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Parameter Estimation for the Exponentiated Weibull Distribution Using SDPWM

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Abstract: In this paper, the parameters of the Exponentiated Weibull distribution were estimated by the methods of probability weighted moments (PWM) and self-determined probability weighted moments (SDPWM). In addition, simulation experiments were conducted to compare the performance of the two methods. The simulation results showed that the SDPWM method was better than PWM in almost all the cases. In addition, the two methods provided consistent estimators for the unknown parameters.

Keywords: Self-Determined probability weighted moments, probability weighted moments, Exponentiated Weibull distribution.

1. Introduction

For the estimation of probability distribution parameters, Greenwood et al [2] introduced the method of probability weighted moments (PWM). This method works very well with small samples. In addition, it was found to possess some desirable statistical properties; it is also computationally simple and robust. Another method of estimation is the method of self-determined probability weighted moments (SDPWM) which was proposed by Haktanir [3] as a refinement on the original method of PWM. It has many advantages over the method of PWM, such as the ability to directly consider outliers and the ability to detect whether a particular distribution is appropriate for describing a sample. The Exponentiated Weibull distribution was introduced by Mudholkar and Srivastava [] as an extension of the Weibull distribution by adding a second shape parameter. The cumulative distribution function for the exponentiated Weibull distribution is:

$$F(x;\alpha,\lambda,k)=\left[1-e^{-\left(\frac{x}{\lambda}\right)^k}\right]^{\alpha};\quad \alpha,\lambda,k>0 \tag{1}$$

for $x>0$ and 0 otherwise. The corresponding density function is:

$$f(x;\alpha,\lambda,k)=\alpha\frac{k}{\lambda}\left[\frac{x}{\lambda}\right]^{k-1}\left[\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)\right]^{\alpha-1}e^{-\left(\frac{x}{\lambda}\right)^k};\quad \alpha,\lambda,k>0 \tag{2}$$

for $x>0$ and 0 otherwise. Here α and k are shape parameters and λ is a scale parameter. If the shape parameter $\alpha=1$, then the Exponentiated Weibull distribution coincides with the Weibull distribution with a scale parameter λ. Also, when the

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shape parameter $k = 1$ then it will be the Exponentiated exponential distribution. The main aim of this paper is to compare the efficiency of the SDPWM and PWM methods in estimating the parameters of the Exponentiated Weibull distribution. In the two methods of estimation, determination of parameter estimates requires solving non-linear equations. Therefore, all estimation algorithms are implemented using the Mathcad (version 14.0) software, which contains iterative procedures that are useful for solving non-linear equations. The organization of this article is as follows. A theoretical overview of the methods of PWM and SDPWM is provided in section 2. In section 3, the detailed procedure for PWM estimation of the parameters of the Exponentiated Weibull distribution is introduced. Then the SDPWM estimation of the Exponentiated Weibull parameters is discussed in section 4. Section 5 contains simulation results and discussions. Finally, conclusions are included in section 6. Results Tables are included in the appendix.

2. Overview

The method of SDPWM was proposed as a modification of the original method of PWM. The PWM method was originally designed to facilitate parameter estimation for distributions that are analytically expressible only in inverse form (for example, Tukey's lambda and Wakeby distributions). The PWMs of a random variable X with cumulative distribution function $F(x) = P(X \leq x)$ and inverse distribution function $x = x(F)$ are the quantities:

$$M_{p,r,s} = E[X^p F^r (1 - F)^s] = \int_0^1 [x(F)]^p F^r [1 - F]^s dF, \tag{3}$$

where p, r, and s are real numbers. The most commonly used PWMs are $M_{1,0,s}$ $(s = 0, 1, 2, ...)$ and $M_{1,r,0}$ $(r = 0, 1, 2, ...)$; for convenience, they are re-expressed as follows:

$$\alpha_s \equiv M_{1,0,s} = \int_0^1 x(F) \cdot [1 - F]^s dF \tag{4}$$

$$\beta_r \equiv M_{1,r,0} = \int_0^1 x(F) \cdot F^r dF \tag{5}$$

The two PWM sets, α_s andβ_r, are linear combinations of each other. Therefore, any one of them can be used; whichever is possible; for parameter estimation without loss of generality. To estimate the parameters of a distribution, a sample estimator of α_s and β_r is needed. Landwehr et al. (1979a) introduced unbiased estimators of α_s and β_r, where s and r are nonnegative integers, which are based on the ordered sample $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ from the distribution F. They are given by:

$$\hat{\alpha}_s = \frac{1}{n} \sum_{i=1}^{n} x_{(i)} \begin{pmatrix} n-i \\ s \end{pmatrix} \Big/ \begin{pmatrix} n-1 \\ s \end{pmatrix} \tag{6}$$

and

$$\hat{\beta}_r = \frac{1}{n} \sum_{i=1}^{n} x_{(i)} \begin{pmatrix} i-1 \\ r \end{pmatrix} \Big/ \begin{pmatrix} n-1 \\ r \end{pmatrix} \tag{7}$$

where $\begin{pmatrix} n-i \\ s \end{pmatrix} \Big/ \begin{pmatrix} n-1 \\ s \end{pmatrix}$ and $\begin{pmatrix} i-1 \\ r \end{pmatrix} \Big/ \begin{pmatrix} n-1 \\ r \end{pmatrix}$ are estimates for the exceedance $[1-F(x)]$ and non-exceedance $F(x)$ probabilities, respectively; and $x_{(i)}$ is the i^{th} observation in the ordered sample. These estimates are not based on the assumed distribution, but are based solely on the position of $x_{(i)}$ within the ordered sample. The method of SDPWM deviates from the method of PWM in that it is assumed that the ordered sample follows a particular distribution. Accordingly, the non-exceedance probabilities are assigned using the corresponding cumulative distribution function of the assumed distribution. Therefore, the SDPWM sample estimators $\tilde{\alpha}_s$ and $\tilde{\beta}_r$ for α_s and β_r, respectively, are defined as follows:

$$\tilde{\alpha}_s = \frac{1}{n}\sum_{i=1}^{n}[1 - F(x_{(i)};\theta_1,\theta_2,\ldots,\theta_k)]^s x_{(i)} \tag{8}$$

and

$$\tilde{\beta}_r = \frac{1}{n}\sum_{i=1}^{n}[F(x_{(i)};\theta_1,\theta_2,\ldots,\theta_k)]^r x_{(i)} \tag{9}$$

Where F is the distribution function of a given distribution; $\theta_1,\theta_2,\ldots,\theta_k$ are the unknown parameters of this distribution; and $x_{(i)}$ is the i^{th} observation in the ordered sample $x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$.

3. Probability Weighted Moments Estimators

For the Exponentiated Weibull distribution, it is more convenient to work with the PWM of the form β_r which is given by:

$$\beta_r = \int_0^1 \lambda\left[-\ln\left(1 - F^{\frac{1}{\alpha}}\right)\right]^{\frac{1}{k}} F^r dF \tag{10}$$

Since the Exponentiated Weibull distribution is a three-parameter distribution, then only the first three PWM; β_0, β_1 and β_2 are needed. They are given as follows:

$$\beta_0 = \int_0^1 \lambda\left[-\ln\left(1 - F^{\frac{1}{\alpha}}\right)\right]^{\frac{1}{k}} dF \tag{11}$$

$$\beta_1 = \int_0^1 \lambda\left[-\ln\left(1 - F^{\frac{1}{\alpha}}\right)\right]^{\frac{1}{k}} F\, dF \tag{12}$$

and

$$\beta_2 = \int_0^1 \lambda\left[-\ln\left(1 - F^{\frac{1}{\alpha}}\right)\right]^{\frac{1}{k}} F^2\, dF \tag{13}$$

Also, the first three PWM sample estimators are needed. They are:

$$\hat{\beta}_0 = \frac{1}{n}\sum_{i=1}^{n} x_{(i)}, \tag{14}$$

$$\hat{\beta}_1 = \frac{1}{n(n-1)}\sum_{i=1}^{n}(i-1)x_{(i)} \tag{15}$$

and

$$\hat{\beta}_2 = \frac{1}{n(n-1)(n-2)}\sum_{i=1}^{n}(i-1)(i-2)x_{(i)} \tag{16}$$

To obtain the PWM estimators $\hat{\alpha}_{PWM}$, $\hat{k}_{PWM}$ and $\hat{\lambda}_{PWM}$; the population PWMs (β_0, β_1, β_2) in equations (11), (12) and (13) are replaced by their sample estimators ($\hat{\beta}_0$, $\hat{\beta}_1$, $\hat{\beta}_2$). Then, the resulting three equations:

$$\frac{1}{n}\sum_{i=1}^{n} x_{(i)} = \int_0^1 \lambda\left[-\ln\left(1 - F^{\frac{1}{\alpha}}\right)\right]^{\frac{1}{k}} dF \tag{17}$$

$$\frac{1}{n\,(n-1)}\sum_{i=1}^{n}(i-1)\,x_{(i)} = \int_0^1 \lambda\left[-\ln\left(1 - F^{\frac{1}{\alpha}}\right)\right]^{\frac{1}{k}} FdF \tag{18}$$

$$\frac{1}{n\,(n-1)\,(n-2)}\sum_{i=1}^{n}(i-1)\,(i-2)\,x_{(i)} = \int_0^1 \lambda\left[-\ln\left(1 - F^{\frac{1}{\alpha}}\right)\right]^{\frac{1}{k}} F^2 dF \tag{19}$$

Can be solved numerically to estimate the parameters.

4. Self-Determined Probability Weighted Moments Estimators

The procedure of SDPWM estimation of the Exponentiated Weibull parameters is the same as the procedure of PWM estimation. The difference is that in the case of SDPWM estimation, the population PWMs (β_0, β_1, β_2) will be replaced by the SDPWM sample estimators, $\tilde{\beta}_0$, $\tilde{\beta}_1$ and $\tilde{\beta}_2$, where:

$$\tilde{\beta}_0 = \frac{1}{n}\sum_{i=1}^{n} x_{(i)} = \bar{x}, \tag{20}$$

$$\tilde{\beta}_1 = \frac{1}{n}\sum_{i=1}^{n} \left[1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right]^{\alpha} x_{(i)}, \tag{21}$$

and

$$\tilde{\beta}_2 = \frac{1}{n}\sum_{i=1}^{n} \left[1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right]^{2\alpha} x_{(i)} \tag{22}$$

Therefore, the SDPWM parameter estimators, $\tilde{\alpha}_{SDPWM}$, $\tilde{k}_{SDPWM}$ and $\tilde{\lambda}_{SDPWM}$, can be obtained by replacing the population PWMs (β_0, β_1, β_2) in equations (11), (12) and (13) by the SDPWM sample estimators ($\tilde{\beta}_0$, $\tilde{\beta}_1$, $\tilde{\beta}_2$). Then, this will make three new equations that are:

$$\bar{x} = \int_0^1 \lambda \left[-\ln\left(1 - F^{\frac{1}{\alpha}}\right)\right]^{\frac{1}{k}} dF \tag{23}$$

$$\frac{1}{n}\sum_{i=1}^{n} \left[1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right]^{\alpha} x_{(i)} = \int_0^1 \lambda \left[-\ln\left(1 - F^{\frac{1}{\alpha}}\right)\right]^{\frac{1}{k}} F dF \tag{24}$$

$$\frac{1}{n}\sum_{i=1}^{n} \left[1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right]^{2\alpha} x_{(i)} = \int_0^1 \lambda \left[-\ln\left(1 - F^{\frac{1}{\alpha}}\right)\right]^{\frac{1}{k}} F^2 dF \tag{25}$$

Which can be solved numerically to obtain the estimates.

5. Comparative Study

It is very difficult to compare the theoretical performances of the two estimators described in the previous sections. Therefore, extensive simulations are performed to compare the performances of the two methods of estimation mainly in terms of bias and mean square error (MSE), for different sample sizes and for different values of the shape parameter α. The simulation experiments are performed using the Mathcad software (version 14.0). Different sample sizes are considered through the experiments that are $n = 15, 20, 30, 50$ and 100. In addition, different values for the shape parameter α are considered, which are α =0.5, 1.5 and 2.5 with the two parameter k and λ taken to be 1 in all the experiments. For each combination of the sample size and the value of α the experiment will be repeated 10000 times. In each experiment, the estimates of α, k and λ will be obtained by the methods of PWM and SDPWM. The biases and MSEs for the two estimators will be reported from these experiments.

The algorithm for obtaining the estimates by the two methods of estimation can be described in details in the following steps:

Step (1): Generate a random sample of size n, $x_1, x_2, \ldots, x_n$, from the Exponentiated Weibull distribution. This can be achieved by firstly generating a random sample from the uniform (0,1) distribution, $u_1, u_2, \ldots, u_n$. Then the uniform random numbers can be transformed to Exponentiated Weibull random numbers using the following transformation:

$$x_i = \lambda \left[-\ln\left(1 - u_i^{\frac{1}{\alpha}}\right)\right]^{\frac{1}{k}}, \; i = 1, 2, \ldots, n \tag{26}$$

Step (2): Sort the random sample $x_1, x_2, \ldots, x_n$ to obtain the ordered complete sample $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$.

Step (3): Substitute the values of the ordered sample $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ in equations (17), (18) and (19) and solve the three equations numerically to obtain the PWM estimates $\hat{\alpha}_{PWM}$, $\hat{k}_{PWM}$ and $\hat{\lambda}_{PWM}$.

Step (4): Substitute the values of the ordered sample $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ in equations (23), (24) and (25) and solve the three equations numerically to obtain the SDPWM estimates $\tilde{\alpha}_{SDPWM}$, $\tilde{k}_{SDPWM}$ and $\tilde{\lambda}_{SDPWM}$.

The biases and MSEs of the three estimators of α, $k$and λ are reported in Tables 1 and 2, respectively. From these tables many observations can be made on the performance of the method of SDPWM versus the method of PWM. These observations are summarized as follows:

(1). It is observed that the biases and MSEs of the two estimators of α, k and λ depend on the value of the shape parameter α. For all the methods considered as α increases the biases and MSEs of the estimators of α and k increase. On the other hand, the biases and MSEs of the estimators of λ decrease as α increases for all the methods.

(2). For all the cases considered the biases and MSEs of two different estimators of α, k and λ decrease as the sample size increases. This indicates that both methods provide consistent estimators for α, k and λ.

(3). Comparing the biases of the different estimators of α, k and λ, it is clear that the method of PWM yields the minimum bias in almost all the cases considered.

(4). Considering the MSEs of the two estimators of α, k and λ it is clear from Table 2 that the SDPWM estimator has the minimum MSE in all of the cases.

6. Conclusions

After analytically deriving SDPWM and PWM parameter estimators for the Exponentiated Weibull distribution, the performance of these two methods was investigated through numerical analysis. The comparative study revealed that the SDPWM works the best in all the cases considered with respect to MSE. As far as the minimum bias is concerned, the method of PWM works the best in most of the cases considered. In addition, both methods considered have provided consistent estimators for α, k and λ.

Appendix

n	Method	**Estimators of α**			**Estimators of λ**			**Estimators of k**		
		$\alpha = 0.5$	$\alpha = 1.5$	$\alpha = 2.5$	$\alpha = 0.5$	$\alpha = 1.5$	$\alpha = 2.5$	$\alpha = 0.5$	$\alpha = 1.5$	$\alpha = 2.5$
15	PWM	0.033	0.152	0.239	0.134	0.044	0.013	0.081	0.097	0.219
	SDPWM	0.055	0.201	0.409	0.193	0.083	0.063	0.116	0.117	0.296
20	PWM	0.022	0.089	0.123	0.102	0.027	0.00103	0.048	0.063	0.12
	SDPWM	0.038	0.147	0.313	0.144	0.063	0.049	0.08	0.089	0.221
30	PWM	0.004	0.033	0.024	0.054	0.003	-0.015	0.022	0.029	0.04
	SDPWM	0.013	0.095	0.223	0.084	0.034	0.029	0.049	0.058	0.154
50	PWM	0.003	-0.005	-0.057	0.023	-0.012	-0.031	0.001	0.005	-0.023
	SDPWM	0.014	0.06	0.146	0.046	0.018	0.013	0.023	0.033	0.097
100	PWM	-0.001	-0.029	-0.112	0.003	-0.022	-0.037	-0.013	-0.01	-0.062
	SDPWM	0.008	0.036	0.1	0.019	0.005	0.005	0.007	0.018	0.063

Table 1. **Biases of the Parameter Estimators Obtained by the Methods of PWM and SDPWM**

n	Method	Estimators of α			Estimators of λ			Estimators of k		
		$\alpha = 0.5$	$\alpha = 1.5$	$\alpha = 2.5$	$\alpha = 0.5$	$\alpha = 1.5$	$\alpha = 2.5$	$\alpha = 0.5$	$\alpha = 1.5$	$\alpha = 2.5$
15	PWM	0.063	0.755	2.287	0.368	0.129	0.09	0.111	0.311	1.517
	SDPWM	0.032	0.378	1.314	0.334	0.106	0.075	0.09	0.147	0.784
20	PWM	0.044	0.467	1.414	0.253	0.095	0.067	0.081	0.199	0.913
	SDPWM	0.022	0.259	0.909	0.225	0.078	0.056	0.066	0.102	0.522
30	PWM	0.005	0.25	0.769	0.142	0.058	0.043	0.049	0.108	0.47
	SDPWM	0.003	0.151	0.541	0.124	0.048	0.035	0.039	0.06	0.305
50	PWM	0.014	0.134	0.407	0.075	0.034	0.026	0.029	0.058	0.253
	SDPWM	0.008	0.084	0.301	0.063	0.027	0.02	0.023	0.033	0.172
100	PWM	0.007	0.062	0.189	0.034	0.016	0.013	0.014	0.026	0.116
	SDPWM	0.004	0.039	0.14	0.028	0.012	0.01	0.011	0.016	0.08

Table 2. **Mean square errors of the Parameter Estimators Obtained by the Methods of PWM and SDPWM**

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