Int. J. Math. And Appl., 6(1–D)(2018), 643–649
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Einstein Criterion for Finsler Space With Special (α, β)-Metrics

K. Chandru[1,*] and S. K. Narasimhamurthy[2]

1 Department of P.G. Studies and Research in Mathematics, Kuvempu University, Shankaraghatta, Shivamogga, Karnataka, India.

Abstract: Einstein-Finsler metrics are very useful to study geometric structure of spacetime and to build applications in theory of relativity. In this paper, we consider the special (α, β)-metric $L = \mu\frac{\alpha^2}{\beta} + \nu\frac{\beta^2}{\alpha}$ and obtained the Riemann curvature. Then we obtained the necessary and sufficient condition for that (α, β)-metric to be Einstein metric, when β is a constant killing form. Finally, we proved that the above metric is Einstein if and only if it is Ricci flat.

MSC: 53B20, 53B40, 53C60.

Keywords: (α, β)-metrics, Riemannian curvature, Ricci curvature, Einstein Finsler space.

1. Introduction

A Finsler space is a manifold M together with positively homogeneous metric function $L(x, y)$. (α, β)-metrics are the special class of Finsler metrics which having a major role in formulating applications in Einstein theory of relativity, Mechanics, Biology, control theory, etc., [1, 2, 10]. C. Robles invented Randers Einstein metrics in 2003, and derived the necessary and sufficient conditions for Randers metrics to be Einstein. In [11], authors proved the Einstein Schur type lemma for (α, β)-metrics. In Finsler geometry Einstein metrics are solutions of Einstein Field equations in general relativity. In order to characterize Einstein-Finsler (α, β)-metrics, it is necessary to compute the Riemann curvature and the Ricci curvature for (α, β)-metrics. For a Finsler space, Riemannian curvature $R_y : T_xM \to T_xM$ is given by $R_y(u) = R^i_k(y)u^k \frac{tial}{tialx^i}$. By this curvature, Ricci scalar defined by $Ric(x, y) = R^i_k$. A Finsler metric is Einstein if the Ricci scalar is of the form $Ric = c(x)F^2(x, y)$ for some function c on manifold M, i.e., the Ricci scalar is a function of x alone [3, 4]. A manifold is called Ricci flat if Ricci tensor vanishes, which represents vacuum solution to Einstein field equations in relativity [3]. In [12], Razaei, Razavi and Sadeghazadeh, consider the (α, β)-metrics such as generalized Kropina metric, Matsumoto metric with β a constant killing form and obtained the necessary and sufficient conditions to be Einstein metrics. In 2012 [10], Rafie and Rezaei proved that the second Schur type lemmafor Finsler-Matsumoto metric. Then, Cheng, Shen and Tian, proved (α, β)-metric is Ricci flat [7]. In [14], authors classified the projectively related Einstein Finsler metrics over compact manifold. In this paper We consider the special (α, β)-metric $L = \mu\frac{\alpha^2}{\beta} + \nu\frac{\beta^2}{\alpha}$, where α is the Riemannian metric, β is a constant killing form and μ, ν are some constants. Then we find the Riemannian curvature for these metrics and we obtained the necessary and sufficient condition for them to be Einstein metrics, when β as a constant killing form. Finally,

* E-mail: kchandru2510@gmail.com

we proved the lemma states that the above mentioned metrics are Einstein if and only if it is Ricci flat. In the entire paper we use the Einstein convention.

2. Preliminaries

Let (M,F) be an n-dimensional Finsler space. We denote the tangent space at $x \in M$ by T_xM and the tangent bundle of M by TM. Each element of TM has the form (x,y), where $x \in M$ and $y \in T_xM$. The formal definition of Finsler space as follows;

Definition 2.1 ([3]). *A Finsler space is a triple $F^n = (M,D,L)$, where M is an n-dimensional manifold, D is an open subset of a tangent bundle TM and L is a Finsler metric defined as a function $L: TM \to [0,\infty)$ with the following properties:*

(i). Regular: L is C^∞ on the entire tangent bundle $TM\backslash\{0\}$.

(ii). Positive homogeneous: $L(x,\lambda y) = \lambda L(x,y)$.

(iii). Strong convexity: The $n \times n$ Hessian matrix $g_{ij} = \frac{1}{2}[L^2]_{y^iy^j}$

is positive definite at every point on $TM\backslash\{0\}$, where $TM\backslash\{0\}$ denotes the tangent vector y is non zero in the tangent bundle TM.

M. Matsumoto introduced the class of (α,β)-metrics [9]. An (α,β)-metric is a scalar function L on TM defined by $L = \alpha\phi(s)$, where $s = \frac{\beta}{\alpha}$. Here $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity, α is a Riemannian metric and β ia a one form on M. Denote the Levi-Civita connection of α by ∇. Recall some geometric quantities of (α,β)-metric:

$$\begin{aligned} r_{ij} &= \nabla_j b_i + \nabla_i b_j,\ s_{ij} = \nabla_j b_i - \nabla_i b_j, \\ r^i_j &= a^{ik} r_{kj},\ r_{00} = r_{ij}y^iy^j,\ r = r_ijb^ib^j, \\ s^i_j &= a^{ik}s_{kj},\ s_j = b^is_{ij},\ s_0 = s_iy^i,\ B = b^ib_j. \end{aligned} \tag{1}$$

In Finsler geometry geodesic spray is defined by $G = y^i \frac{tial}{tialx^i} - 2G^i(x,y)\frac{tial}{tialy^i}$, where G^i are the spray coefficients given by,

$$G^i(x,y) = \frac{1}{4}g^{ij}(x,y)\{2\frac{tialg_{jl}}{tialx^k}(x,y) - \frac{tialg_{jk}}{tialx^l}(x,y)\}y^jy^k, \tag{2}$$

where $(g^{ij})$is the inverse matrix of (g_{ij}). For the Berwald connection the coefficients G^i_j, G^i_{jk} of spray G^i defined as,

$$G^i = \frac{tialG^i}{tialy^j},\ \ G^i_{jk} = \frac{tialG^i_j}{tialy^k}.$$

In Finsler space, Riemannian curvature tensor R_y is the function $R_y = R^i_k(y)dx^k \otimes \frac{tial}{tialx^i}|_x : T_xM \to T_xM$ is defined as,

$$R^i_k(y) = 2\frac{tialG^i}{tialx^k} - \frac{tial^2G^i}{tialx^itialy^k}y^j + 2G^j\frac{tialG^i}{tialy^jtialy^k} - \frac{tialG^i}{tialy^j}\frac{tialG^j}{tialy^k}. \tag{3}$$

If $L = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric, then $R^i_k = R^i_{jkl}(x)y^jy^l$, where $R^i_{jkl}(x)$ denote the coefficients of Riemannian curvature tensor. Thus, R_y is called Riemannian curvature in Finsler geometry. With respect to the Riemannian curvature, Ricci scalar function for the Finsler metric defined by $\rho = \frac{1}{L^2}R^i_i$, which is positive homogeneous function of degree 0 in y [3]. It shows that $\rho(x,y)$ depends on the direction of the flag pole y, but not its length. Then the Ricci tensor given by,

$$Ric_{ij} = \{\frac{1}{2}R^i_j\}_{y^iy^j}. \tag{4}$$

If the Ricci tensor on a manifold becomes zero, then such manifold called as Ricci-flat [3]. The Ricci tensor plays major role Finsler geometry to study the Einstein criterion for Finsler spaces. A Finsler metric becomes Einstein metric if the Ricci scalar function is a function of x-alone [3]. i.e.,

$$Ric_{ij} = \rho(x) g_{ij}. \tag{5}$$

Let (M, L) be an n-dimensional Finsler space equipped with an (α, β)-metric L, where $\alpha(y) = \sqrt{a_{ij} y^i y^j}$, $\beta(y) = b_i(x) y^i$. In [8], M. Matsumoto, proved that G^i of (α, β)-metric space are given by,

$$2G^i = \gamma^i_{00} + 2B^i, \tag{6}$$

where,

$$\begin{aligned}
&B^i = (E/\alpha) y^i = (\alpha L_\beta / L_\alpha) s^i_0 - (\alpha L_{\alpha\alpha}/L_\alpha) C\{(y^i/\alpha) - (\alpha/\beta) b^i\}, \qquad (7)\\
&E = (\beta L_\beta / L) C, \quad C = \alpha\beta(r_{00} L_\alpha - 2\alpha s_0 L_\beta)/2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}),\\
&b^i = a^{ir} b_r, \quad b^2 = b^r b_r, \quad \gamma^2 = b^2\alpha^2 - \beta^2,\\
&r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}),\\
&s^i_j = a^{ih} s_{hj}, \quad s_j = b_i s^i_j.
\end{aligned}$$

where "|" in the above formula stands for the h-covariant derivation with respect to the Riemannian connection in the space (M, α), and the matrix (a^{ij}) denotes the inverse of matrix (a_{ij}). The functions γ^i_{jk} stand for the Christoffel symbols in the space (M, α). Now (3) is re-written as

$$B^i = (\tilde{p} r_{00} + \tilde{q}_0 s_0) y^i + \tilde{r} s^i_0 + (\tilde{s}_0 r_{00} + \tilde{t} s_0) b^i, \tag{8}$$

where

$$\tilde{p} = \beta(\beta L_\alpha L_\beta - \alpha L L_{\alpha\alpha})/2L(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}), \tag{9}$$

$$\tilde{q} = -\alpha\beta L_\beta(\beta L_\alpha L_\beta - \alpha L L_{\alpha\alpha})/L L_\alpha(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}), \tag{10}$$

$$\tilde{r} = \alpha L_\beta / L_\alpha \tag{11}$$

$$\tilde{s}_0 = \alpha^3 L_{\alpha\alpha}/2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}), \tag{12}$$

$$\tilde{t} = -\alpha^4 L_{\alpha\alpha} L_\beta / L_\alpha(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}). \tag{13}$$

Substituting (4) in (2) and (1), we obtain Berwalds formula for Riemannian curvature tensor as follows:

$$K^i_k(y) = \bar{K}^i_k + \{2B^i_{|k} - y^j (B^i_{|j})_{y^k} - (B^i)_{y^j} (B^j)_{y^k} + 2B^j (B^i)_{y^j y^k}\}. \tag{14}$$

The 1-form β is said to be Killing (closed) 1-form if $r_{ij} = 0$ ($s_{ij} = 0$ respectively). β is said to be a constant Killing form if it is Killing and has constant length with respect to α, equivalently $r_{ij} = 0$, $s_i = 0$.

3. Riemannian Curvature of Finsler Space with Special (α, β)-metrics

In this section, we consider Finsler space with special (α,β)-metric $L=\mu\frac{\alpha^2}{\beta}+\nu\frac{\beta^2}{\alpha}$. For this metric we derive the Riemannian curvature. The partial derivatives with respect to both α and β respectively given by,

$$L_\alpha = 2\mu\frac{\alpha}{\beta}-\nu\frac{\beta^2}{\alpha^2},\;\; L_\beta = -\mu\frac{\alpha^2}{\beta}+2\nu\frac{\beta}{\alpha}. \tag{15}$$

Now by equation (11) we have,

$$\tilde{r}=\frac{\alpha^2(-\mu\alpha^3+2\nu\beta^3)}{\beta(2\mu\alpha^3-\nu\beta^3)}. \tag{16}$$

Suppose that β is a constant Killing form, then by substituting (16) in (8), we get

$$B^i=\frac{(-\mu\alpha^5+2\nu\alpha^2\beta^3)}{2\mu\alpha^3\beta-\nu\beta^4}s^i_0. \tag{17}$$

Now, by Covariant and contravariant differentiation of (17), we obtained that,

$$B^i_{.j} = \frac{C_1y_j}{(2\mu\alpha^3\beta-\nu\beta^4)^2}s^i_0+\frac{C_2b_j}{(2\mu\alpha^3\beta-\nu\beta^4)^2}s^i_0+\frac{(-\mu\alpha^5+2\nu\alpha^2\beta^3)}{(2\mu\alpha^3\beta-\nu\beta^4)}s^i_j, \tag{18}$$

$$B^i_{|j} = \frac{C_3b_{0|j}}{(2\mu\alpha^3\beta-\nu\beta^4)^2}s^i_0+\frac{(-\mu\alpha^5+2\nu\alpha^2\beta^3)}{(2\mu\alpha^3\beta-\nu\beta^4)}s^i_{0|j}, \tag{19}$$

where

$$\begin{aligned} B^i_{.j} &= B^i_{y^j},\\ C_1 &= 3\mu^2\alpha^5\beta-5\mu^2\alpha^3\beta+4\mu\nu\alpha^3\beta^4-6\mu\nu\alpha^2\beta^4+\frac{5}{2}\mu\nu\beta^4-2\nu^2\beta^7,\\ C_2 &= 8\mu\alpha^5\beta^2-4\mu\nu\alpha^5\beta^2-6\nu\alpha^2\beta^6+\nu^2\alpha^2\beta^5+2\mu^2\beta^8,\\ C_3 &= 12\mu\nu\alpha^5\beta^2-8\mu\nu\alpha^5\beta^3-6\nu^2\alpha^2\beta^5+4\nu^2\alpha^2\beta^6+2\mu^2\alpha^8 \end{aligned}$$

From (18), we have,

$$B^iB^i_{.j.i} = 0, \tag{20}$$

$$B^i_{.j}B^j_{.i} = \frac{C_1^2\alpha^2}{(2\mu\alpha^3\beta-\nu\beta^4)^4}s^i_0s_{0i}+\frac{(-\mu\alpha^5+2\nu\alpha^2\beta^3)^2}{(2\mu\alpha^3\beta-\nu\beta^4)^2}s^{ij}s_{ij}. \tag{21}$$

And differentiate (19) with respect to y^i and transvecting by y^j, we get

$$y^j(B^i_{|j})_{.i}=0. \tag{22}$$

Finally by substituting (18) to (22) in (14), we obtain,

$$\begin{aligned} R^i_i &= \overline{R}^i_i+\frac{2(-\mu\alpha^5+2\nu\alpha^2\beta^3)}{(2\mu\alpha^3\beta-\nu\beta^4)}s^i_{0|i}-\frac{C_3\alpha^2}{(2\mu\alpha^3\beta-\nu\beta^4)^4}s^i_0s_{0i}\\ &\quad -\frac{(-\mu\alpha^5+2\nu\alpha^2\beta^3)^2}{(2\mu\alpha^3\beta-\nu\beta^4)^2}s^{ij}s_{ij}, \end{aligned} \tag{23}$$

where

$$C_3 = 9\mu^4\alpha^{10}\beta^2-30\mu^4\alpha^8\beta^2+24\mu^3\nu\alpha^8\beta^5-30\mu^3\nu\alpha^7\beta^5+(25\mu^4\beta^2+16\mu^2\nu^2\beta^8)\alpha^6$$

$$
\begin{aligned}
&+ \ (\frac{30}{4}\mu^3\nu\beta^5 + 60\mu^3\nu\beta^5 - 60\mu^2\nu^2\beta^8)\alpha^5 s^{ij}s_{ij} + 36\mu^2\nu^2\beta^8\alpha^4 \\
&+ \ (40\mu^2\nu^2\beta^8 - \frac{50}{4}\mu^3\nu\beta^5 - 16\mu\nu^3\beta^{11})\alpha^3 + (24\mu\nu^3\beta^{11} - 30\mu^2\nu^2\beta^8)\alpha^2 \\
&+ \ (\frac{25}{4}\mu^2\nu^2\beta^8 - 10\mu\nu^3\beta^{11} + 4\nu^{14}).
\end{aligned}
$$

In the above equation $\overline{R}^i_i$ is the Riemannian curvature of the Finsler space, thus we state the following,

Theorem 3.1. *The Riemannian curvature of the Finsler space with special (α,β)-metric $L = \mu\frac{\alpha^2}{\beta} + \nu\frac{\beta^2}{\alpha}$, with β as constant Killing form is of the form (23).*

4. Einstein Criterion for Finsler Space with Special (α, β)-metrics

A Finsler metric $L = L(x, y)$ on an n-dimensional manifold M is called an Einstein metric if the Ricci scalar satisfies the following condtion,

$$
Ric = (n-1)\lambda L^2, \tag{24}
$$

where $\lambda = \lambda(x)$ is a scalar function on M. L is Ricci constant if λ is constant [3]. In this section we consider the Finsler space with special (α,β)-metric $L = \mu\frac{\alpha^2}{\beta} + \nu\frac{\beta^2}{\alpha}$ and characterize the Einstein criterion. Now, we suppose the Ricci scalar of the mentioned (α,β)-metric is the function of x alone, i.e., L is Einstein, then we have $L^2 Ric(x) = R^i_i$, so we can derive the necessary and sufficient conditions for this to be Einstein. From(23), we have,

$$
\begin{aligned}
0 \ = \ & \overline{Ric}_{00} + \frac{2(-\mu\alpha^5 + 2\nu\alpha^2\beta^3)}{(2\mu\alpha^3\beta - \nu\beta^4)} s^i_{0|i} - \frac{C_3\alpha^2}{(2\mu\alpha^3\beta - \nu\beta^4)^4} s^i_0 s_{0i} \\
- \ & \frac{(-\mu\alpha^5 + 2\nu\alpha^2\beta^3)^2}{(2\mu\alpha^3\beta - \nu\beta^4)^2} s^{ij}s_{ij} - \left(\mu\frac{\alpha^2}{\beta} + \nu\frac{\beta^2}{\alpha}\right)^2 Ric(x).
\end{aligned} \tag{25}
$$

Multiplying (27) by $\alpha^2\beta^2(2\mu\alpha^3\beta - \nu\beta^4)^4$ removes y from the denominators and after simplification we obtained as follows:

$$
Rat + \alpha Irrat = 0, \tag{26}
$$

where

$$
\begin{aligned}
Rat \ = \ & (16\mu^4\alpha^{14}\beta^6 + 24\mu^2\nu^2\alpha^8\beta^{12} + \nu^4\alpha^2\beta^{12})\overline{Ric}_{00} \\
- \ & (48\mu^2\nu^2\alpha^{10}\beta^9 + 4\nu^4\alpha^4\beta^{17} + 16\mu^4\alpha^{16}\beta^5 + 12\mu^2\nu^2\alpha^{10}\beta^{11})s_{0|i} \\
- \ & 4[9\mu^4\alpha^{14}\beta^4 - 30\mu^4\alpha^{12}\beta^4 + 24\mu^3\nu\alpha^{12}\beta^7 + (25\mu^4\beta^4 + 16\mu^2\nu^2\beta^{10})\alpha^{10} \\
+ \ & 36\mu^2\nu^2\alpha^8\beta^{10} + (24\mu\nu^3\beta^{13} - 30\mu^2\nu^2\beta^{10})\alpha^6 + (\frac{25}{4}\mu^2\nu^2\beta^{10} - 10\mu\nu^3\beta^{13} + 4\nu^4\beta^{16})\alpha^4]s^i_0 s_{0i} \\
- \ & (4\mu^4\alpha^{10}\beta^4 + 33\mu^2\nu^2\alpha^{12}\beta^{10} + 4\nu^4\alpha^6\beta^{16})s^{ij}s_{ij} - (4\mu^4\alpha^{12}\beta^2 - 6\mu^2\nu^2\alpha^6\beta^8 + \nu^4\beta^{14})Ric(x)
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
Irrat \ = \ & (-32\mu^3\nu\alpha^{10}\beta^9 - 2\mu\nu^3\alpha^4\beta^{15})\overline{Ric}_{00} + (32\mu^3\nu\alpha^{12}\beta^8 + 24\mu^3\nu\alpha^{12}\beta^6 + 22\mu\nu^3\alpha^6\beta^{14})s^i_{0|i} \\
- \ & \left(-30\mu^3\nu\alpha^{10}\beta^7 + (\frac{270}{4}\mu^3\nu\beta^7 - 60\mu^2\nu^2\beta^{10})\alpha^8 + (40\mu^2\nu^2\beta^{10} - 16\mu\nu^3\beta^{13} - \frac{50}{4}\mu^2\nu\beta^7)\alpha^6\right) s^i_0 s_{0i} \\
+ \ & (20\mu^3\nu\alpha^{14}\beta^7 + 20\mu\nu^3\alpha^8\beta^{13})s^{ij}s_{ij} - (4\mu^3\nu\alpha^8\beta^5 - 2\mu\nu^3\alpha^2\beta^{11})Ric(x).
\end{aligned} \tag{28}
$$

Here Rat and $Irrat$ are polynomials of degree 12 and 10 in y respectively. According to the above we state the Einstein criterion as follows,

Lemma 4.1. *A Finsler space with special (α,β)-metric $L = \alpha + \beta - \frac{\beta^2}{\alpha}$ with constant Killing form β is Einstein if and only if both $Rat = 0$ and $Irrat = 0$ hold.*

Proof. Let $Rat = P(y)$ and $Irrat = Q(y)$. We know that α can never be polynomial in y. Otherwise, the quadratic $\alpha^2 = a_{ij}(x)y^i y^j$ would have been factored into linear term. It's zero set would then consist of a hyperplane, contradicting the positive definiteness of a_{ij}. Now, suppose the polynomial Rat is not zero. Then the above equation would imply that it is the product of polynomial $Irrat$ with a non-polynomial factor α, this is not possible. So Rat must vanish and, since α is positive at all $y \neq 0$, we see that $Irrat$ also must be zero. Hence the proof. Now if $Rat = 0$, then we have

$$0 = \alpha^2 C_1 + C_2 \tag{29}$$

Where C_1 and C_2 are as follows:

$$\begin{aligned}
C_1 &= (16\mu^4\alpha^{12}\beta^6 + 24\mu^2\nu^2\alpha^6\beta^{12} + \nu^4\beta^{12} + 16\mu^2\nu^2\alpha^8\beta^{12})\overline{Ric}_{00} \\
&- (48\mu^2\nu^2\alpha^8\beta^9 + 4\nu^4\alpha^2\beta^{17} + 16\mu^4\alpha^{14}\beta^5 + 12\mu^2\nu^2\alpha^8\beta^{11})s_{0|i} \\
&- 4[9\mu^4\alpha^{12}\beta^4 - 30\mu^4\alpha^{10}\beta^4 + 24\mu^3\nu\alpha^{10}\beta^7 + (25\mu^4\beta^4 + 16\mu^2\nu^2\beta^{10})\alpha^8 \\
&+ 36\mu^2\nu^2\alpha^6\beta^{10} + (24\mu\nu^3\beta^{13} - 30\mu^2\nu^2\beta^{10})\alpha^4 + (\frac{25}{4}\mu^2\nu^2\beta^{10} - 10\mu\nu^3\beta^{13} + 4\nu^4\beta^{16})\alpha^2]s^i_0 s_{0i} \\
&- (4\mu^4\alpha^8\beta^4 + 33\mu^2\nu^2\alpha^{10}\beta^{10} + 4\nu^4\alpha^4\beta^{16})s^{ij}s_{ij} - (4\mu^4\alpha^{10}\beta^2 - 6\mu^2\nu^2\alpha^4\beta^8)Ric(x) \\
C_2 &= -\nu^4\beta^{14}Ric(x).
\end{aligned}$$

Thus, by (29) we conclude that α^2 divides C_2 and so $\beta = 0$. Then the Finsler metric is Riemannian. □

Thus we state that

Theorem 4.2. *An Einstein Finsler metric $L = \mu\frac{\alpha^2}{\beta} + \nu\frac{\beta^2}{\alpha}$ with constant killing form β. Then L is Einstein if and only if L is Riemannian Einstein metric, i.e., Ricci flat.*

5. Conclusion

The Einstein metrics comprise a major focus in differential geometry and mainly connect with gravitation in general relativity. In particular, Einstein metric are solutions to Einstein field equations in general relativity containing the Ricci-flat metric. Einstein Finsler metric which represent a non Riemannian stage for the extensions of metric gravity provide an interesting source of geometric issues and the (α, β)-metric is an important class of Finsler metric appearing frequently in the study of applications in Physics. In this paper we consider a special (α, β)-metric such as $L = \mu\frac{\alpha^2}{\beta} + \nu\frac{\beta^2}{\alpha}$. For this (α, β)-metric, we obtain Riemannian curvature. Further we find the necessary and sufficient conditions for this (α, β)-metric to be Einstein metric, when β is a constant Killing form. Finally we prove that the above mentioned Einstein metric must be Riemannian or Ricci flat.

References

[1] P.L.Antonelli, A.Bna and M.A.Slawiski, *Seismic rays as Finsler geodesics*, Nonlinear Anal. RWA, 4(2003), 711722.

[2] D.Bao, S.S.Chern and Z.Shen, *An introduction to Riemann-Finsler geometry*, Springer, (2000).

[3] D.Bao and C.Robles, *On Ricci and flag curvatures in Finsler geometry*, MSRI Series. Cambridge University Press, (2003).

[4] J.P.Bourguignon, *Ricci curvature and Einstein metrics*, In Global Differential Geometry and Global Analysis, (1979), 42-63.

[5] K.Chandru and S.K.Narasimhamurthy, *On curvatures of homogeneous Finsler-Kropina space*, Gulf Journal of Mathematics, 5(1)(2017), 73-83.

[6] X.Cheng, *A class of Einstein* (α, β)*-metrics*, Israel Journal of Mathematics, 192(2012), 221-249.

[7] X.Cheng, Z.Shen and Y.Tian, *A class of Einstein* (α, β)*-metrics*, preprint, (2010).

[8] M.Matsumoto, *Foundations of Finsler geometry and Finsler spaces*, Japan: Kaiseisha Press, (1986).

[9] M.Matsumoto, *The Berwald connection of Finsler with an* (α, β)*-metric,*, Tensor N.S., 50(1991), 1821.

[10] M.Rafie-Rad and B.Razavi, *On Einstein Matsumoto metrics*, Elsevier, Nonlinear Analysis: Real World Applications, 13(2012), 882-886.

[11] M.Rafie-Rad, *Time-optimal solutions of parallel navigation and Finsler geodesics*, Nonlinear Anal. RWA, 11(2010), 38093814.

[12] B.Rezaei, A.Razavi and N.Sadeghzadeh, *On Einstein* (α, β) *-metrics*, Iranian Journal of Science and Technology, 31(A4), 403-412.

[13] C.Robles, *Einstein Metrics of Randers type*, Ph.D Thesis, British Columbia, (2003).

[14] Z.Shen, *On Projectively related Einstein metrics in Riemann-Finsler geometry*, Math. Ann., (2001), 625-647.

[15] Z.Shen, *Differential Geometry of Spray and Finsler Spaces*, Dordrecht, Kluwer Academic Publishers, (2001).