] certain new subclasses of the bi-univalent function class Σ are introduced by unifying the classes $\mathcal{S}_s^*(\phi),\ \mathcal{C}_s(\phi)$ and obtained bounds on $|a_2|$ and $|a_3|$.

In the present paper we extend the work of Lashin [16] by introducing certain Sakaguchi type subclasses of the bi-univalent functions class Σ associated with quasi-subordination in the open disk $\mathbb{U}$. We also obtain bounds on the initial Taylor-MacLaurin coefficients for the functions belonging to these new subclasses. Further we discuss some special cases of these new classes. In this investigation through out the paper, we assume that:

$$\psi(z) = A_0 + A_1 z + A_2 z^2 + ... \quad (|\psi(z)| \leq 1,\ z \in \mathbb{U}) \tag{9}$$

and $\phi(z)$ is an analytic function in $\mathbb{U}$ with the form:

$$\phi(z) = 1 + B_1 z + B_2 z^2 + ... \ (B_1 > 0). \tag{10}$$

In proving our results, we use an inequality of Keogh and Merkes [26] which is given in the following Lemma.

Lemma 1.1. *Let the Schwarz function $\omega(z)$ be given by*

$$\omega(z) = \omega_1 z + \omega_2 z^2 + \omega_3 z^3 + ... \ (z \in \mathbb{U}), \tag{11}$$

then $|\omega| \leq 1$, $|\omega_2 - t\omega_1^2| \leq 1 + (|t| - 1)|\omega_1^2| \leq \max\{1, |t|\}$, where $t \in \mathbb{C}$. The result is sharp for the function $\omega(z) = z$ or $\omega(z) = z^2$.

Definition 1.2. *For $0 \leq \alpha \leq 1$, a function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{S}_{\Sigma}^{\psi,\phi}(\alpha)$, if the following quasi-subordinations hold:*

$$(1-\alpha)\frac{2zf'(z)}{f(z)-f(-z)} + \alpha\frac{2(zf'(z))'}{(f(z)-f(-z))'} - 1 \prec_q (\phi(z) - 1) \quad (z \in \mathbb{U}) \tag{12}$$

and

$$(1-\alpha)\frac{2\omega g'(\omega)}{g(\omega)-g(-\omega)} + \alpha\frac{2(\omega g'(\omega))'}{(g(\omega)-g(-\omega))'} - 1 \prec_q (\phi(\omega) - 1) \quad (\omega \in \mathbb{U}), \tag{13}$$

where $g(\omega) = f^{-1}(\omega)$ and the function ϕ is given by (10).

It follows that a function f is in the class $\mathcal{S}_{\Sigma}^{\psi,\phi}(\alpha)$ if and only if there exists an analytic function ψ with $|\psi(z)| \leq 1$, $(z \in \mathbb{U})$ such that

$$\frac{(1-\alpha)\frac{2zf'(z)}{f(z)-f(-z)} + \alpha\frac{2(zf'(z)')}{(f(z)-f(-z))'} - 1}{\psi(z)} \prec (\phi(z) - 1) \tag{14}$$

and

$$\frac{(1-\alpha)\frac{2\omega g'(\omega)}{g(\omega)-g(-\omega)} + \alpha\frac{2(\omega g'(\omega)')}{(g(\omega)-g(-\omega))'} - 1}{\psi(\omega)} \prec (\phi(\omega) - 1), \tag{15}$$

where $g = f^{-1}$ and ϕ is given by (10) and $z, \omega \in \mathbb{U}$.

2. Coefficient Estimates for the Function Class $\mathcal{S}_{\Sigma}^{\psi,\phi}(\alpha)$

Theorem 2.1. *Let the function $f \in \Sigma$ given by (1) be in the class $\mathcal{S}_{\Sigma}^{\psi,\phi}(\alpha)$. Then*

$$|a_2| \leq \min\left\{\frac{|A_0|B_1}{2(1+\alpha)},\ \sqrt{\frac{|A_0|(B_1+|B_2|)}{2(1+2\alpha)}}\right\} \tag{16}$$

and

$$|a_3| \leq \left\{\frac{1}{2(1+2\alpha)}\{|A_0|(B_1+|B_2|) + |A_1|B_1\}\right\}. \tag{17}$$

Proof. Since $f \in \mathcal{S}_{\Sigma}^{\psi,\phi}(\alpha)$, there exist two analytic functions u, v $\mathbb{U} \to \mathbb{U}$ with $u(0) = v(0) = 0$, $|u(z)| \leq 1$ and $|v(z)| \leq 1$ and a function ψ in $\mathbb{U}$ defined by (9) satisfying

$$(1-\alpha)\frac{2zf'(z)}{f(z)-f(-z)} + \alpha\frac{2(zf'(z))'}{(f(z)-f(-z))'} - 1 = \psi(z)\left[\phi(u(z)) - 1\right] \quad (z \in \mathbb{U}) \tag{18}$$

and

$$(1-\alpha)\frac{2\omega g'(\omega)}{g(\omega)-g(-\omega)} + \alpha\frac{2(\omega g'(\omega))'}{(g(\omega)-g(-\omega))'} - 1 = \psi(\omega)\left[\phi(v(\omega)) - 1\right] \quad (\omega \in \mathbb{U}). \tag{19}$$

Define the function p and q such that

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + b_1 z + b_2 z^2 + ...$$

and

$$q(z) = \frac{1+v(z)}{1-v(z)} = 1 + c_1 z + c_2 z^2 + ...$$

Equivalently

$$u(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[b_1 z + \left(b_2 - \frac{b_1^2}{2}\right) z^2 + ...\right] \tag{20}$$

and

$$v(z) = \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + ...\right]. \tag{21}$$

Here p and q are analytic in $\mathbb{U}$ with $p(0) = 1 = q(0)$ and since $u,\ v : \mathbb{U} \to \mathbb{U}$, the functions p and q have a positive real parts in $\mathbb{U}$ such that $|b_i| \leq 2$ and $|c_i| \leq 2$ $(i = 1, 2)$. In view of (20) and (21), we have

$$(1-\alpha)\frac{2zf'(z)}{f(z)-f(-z)} + \alpha\frac{2(zf'(z))'}{(f(z)-f(-z))'} - 1 = \psi(z)\left[\phi\left(\frac{p(z)-1}{p(z)+1}\right) - 1\right] \quad (z \in \mathbb{U}) \tag{22}$$

and

$$(1-\alpha)\frac{2\omega g'(\omega)}{g(\omega)-g(-\omega)} + \alpha\frac{2(\omega g'(\omega))'}{(g(\omega)-g(-\omega))'} - 1 = \psi(\omega)\left[\phi\left(\frac{q(\omega)-1}{q(\omega)+1}\right) - 1\right] \quad (\omega \in \mathbb{U}). \tag{23}$$

Using (20) and (21) together with (9) and (10), it is evident that

$$\psi(z)\left[\phi\left(\frac{p(z)-1}{p(z)+1}\right) - 1\right] = \frac{1}{2}A_0B_1b_1z + \left[\frac{1}{2}A_1B_1b_1 + \frac{1}{2}A_0B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{A_0B_2}{4}b_1^2\right]z^2 + ... \tag{24}$$

and

$$\psi(\omega)\left[\phi\left(\frac{q(\omega)-1}{q(\omega)+1}\right) - 1\right] = \frac{1}{2}A_0B_1c_1\omega + \left[\frac{1}{2}A_1B_1c_1 + \frac{1}{2}A_0B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{A_0B_2}{4}c_1^2\right]\omega^2 + ... \tag{25}$$

Since the function f and its inverse g are given by (1) and (4) respectively, we have

$$(1-\alpha)\frac{2zf'(z)}{f(z)-f(-z)} + \alpha\frac{2(zf'(z))'}{(f(z)-f(-z))'} - 1 = 2(1+\alpha)a_2z + 2(1+2\alpha)a_3z^2 + ... \tag{26}$$

and

$$(1-\alpha)\frac{2\omega g'(\omega)}{g(\omega)-g(-\omega)} + \alpha\frac{2(\omega g'(\omega))'}{(g(\omega)-g(-\omega))'} - 1 = -2(1+\alpha)a_2\omega + 2(1+2\alpha)(2a_2^2 - a_3)\omega^2 + ... \tag{27}$$

Now using (24) and (26) in (22) and comparing the coefficients of z and z^2, we get

$$2(1+\alpha)a_2 = \frac{1}{2}A_0B_1b_1, \tag{28}$$

$$2(1+2\alpha)a_3 = \frac{1}{2}A_1B_1b_1 + \frac{1}{2}A_0B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{A_0B_2}{4}b_1^2. \tag{29}$$

Similarly from (25) and (27), on comparing the coefficient of ω and ω^2, we get

$$-2(1+\alpha)a_2 = \frac{1}{2}A_0B_1c_1 \tag{30}$$

$$2(1+2\alpha)(2a_2^2 - a_3) = \frac{1}{2}A_1B_1c_1 + \frac{1}{2}A_0B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{A_0B_2}{4}c_1^2. \tag{31}$$

From (28) and (30), we find that

$$b_1 = -c_1, \tag{32}$$

$$a_2 = \frac{A_0B_1b_1}{4(1+\alpha)} = -\frac{A_0B_1c_1}{4(1+\alpha)}. \tag{33}$$

Adding (29) and (31), we obtain

$$4(1+2\alpha)a_2^2 = \frac{1}{2}A_0B_1(b_2 - \frac{b_1^2}{2}) + \frac{1}{2}A_0B_1(c_2 - \frac{c_1^2}{2}) + \frac{A_0B_2}{4}(b_1^2 + c_1^2). \tag{34}$$

Applying Lemma 1.1 with $|b_i| \leq 2$ and $|c_i| \leq 2$ in (33), (34) and (29), we find that

$$|a_2| \leq \frac{|A_0|B_1}{2(1+\alpha)} \tag{35}$$

$$|a_2|^2 \leq \frac{|A_0|(B_1 + |B_2|)}{2(1+2\alpha)} \tag{36}$$

and

$$|a_3| \leq \frac{|A_0|(B_1 + |B_2|) + |A_1|B_1}{2(1+2\alpha)}. \tag{37}$$

Similarly from (31) and (33), we obtain

$$|a_3| \leq \frac{|A_0|^2B_1^2}{2(1+\alpha)^2} + \frac{|A_0|(B_1 + |B_2|) + |A_1|B_1}{2(1+2\alpha)}. \tag{38}$$

In view of (35)-(38), we find the desired assertions for $|a_2|$ and $|a_3|$. This completes the proof of Theorem 2.1. ☐

If we set $\phi(z) = \left(\frac{1+z}{1-z}\right)^{\gamma} = 1 + 2\gamma z + 2\gamma^2 z^2 + \ldots (0 < \gamma \leq 1, z \in \mathbb{U})$ in Definition 1.2, we obtain a new sub class $\mathcal{S}_{\Sigma}^{\psi}(\alpha, \gamma)$ of the bi-univalent function class Σ.

Definition 2.2. *For $0 \leq \alpha \leq 1$ and $0 < \gamma \leq 1$, a function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{S}_{\Sigma}^{\psi}(\alpha, \gamma)$ if the following subordinations are satisfied*

$$(1-\alpha)\frac{2zf'(z)}{f(z) - f(-z)} + \alpha\frac{2(zf'(z))'}{(f(z) - f(-z))'} - 1 \prec_q \left[\left(\frac{1+z}{1-z}\right)^{\gamma} - 1\right]$$

and

$$(1-\alpha)\frac{2\omega g'(\omega)}{g(\omega) - g(-\omega)} + \alpha\frac{2(\omega g'(\omega))'}{(g(\omega) - g(-\omega))'} - 1 \prec_q \left[\left(\frac{1+\omega}{1-\omega}\right)^{\gamma} - 1\right],$$

where $g(\omega) = f^{-1}(\omega)$ and $z, \omega \in \mathbb{U}$.

Using the parameter setting of Definition 2.2 in Theorem 2.1, we get the following corollary:

Corollary 2.3. *For $0 \le \alpha \le 1$ and $0 < \gamma \le 1$, let the function $f \in \mathcal{S}_{\Sigma}^{\psi}(\alpha,\gamma)$ be of the form (1). Then*

$$|a_2| \le \min\left\{\frac{|A_0|\gamma}{(1+\alpha)}, \sqrt{\frac{|A_0|\gamma(1+\gamma)}{1+2\alpha}}\right\}$$

and

$$|a_3| \le \left\{\frac{(|A_0|(1+\gamma)+|A_1|)\gamma}{(1+2\alpha)}\right\}.$$

If we set $\phi(z) = \frac{1+(1-2\nu)z}{1-z} = 1+2(1-\nu)z+2(1-\nu)z^2+...(0 \le \nu < 1, z \in \mathbb{U})$ in Definition 1.2, we obtain a new subclass $\mathcal{S}_{\Sigma,\nu}^{\psi}(\alpha)$.

Definition 2.4. *For $0 \le \alpha \le 1$ and $0 \le \nu < 1$, a function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{S}_{\Sigma,\nu}^{\psi}(\alpha)$ if the following quasi-subordinations hold*

$$(1-\alpha)\frac{2zf'(z)}{f(z)-f(-z)} + \alpha\frac{2(zf'(z))'}{(f(z)-f(-z))'} - 1 \prec_q \left[\left(\frac{1+(1-2\nu)z}{1-z}\right) - 1\right] \quad (z \in \mathbb{U})$$

and

$$(1-\alpha)\frac{2\omega g'(\omega)}{g(\omega)-g(-\omega)} + \alpha\frac{2(\omega g'(\omega))'}{(g(\omega)-g(-\omega))'} - 1 \prec_q \left[\left(\frac{1+(1-2\nu)\omega}{1-\omega}\right) - 1\right], \quad (\omega \in \mathbb{U}),$$

where $g(\omega) = f^{-1}(\omega)$.

Using the parameter setting of Definition 2.4 in Theorem 2.1, we get the following corollary.

Corollary 2.5. *For $0 \le \alpha \le 1$ and $0 \le \nu < 1$, let the function $f \in \mathcal{S}_{\Sigma,\nu}^{\psi}(\alpha)$ be of the form (1). Then*

$$|a_2| \le \min\left\{\frac{|A_0|(1-\nu)}{1+\alpha}, \sqrt{\frac{2|A_0|(1-\nu)}{1+2\alpha}}\right\}$$

and

$$|a_3| \le \frac{(1-\nu)}{1+2\alpha}\left(2|A_0|+|A_1|\right).$$

3. Coefficient Estimates for the Function Class $\mathcal{C}_{\Sigma}^{\psi,\phi}(\alpha)$

Definition 3.1. *For $0 \le \alpha \le 1$, a function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{C}_{\Sigma}^{\psi,\phi}$ if the following quasi-subordinations hold:*

$$\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2(zf'(z))'}{(f(z)-f(-z))'}\right)^{1-\alpha} - 1 \prec_q (\phi(z)-1) \quad (z \in \mathbb{U})$$

and

$$\left(\frac{2\omega g'(\omega)}{g(\omega)-g(-\omega)}\right)^{\alpha}\left(\frac{2(\omega g'(\omega))'}{(g(\omega)-g(-\omega))'}\right)^{1-\alpha} - 1 \prec_q (\phi(\omega)-1) \quad (\omega \in \mathbb{U}),$$

where $g(\omega) = f^{-1}(\omega)$ and the functon ϕ is given by(10).

Theorem 3.2. *Let the function $f \in \Sigma$ given by (1) be in the class $\mathcal{C}_{\Sigma}^{\psi,\phi}(\alpha)$. Then*

$$|a_2| \le \min\left\{\frac{|A_0|B_1}{2(2-\alpha)}, \sqrt{\frac{|A_0|(B_1+|B_2|)}{2(\alpha^2-3\alpha+3)}}\right\} \tag{39}$$

and

$$|a_3| \le \frac{1}{2}\left\{\frac{|A_0|(B_1+|B_2|)}{\alpha^2-3\alpha+3} + \frac{|A_1|B_1}{3-2\alpha}\right\}. \tag{40}$$

Proof. Since $f \in \mathcal{C}_{\Sigma}^{\psi,\phi}(\alpha)$, there exist two analytic functions u, v $\mathbb{U} \to \mathbb{U}$ with $u(0) = v(0) = 0$, $|u(z)| \leq 1$ and $|v(z)| \leq 1$ and a function ψ in $\mathbb{U}$ defined by (9) satisfying

$$\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2(zf'(z))'}{(f(z)-f(-z))'}\right)^{1-\alpha} - 1 = \psi(z)\phi\left[u(z)-1\right] \quad (z \in \mathbb{U}) \tag{41}$$

and

$$\left(\frac{2\omega g'(\omega)}{g(\omega)-g(-\omega)}\right)^{\alpha}\left(\frac{2(\omega g'(\omega))'}{(g(\omega)-g(-\omega))'}\right)^{1-\alpha} - 1 = \psi(\omega)\phi\left[v(\omega)-1\right] \quad (\omega \in \mathbb{U}). \tag{42}$$

Define the function p and q analytic in $\mathbb{U}$ as in (20) and (21) and then proceed similarly up to (25). Also on expanding L.H.S. of (41) and (42), we get

$$\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2(zf'(z))'}{(f(z)-f(-z))'}\right)^{1-\alpha} - 1 = 2(2-\alpha)a_2 z + 2\left[(3-2\alpha)a_3 - \alpha(1-\alpha)a_2^2\right]z^2 + ... \tag{43}$$

and

$$\left(\frac{2\omega g'(\omega)}{g(\omega)-g(-\omega)}\right)^{\alpha}\left(\frac{2(\omega g'(\omega))'}{(g(\omega)-g(-\omega))'}\right)^{1-\alpha} - 1 = -2(2-\alpha)a_2\omega + 2\left[(3-2\alpha)(2a_2^2-a_3) - \alpha(1-\alpha)a_2^2\right]\omega^2 + \dots \tag{44}$$

It follows from (41) and (43) in view of (24) that,

$$2(2-\alpha)a_2 = \frac{1}{2}A_0B_1b_1 \tag{45}$$

$$2\left[(3-2\alpha)a_3 - \alpha(1-\alpha)a_2^2\right] = \frac{1}{2}A_1B_1b_1 + \frac{1}{2}A_0B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{A_0B_2}{4}b_1^2. \tag{46}$$

Similarly from (42) and (44), on comparing the coefficient of ω and ω^2, we get

$$-2(2-\alpha)a_2 = \frac{1}{2}A_0B_1c_1. \tag{47}$$

$$2\left[(3-2\alpha)(2a_2^2-a_3) - \alpha(1-\alpha)a_2^2\right] = \frac{1}{2}A_1B_1c_1 + \frac{1}{2}A_0B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{A_0B_2}{4}c_1^2. \tag{48}$$

From (45) and (47), we obtain

$$b_1 = -c_1 \tag{49}$$

and

$$a_2 = \frac{A_0B_1b_1}{4(2-\alpha)} = -\frac{A_0B_1c_1}{4(2-\alpha)}. \tag{50}$$

Adding (46) and (48), we get

$$4(\alpha^2-3\alpha+3)a_2^2 = \frac{1}{2}A_0B_1(b_2 - \frac{b_1^2}{2}) + \frac{1}{2}A_0B_1(c_2 - \frac{c_1^2}{2}) + \frac{A_0B_2}{4}(b_1^2 + c_1^2). \tag{51}$$

Applying Lemma 1.1 with $|b_i| \leq 2$ and $|c_i| \leq 2$ in (50) and (51), we obtain

$$|a_2| \leq \frac{|A_0|B_1}{2(2-\alpha)}. \tag{52}$$

and

$$|a_2|^2 \leq \frac{|A_0|(B_1 + |B_2|)}{2(\alpha^2-3\alpha+3)} \tag{53}$$

Similarly from (46) and (48), we can obtain

$$|a_3| \leq \frac{1}{2}\left\{\frac{|A_0|(B_1+|B_2|)}{\alpha^2-3\alpha+3} + \frac{|A_1|B_1}{3-2\alpha}\right\}. \tag{54}$$

One more bound on $|a_3|$ can also be obtained from (45) and (46), but that will be greater than that of (54). This completes the proof of Theorem 3.2. □

If we set $\phi(z) = \left(\frac{1+z}{1-z}\right)^{\gamma} = 1 + 2\gamma z + 2\gamma^2 z^2 + ...(0 < \gamma \leq 1, z \in \mathbb{U})$ in Definition 3.1, we obtain a new class $\mathcal{C}_{\Sigma}^{\psi}(\alpha, \gamma)$ of the bi-univalent function class Σ as given below.

Definition 3.3. *For $0 \leq \alpha < 1$ and $0 < \gamma \leq 1$, a function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{C}_{\Sigma}^{\psi}(\alpha, \gamma)$ if the following quasi-subordinations hold*

$$\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2(zf'(z))'}{(f(z)-f(-z))'}\right)^{1-\alpha} - 1 \prec_q \left[\left(\frac{1+z}{1-z}\right)^{\gamma} - 1\right] \quad (z \in \mathbb{U})$$

and

$$\left(\frac{2\omega g'(\omega)}{g(\omega)-g(-\omega)}\right)^{\alpha}\left(\frac{2(\omega g'(\omega))'}{(g(\omega)-g(-\omega))'}\right)^{1-\alpha} - 1 \prec_q \left[\left(\frac{1+\omega}{1-\omega}\right)^{\gamma} - 1\right] \quad (\omega \in \mathbb{U}),$$

where $g(\omega) = f^{-1}(\omega)$.

Using the parameter setting of Definition 3.3 in Theorem 3.2, we get the following corollary.

Corollary 3.4. *For $0 \leq \alpha \leq 1$ and $0 < \gamma \leq 1$, let the function $f \in \mathcal{C}_{\Sigma}^{\psi}(\alpha, \gamma)$ be of the form (1), then*

$$|a_2| \leq \min\left\{\frac{|A_0|\gamma}{(2-\alpha)}, \sqrt{\frac{|A_0|\gamma(1+\gamma)}{(\alpha^2 - 3\alpha + 3)}}\right\}$$

and

$$|a_3| \leq \left\{\frac{\gamma(1+\gamma)|A_0|}{(\alpha^2 - 3\alpha + 3)} + \frac{\gamma|A_1|}{3 - 2\alpha}\right\}.$$

If we set $\phi(z) = \frac{1+(1-2\nu)z}{1-z} = 1 + 2(1-\nu)z + 2(1-\nu)z^2 + ...(0 \leq \nu < 1, z \in \mathbb{U})$ in Definition 3.1, we obtain a new subclass $\mathcal{C}_{\Sigma,\nu}^{\psi}(\alpha)$ of the bi-univalent function class Σ.

Definition 3.5. *For $0 \leq \alpha < 1$ and $0 \leq \nu < 1$, a function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{C}_{\Sigma,\nu}^{\psi}(\alpha)$ if the following quasi-subordinations hold*

$$\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2(zf'(z))'}{(f(z)-f(-z)')}\right)^{1-\alpha} - 1 \prec_q \left[\left(\frac{1+(1-2\nu)z}{1-z}\right) - 1\right] \quad (z \in \mathbb{U})$$

and

$$\left(\frac{2\omega g'(\omega)}{g(\omega)-g(-\omega)}\right)^{\alpha}\left(\frac{2(\omega g'(\omega))'}{(g(\omega)-g(-\omega)')}\right)^{1-\alpha} - 1 \prec_q \left[\left(\frac{1+(1-2\nu)\omega}{1-\omega}\right) - 1\right], \quad (\omega \in \mathbb{U})$$

where $g(\omega) = f^{-1}(\omega)$.

Using the parameter setting of Definition 3.5 in Theorem 3.2, we get the following corollary.

Corollary 3.6. *For $0 \leq \alpha \leq 1$ and $0 \leq \nu < 1$, let the function $f \in \mathcal{C}_{\Sigma,\nu}^{\psi}(\alpha)$ be of the form (1), then*

$$|a_2| \leq \min\left\{\frac{|A_0|(1-\nu)}{(2-\alpha)}, \sqrt{\frac{2|A_0|(1-\nu)}{\alpha^2 - 3\alpha + 3}}\right\}$$

and

$$|a_3| \leq (1-\nu)\left\{\frac{2|A_0|}{\alpha^2 - 3\alpha + 3} + \frac{|A_1|}{3 - 2\alpha}\right\}.$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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