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Generating Function Associated with the Product of Gegenbauer Polynomials

N.U. Khan[1], M. Kamarujjama[1] and T. Kashmin[1,*]

1 Department of Applied Mathematics, Faculty of Engineering and Technology, Aligarh Muslim University, Aligarh, India.

Abstract: This paper deals with general expansions which gives as special cases involving Legendre, Ultraspherical, and Tchebcheff polynomials, Appell, Bessel's and generalized hypergeometric functions of mathematical physics. The results extended Pathan and Kamarujjama expansion [7], Exton's generating function [5] and Feldheim's expansion [3]. A number of known and new generating functions are shown to follow as an application of the main result.

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1. Introduction

We considered a generating function [4]

$$\sum_{n=0}^{\infty} \frac{C_n^{\nu}(x)\ t^n}{(2\nu)_n} = e^{xt}\ {}_0F_1\left[\begin{matrix} -; & \frac{t^2(x^2-1)}{4} \\ \nu+\frac{1}{2}; & \end{matrix}\right] \tag{1}$$

where $C_n^{\nu}(x)$ is the Gegenbauer polynomial defined by [4]

$$C_n^{\nu}(x) = \frac{(2\nu)_n}{n!}\ {}_2F_1\left[\begin{matrix} -n\quad , & 2\nu+n; & \frac{1}{2}(1-x) \\ \nu+\frac{1}{2}; & & \end{matrix}\right] \tag{2}$$

${}_pF_q$ denote generalized hypergeometric function of one variable with p numerator parameters and q denominator parameters defined by [6]

$${}_pF_q\left[\begin{matrix} a_1, a_2, ..., a_p; & x \\ b_1, b_2, ..., b_q; & \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n ....(a_p)_n x^n}{(b_1)_n (b_2)_n ....(b_q)_n n!} = {}_pF_q(a_1, a_2, ..., a_p; b_1, b_2, ..., b_q; x) \tag{3}$$

where $(a)_n$ is the pochhammer symbol, defined by

$$(a)_n = \left\{\begin{matrix} 1 & \text{if } n=0 \\ a(a+1)(a+2)...(a+n-1) & \text{if } n=1,2,3,... \end{matrix}\right\}. \tag{4}$$

* E-mail: kashminalig7384@gmail.com

The denominator parameters are neither zero nor negative integers the numerator parameters may be zero and negative integers. We have some important polynomials which can be expressed in terms of Gegenbauer polynomials for different values of ν, as follows:

$$C_n^{\frac{1}{2}}(x) = P_n(x) \tag{5}$$

$$Cn^1(x) = \frac{(n+1)!}{(\frac{3}{2})_n}\, P_n^{(\frac{1}{2},\frac{1}{2})}(x) = U_n(x) \tag{6}$$

$$C_n^{\nu}(x) = \frac{(2\nu)_n}{(\nu+\frac{1}{2})_n} P_n^{(\nu-\frac{1}{2},\ \nu-\frac{1}{2})}(x) \tag{7}$$

where $P_n(x)$, $P_n^{(\alpha,\alpha)}(x)$ and $U_n(x)$ are Legendre, Ultraspherical, and Tchebcheff polynomials of second kind respectively. The Gegenbauer polynomial is an important class of orthogonal polynomial which is the generalization of Legendre, and Tchebcheff polynomials of second kind $U_n(x)$. It is also known that the Gegenbauer and Ultraspherical polynomials are essentially equivalent (see [4])

$$C_n^{\nu}(x) = \frac{(2\nu)_n}{(\nu+\frac{1}{2})_n} P_n^{(\nu-\frac{1}{2},\ \nu-\frac{1}{2})}(x), \tag{8}$$

$$P_n^{(\alpha,\alpha)}(x) = \frac{(1+\alpha)_n C_n^{\alpha+\frac{1}{2}}(x)}{(1+2\alpha)_n}. \tag{9}$$

Recently Pathan and Kamarujjama [7] introduced a generating relation involving product of three Laguerre polynomials in the form.

$$\exp\left[-\left(u+v-\frac{wv}{u}\right)x\right](1+u)^{\alpha}(1+v)^{\beta}\left(1-\frac{wv}{u}\right)^{\gamma} = \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} u^m v^n \sum_{r=0}^{\infty} L_{(m+r)}^{\alpha-(m+r)}(x) L_{(n-r)}^{\beta-(n-r)}(x) L_r^{\alpha-(\gamma-r)}(x)(-w)^r. \tag{10}$$

where $m^* = \max(0,-m)$, so that all factorials of negative integers have meaning. Equation (10) is in fact generalization of number of results due to Feldheim [3].

2. Generating Relation for the Product of Gegenbauer Polynomials

In this paper we drive a generating relation involving the product of three Gegenbauer polynomials which generalize many known results of Pathan and Kamarujjam [7], Feldhiem [3] and Exton [5]. To obtain our main results, consider the product.

$$S(u,v,w) = \exp\left[x\left(u+v-\frac{wv}{u}\right)\right] {}_0F_1\left[\begin{matrix} -; & \frac{u^2(x^2-1)}{4} \\ \gamma_1+\frac{1}{2}; & \end{matrix}\right] {}_0F_1\left[\begin{matrix} -; & \frac{v^2(x^2-1)}{4} \\ \gamma_2+\frac{1}{2}; & \end{matrix}\right] {}_0F_1\left[\begin{matrix} -; & \frac{\frac{w^2v^2}{u^2}(x^2-1)}{4} \\ \gamma_3+\frac{1}{2}; & \end{matrix}\right]. \tag{11}$$

Now expanding the right hand member of (11) as a multiple series with the help (1), we get

$$S(u,v,w) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty} \frac{C_p^{\gamma_1}(x) C_q^{\gamma_2}(x) C_r^{\gamma_3}(x) u^{p-r} v^{q+r}(-w)^r}{(2\gamma_1)_p(2\gamma_2)_q(2\gamma_3)_r} \tag{12}$$

Now replacing $p-r$ and $q+r$ by m and n respectively, then after rearrangement justified by the absolute convergence of the above series, we have

$$S(u,v,w) = \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} u^m v^n \sum_{r=0}^{\infty} \frac{C_{m+r}^{\gamma_1}(x) C_{n-r}^{\gamma_2}(x) C_r^{\gamma_3}(x)(-w)^r}{(2\gamma_1)_{m+r}(2\gamma_2)_{n-r}(2\gamma_3)_r} \tag{13}$$

where $m^* = \max(0,-m)$, so that all factorials of negative integers have meaning.

3. Special Cases

Equation (13) gives many generating functions for well known polynomials. We are presenting only some interesting special cases here.

(1). Setting $x = 1$ in (13), we get a known result [4]

$$C_n^{\gamma_1}(1) = \frac{(2\gamma)_n}{n!}, \tag{14}$$

then we obtain a modified result of Exton [2] due to Pathan and Yasmeen [6]

$$\exp\left[x\left(u+v-\frac{wv}{u}\right)\right] = \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{u^m v^n}{m!n!}\,{}_1F_1\left[-n; 1+m; w\right]. \tag{15}$$

(2). If $\gamma_1 = \gamma_2 = \gamma_3 = 1$ in (13) and using the result [2]

$${}_0F_1\left[\begin{matrix}-; \\ \frac{3}{2};\end{matrix}\quad z\right] = \frac{sh\ 2\sqrt{z}}{2\sqrt{z}},$$

then (13) reduces to

$$\begin{aligned}\exp\left[x\left(u+v-\frac{wv}{u}\right)\right]\frac{sh(2u\sqrt{x^2-1})}{(2u\sqrt{x^2-1})}\frac{sh(2v\sqrt{x^2-1})}{(2v\sqrt{x^2-1})}\frac{sh(\frac{2wv}{u}\sqrt{x^2-1})}{(\frac{2wv}{u}\sqrt{x^2-1})}\\ = \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}u^m v^n\sum_{r=0}^{\infty}\frac{U_{m+r}(x)U_{n-r}(x)U_r(x)(-w)^r}{(m+r+1)!\ (n-r+1)!\ r!}\end{aligned} \tag{16}$$

where $U_n(x)$ is called the Tchebicheff polynomials of second kind [6]

(3). If $\gamma_1 = \gamma_2 = \gamma_3 = \frac{1}{2}$ in (13), we get

$$\begin{aligned}\exp\left[x\left(u+v-\frac{wv}{u}\right)\right]\,{}_0F_1\left[\begin{matrix}-; \\ 1;\end{matrix}\quad \frac{u^2(x^2-1)}{4}\right]{}_0F_1\left[\begin{matrix}-; \\ 1;\end{matrix}\quad \frac{v^2(x^2-1)}{4}\right]{}_0F_1\left[\begin{matrix}-; \\ 1;\end{matrix}\quad \frac{\frac{w^2v^2}{u^2}(x^2-1)}{4}\right]\\ = \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}u^m v^n\sum_{r=0}^{\infty}\frac{P_{m+r}(x)P_{n-r}(x)P_r(x)(-w)^r}{(m+r)!\ (n-r)!\ (r)!}.\end{aligned} \tag{17}$$

In view of known generating relation [4]

$${}_0F_1\left[-;\quad \frac{t^2(x^2-1)}{4}\right] = \sum_{n=0}^{\infty}\frac{P_n(x)t^n}{n!},$$

or

$$e^{xt}J_0\left(t\sqrt{1-x^2}\right) = \sum_{n=0}^{\infty}\frac{P_n(x)t^n}{n!}$$

thus (17) reduces to

$$\begin{aligned}\exp\left[x\left(u+v-\frac{wv}{u}\right)\right]J_0\left(u\sqrt{1-x^2}\right)J_0\left(v\sqrt{1-x^2}\right)J_0\left(\frac{wv}{u}\sqrt{1-x^2}\right)\\ = \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}u^m v^n\sum_{r=0}^{\infty}\frac{P_{m+r}(x)P_{n-r}(x)P_r(x)(-w)^r}{(m+r)!\ (n-r)!\ (r)!}.\end{aligned} \tag{18}$$

where $J_0(x)$ is called the Bessel function of index zero and $P_n(x)$ is called the Legendre polynomials [8]

(4). If we set $w = 0$ in (13), we get

$$\exp[x(u+v)]\ {}_0F_1\left[\begin{matrix} -; & \frac{u^2(x^2-1)}{4} \\ \gamma_1+\frac{1}{2}; & \end{matrix}\right]\ {}_0F_1\left[\begin{matrix} -; & \frac{v^2(x^2-1)}{4} \\ \gamma_2+\frac{1}{2}; & \end{matrix}\right] = \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} u^m v^n \frac{C_m^{\gamma_1}(x)C_n^{\gamma_2}(x)}{(2\gamma_1)_m\ (2\gamma_2)_n}. \tag{19}$$

(5). If $x = 0$ in equation (19), we get

$$\begin{aligned} {}_0F_1\left[\begin{matrix} -; & \frac{-u^2}{4} \\ \gamma_1+\frac{1}{2}; & \end{matrix}\right]\ {}_0F_1\left[\begin{matrix} -; & \frac{-v^2}{4} \\ \gamma_2+\frac{1}{2}; & \end{matrix}\right] &= \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} u^m v^n\ {}_2F_1\left[\begin{matrix} -m & ; & 2\gamma_1+m; & \frac{1}{2} \\ \gamma_1+\frac{1}{2}; & & & \end{matrix}\right] \\ &\quad {}_2F_1\left[\begin{matrix} -n & ; & 2\gamma_2+n; & \frac{1}{2} \\ \gamma_2+\frac{1}{2}; & & & \end{matrix}\right]. \end{aligned} \tag{20}$$

(6). If $\gamma_1 = \gamma_2 = 1$ in (20), and using the relations [8]

$$\frac{\sin(u)}{u}\frac{\sin(v)}{v} = \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} u^m v^n\ {}_2F_1\left[\begin{matrix} -m\ ; & 2+m; & \frac{1}{2} \\ \frac{3}{2}\ \ ; & & \end{matrix}\right]\ {}_2F_1\left[\begin{matrix} -n\ ; & 2+n; & \frac{1}{2} \\ \frac{3}{2}\ \ ; & & \end{matrix}\right]. \tag{21}$$

(7). On replacing u and v by ut and vt in (20), and multiplying both sides by $e^{-t}t^{\mu-1}$, and taking Laplace transform (see [1]), we get

$$\begin{aligned} F_4\left[\frac{\mu}{2}, \frac{\mu+1}{2}; \gamma_1+\frac{1}{2}, \gamma_2+\frac{1}{2}; -u^2, -v^2\right] &= \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} \frac{u^m v^n}{m!n!}(\mu)_{m+n}\ {}_2F_1\left[\begin{matrix} -m & ; & 2\gamma_1+m; & \frac{1}{2} \\ \gamma_1+\frac{1}{2}; & & & \end{matrix}\right] \\ &\quad {}_2F_1\left[\begin{matrix} -n & ; & 2\gamma_2+n; & \frac{1}{2} \\ \gamma_2+\frac{1}{2}; & & & \end{matrix}\right]. \end{aligned} \tag{22}$$

where F_4 is Appell function of two variables defined by [6].

(8). In view of the relation [6]

$$C_n^{\gamma}(x) = (-2)^n (x^2-1)^{(1/2)n} P_n^{(-\gamma-n,-\gamma-n)}\left(\frac{x}{\sqrt{x^2-1}}\right), \tag{23}$$

equation (13) yields an interesting result

$$\begin{aligned} &\exp\left[x\left(u+v-\frac{wv}{u}\right)\right]\ {}_0F_1\left[\begin{matrix} -; & \frac{u^2(x^2-1)}{4} \\ \gamma_1+\frac{1}{2}; & \end{matrix}\right]\ {}_0F_1\left[\begin{matrix} -; & \frac{v^2(x^2-1)}{4} \\ \gamma_2+\frac{1}{2}; & \end{matrix}\right]\ {}_0F_1\left[\begin{matrix} -; & \frac{\frac{w^2v^2}{u^2}(x^2-1)}{4} \\ \gamma_3+\frac{1}{2}; & \end{matrix}\right] \\ &= \sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty} u^m v^n \sum_{r=0}^{\infty} \frac{(-2)^{m+n+r}(x^2-1)^{(1/2)_{m+r}} P_{m+r}^{(-\gamma_1-(m+r),-\gamma_1-(m+r))}\left(\frac{x}{\sqrt{x^2-1}}\right)}{(2\gamma_1)_{m+r}} \\ &\quad .\frac{(x^2-1)^{(1/2)_{n-r}} P_{n-r}^{(-\gamma_1-(n-r),-\gamma_1-(n-r))}\left(\frac{x}{\sqrt{x^2-1}}\right) .(x^2-1)^{(1/2)_r} P_r^{(-\gamma_3-r,-\gamma_3-r)}\left(\frac{x}{\sqrt{x^2-1}}\right)(-w)^r}{(2\gamma_2)_{n-r}\ (2\gamma_3)_r}. \end{aligned} \tag{24}$$

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