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Fuglede Putnam Theorem on Class $p - wA(s, t)$ Operator

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Abstract: In this paper we charcterize Fuglede putnam theorem for class $p - wA(s, t)$ operator.

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1. Introduction

Let B(H) denote the algebra of all bounded linear operator on a complex Hilbert space H. Aluthge [1] found p-hyponormal T which is defined as $(T^*T)^p \geq (TT^*)^p, 0 < p \leq 1$. If p=1, T is called hyponormal. This is a generalization of hyponormal operator. This class of operator have many interesting properties, for example, Putnam's inequality, Fuglede-Putnam type theorem, Bishop's property(β), Weyl's theorem, polaroid. After this discovery, many authors are investigating new generalizations of hyponormal operator. We summarize several class of operators. Let $T = U|T|$ be the polar decomposition of T. Then the Aluthge transformation $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ was introduced by Aluthge [1]. An operator T is called w-hyponormal if $|\tilde{T}| \geq |T| \geq |\tilde{T^*}|$. The class of w-hyponormal operators was introduced and studied by Aluthge and Wang [2, 3]. It is well known that the class of w-hyponormal operators contains p-hyponormal operators. An operator T is called Class A if $|T^2| \geq |T|^2$. Class A operators has been Introduced and studied by Furuta [8]. An operator T is called $p - w$ hyponormal if $|\tilde{T}|^p \geq |T|^p \geq |\tilde{T^*}|^p$. If $p = 1$, then $p - w$ hyponormal operator is w-hyponormal. An operator T is called class $A(s, t)$ operator. If $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$. An operator T is called class $wA(s, t)$ operator. If $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$ and $|T|^{2s} \leq (|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}}$.

Definition 1.1. *Let $T = U|T|$ be the polar decomposition of T and let $s, t \geq 0$ and $0 \geq p \geq 1$. T is called $p - wA(s, t)$ if*

(1). $(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp}$

(2). $|T|^{2sp} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{sp}{s+t}}$.

We remark that if $p = 1$, T is $wA(s, t)$ and Class $1 - wA(1, 1)$ is called class A. Now we define class $p - A$ and class $p - A(s, t)$ as generalizations of class A and class $A(s, t)$.

Proposition 1.2 (Fuglede-Putnam)**.** *Let $S \in B(H)$ and $T^* \in B(K)$ be normal operators and $SX = XT$ for some operator $X \in B(H, K)$. Then $S^*X = XT^*$, [ran X] reduces S, ker$(X)^{\perp}$ reduces T.*

In this paper, we characterize fuglede-putnam theorem for class $p - wA(s, t)$ operator are proved.

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2. Fuglede Putnam Theorem on Class $p - wA(s,t)$ Operator

Theorem 2.1. *Let T be a class p-w class $A(s,t)$ operator for some $s,t \in (0,1]$ and M is an invariant subspace of T. Then the restriction $T|_M$ also p-w class $A(s,t)$ operator.*

Proof. Let $T = \begin{pmatrix} T_1 & S \\ 0 & 0 \end{pmatrix}$ on $H = M \oplus M^{\perp}$ and P-be the orthogonal projection onto M. Let $T_0 = TP = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$.

$$T_0 = TP \geq (P|T|^{2s}P)$$

By Hansens Inequality. Now, $|T^*|^2 = TT^* \geq TPT^* = |T_0^*|^2$. Hence T is $p - wA(s,t)$ operator.

$$(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{tp}{s+t}} \geq |T^*|^{2tp}$$
$$(|T_0^*|^t|T|^{2s}|T_0^*|^t)^{\frac{tp}{s+t}} \geq |T_0^*|^{2tp}$$
$$(|T_0^*|^t|T_0|^{2s}|T_0^*|^t)^{\frac{tp}{s+t}} \geq |T_0^*|^{2tp}$$

Since $|T_0^*| = |T_0|^*P = P|T_0^*|$. Similarly,

$$(|T|^s|T^*|^{2t}|T|^s)^{\frac{sp}{s+t}} \leq |T|^{2sp}$$
$$(|T|^s|T_0^*|^{2t}|T|^s)^{\frac{sp}{s+t}} \leq |T|^{2sp}$$
$$(|T_0|^s|T_0^*|^{2t}|T_0|^s)^{\frac{sp}{s+t}} \leq |T_0^*|^{2sp}$$

Hence $T|_M$ is a class $p - wA(s,t)$ operator. □

Theorem 2.2. *Let $T \in B(H)$ be a class $p - wA(s,t)$ operator. Let M be an invariant subspace of T and $\begin{pmatrix} T_1 & S \\ 0 & T_2 \end{pmatrix}$ on $H = M \oplus M^{\perp}$. If $T_1 = T|_M$ is quasinormal, then $ranS \subset kerT^*$. Moreover, $kerT \subset kerT^*T_1 = T|_M$ is normal, then M reduces T.*

Proof. We may assume that, $p = s = t = 1$. Then T becomes class A operator. Let P be the orthogonal projection onto M. Then we have, $T = \begin{pmatrix} T_1 & S \\ 0 & T_2 \end{pmatrix}$ on $H = M \oplus\ M^{\perp}$.

$$\begin{pmatrix} T_1 & S \\ 0 & T_2 \end{pmatrix} = PT^*TP = P|T|^2P$$
$$\leq \begin{pmatrix} (T_1^{*2}T_1^2)^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} T_1^*T_1 & 0 \\ 0 & 0 \end{pmatrix}$$

Since T_1 is quasinormal. Let $|T^2| = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}$. Then $X = T_1^*T_1$ by using above inequality. Since $|T^2|^2 = (T^*)^2(T^2)$

$$|T^2|^2 = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}$$

$$= \begin{pmatrix} X^2 + YY^* & XY^* + YZ \\ Y^*X + ZY & Y^*Y + Z^2 \end{pmatrix}$$
$$= \begin{pmatrix} T_1^{*2}T_1^2 & T_1^{*2}(T_1S + ST_2) \\ S^*T_1^* + T_2^*S^* & (S^*T_1^* + T_2^*S^*(T_1S + ST_2) + T_2^{*2}T_2^2 \end{pmatrix}$$
$$X^2 + YY^* = (T_1^*)^2T_1^2 = (T_1^*)T_1)^2$$
$$X^2 + YY^* = X^2$$

This implies that $Y = 0$. Then

$$|T^2| = \begin{pmatrix} T_1^*T_1 & 0 \\ 0 & Z \end{pmatrix}$$
$$\geq |T|^2$$
$$= T^*T$$
$$= \begin{pmatrix} T_1^*T & T_1^*S \\ S^*T_1 & S^*S + T_2^*T \end{pmatrix}$$

and $T^*S = 0$ This implies that, $ranS \subset kerT^*$. Moreover, assume T_1 is normal. Then $S(M^\perp) \subset kerT_1^* = kerT_1 \subset kerT^*$

$$0 = T^*Sx$$
$$= \begin{pmatrix} T_1^* & 0 \\ S^* & T_2^* \end{pmatrix} \begin{pmatrix} Sx \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} T_1^*Sx \\ S^*Sx \end{pmatrix}$$

Thus $ranT \subset kerT^*T_1 = T|_M$ is normal, then M reduces T. Hence the proof. □

Theorem 2.3. *Let $T \in B(H)$ be a class $p - wA(s,t)$ operator with $s + t \leq 1$ and $kerT \subset kerT^*$. If L is the self adjoint and $TL = LT^*$. Then $T^*L = LT^*$.*

Proof. Given that $T \in B(H)$ be a class $p - wA(s,t)$ operator with $s + t \leq 1$ and $kerT \subset kerT^*$ Assume that L is the self adjoint and $TL = LT^*$. We may assume that $s + t = 1$, since $kerT \subset kerT^*$ and $TL = LT^*$. $KerT$ reduces T and L. Hence $T = T_1 \oplus 0$, $L = L_1 \oplus L_2$ on $H = [ranT^*] \oplus kerT$. Then $T_1L_1 = L_1T_1^*$ and $0 = kerT_1 \subset kerT_1^*$. Let $[ranL_1]$ is invariant under T_1 and reduce L_1. $T_1 = \begin{pmatrix} T_{11} & S \\ 0 & T_{22} \end{pmatrix}$ and $L_1 = L_{11} \oplus 0$ on $ranT^* = [ranL_1 \oplus kerL_1]$ since T_{11} is injective class $p - wA(s,t)$ operator. By Lemma 2.1 and also given that L is self adjoint operator(hence it has dense range) (ie) $L = L^*$ such that $T_{11}L_{11} = L_{11}T_{11}^*$. Let $T_{11} = V_{11}|T_{11}|$ be the polar decomposition of T_{11}.

$$T_{11}(s,t) = |T_{11}|^sV_{11}|T_{11}|^t$$
$$W = |T_{11}|^sL_{11}|T_{11}|^t$$

Then,

$$T_{11}(s,t)W = |T_{11}|^sV_{11}|T_{11}|^t|T_{11}|^sL_{11}|T_{11}|^s$$

$$= |T_{11}|^s L_{11} |T_{11}|^* |T_{11}|^s V_{11}^* |T_{11}|^s$$
$$= WT_{11}(s,t)^*$$

Since T_{11} is min $\{s,t\}$ hyponormal and $ranW$-is dense (because $kerW = 0$). T_{11} is normal by [4] and $T_{11} = T_{11}(s,t)$ by [6] Then $ranT_1$ reduces T_1 by Theorem 2.2, $T_{11}^* L_{11} = L_{11} T_{11}$. By Proposition 1.2.

$$T = T_{11} \oplus T_{11} \oplus 0$$
$$L = L_{11} \oplus 0 \oplus L_{22}$$
$$T^* L = T_{11}^* L_{11} \oplus 0 \oplus 0$$
$$T^* L = LT$$

□

Theorem 2.4. *Let $T \in B(H)$ be a class $p - wA(s,t)$ operator with $s + t \leq 1$ and $kerT \subset kerT^*$. If $TX = XT^*$ for some operator $X \in B(H)$. Then $T^*X = XT$.*

Proof. Let $X = L + iK$ be the cartesian decomposition of X. Then we have $TL = LT^*$ and $TJ = JT^*$ by assumption. By Theorem 2.3 It follows that $T^*L = LT$ and $T^*J = JT$.

$$\Rightarrow T^*(L + iK) = (L + iK)T$$
$$T^*X = XT$$

□

Theorem 2.5. *Let $S \in B(K), T^* \in B(H)$ be a class $p - wA(s,t)$ operator with $s + t \leq 1$ and $kerS \subset kerS^*, kerT^* \subset kerT$. If $SX = XT$ for some operator $X \in B(K,H)$, then $S^*X = XT^*$.*

Proof. Let $A = \begin{pmatrix} T^* & 0 \\ 0 & S \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix}$ on $H \oplus K$. Then A is the class $p - wA(s,t)$ operator, $kerA \subset kerA^*$ which satisfies $AB = BA^*$. Hence we have $A^*B = BA$ by Theorem 2.4, then $S^*X = XT^*$. □

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